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# BIREGULAR SEMIGROUPS II

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In this paper we continue the investigation of biregular semigroups started in [3]. In section one we consider two types of prime ideals and using M. PETRICH's results of [6], we show that in a biregular semigroup these two types coincide if and only if the semigroup is a semilattice of 0-simple semigroups. In the following section 0-minimal one-sided and maximal ideals are considered and it is shown that they correspond to primitive and maximal idempotents, respectively. Finally, it is shown that if a biregular semigroup without zero admits a faithful transitive representation as the semigroup of mappings on a set, then it is simple.

A semigroup $S$ is said to be *biregular* if every principal two-sided ideal of $S$ is generated by an idempotent in the center of $S$. Throughout this paper we shall use, ideal, for two-sided ideal. The standard terminology and notation used is that of [2]; and for biregular semigroups that of [3].

If $S$ is a biregular semigroup then $E = E\,Z(S)$ will denote the semilattice of central idempotents of $S$ and for each element $a$ of $S$, $e(a)$ will denote the unique central idempotent generator of $J(a)$.

The most natural examples of biregular semigroups are simple semigroups with identity and semilattices.

We shall now consider a class of biregular semigroups built from these two natural ones.

## 1. BIREGULAR $p$-SEMIGROUPS

A proper ideal $I$ of a semigroup $S$ is said to be *completely prime* if $S - I$ is a subsemigroup of $S$. A proper ideal $I$ of $S$ is said to be *prime* if whenever $AB \subseteq I$ then $A \subseteq I$ or $B \subseteq I$ for any two ideals $A$ and $B$ of $S$.

It is clear that a completely prime ideal is prime; and if $S$ is commutative, these two concepts coincide. We shall say that a semigroup $S$ is a *p-semigroup* if every prime ideal of $S$ is completely prime.

The following can be easily proved for arbitrary semigroups; the proof of the corresponding results for rings can be seen e.g. in N. H. McCoy's *Prime ideals in general rings*, Amer. J. Math. *71* (1949).

**Lemma 1.1.** *The following conditions are equivalent for a proper ideal $I$ of a semigroup $S$.*

(1) *$I$ is prime.*

(2) *If $aSb \subseteq I$ then $a \in I$ or $b \in I$ for all $a, b \in S$.*

(3) *If $J(a)\,J(b) \subseteq I$ then $a \in I$ or $b \in I$ for all $a, b \in S$.*

Let us note that if $S$ is a biregular semigroup then $E$ is commutative and thus in $E$ the concepts of prime and completely prime coincide.

**Theorem 1.1.** *If $S$ is a biregular semigroup and $E$ is its semilattice of central idempotents then an ideal $P$ of $E$ is prime if and only if $SP$ is a prime ideal of $S$.*

Proof. Suppose $P$ is a prime ideal of $E$. Let $A$ and $B$ be ideals of $S$ such that $AB \subseteq SP$ and $A \nsubseteq SP$. Let $a \in A$ be such that $a \notin SP$; thus $e(a) \in A$ and $e(a) \notin P$. Let $b \in B$, then $e(b) \in B$ and $e(a)\,e(b) \in AB \subseteq SP$, say, $e(a)\,e(b) = sg$ for some $s \in S$ and $g \in P$. Then, $e(a)\,e(b) = sg = sg^2 = e(a)\,e(b)\,g \in EP \subseteq P$ and since $P$ is prime in $E$ and $e(a)\,T \notin P$, we have $e(b) \in P$; hence $b = b\,e(b) \in SP$; thus $B \subseteq SP$ and so $SP$ is a prime ideal of $S$.

Conversely, suppose that $P$ is a prime ideal of $S$ and let $J = E \cap P$. Clearly $J$ is a proper subset of $E$. Let $e \in E$ and $f \in J$, then $ef = fe \in E$ and also $ef \in P$, so $J$ is an ideal of $E$. Suppose $A$ and $B$ are ideals of $E$ such that $AB \subseteq J$ and $A \nsubseteq J$; say $e \in A$ and $e \notin J$. If $f \in B$, then $ef \in AB \subseteq J$, thus $ef \in P$, therefore $(eS)(fS) = efS \subseteq P$ and since $e \notin J$, $e \notin P$ and so $eS \nsubseteq P$. Hence since $P$ is prime in $S$, $fS \subseteq P$, thus $f \in P$ and so $f \in J$ and therefore $J$ is a prime ideal of $E$.

Moreover, $SJ \subseteq P$ since $J \subseteq P$; if $x \in P$, then $e(x) \in J$ and so $x = x\,e(x) \in SJ$; hence $P = SJ$.

In the remainder of this section we will assume familiarity with the notation and results of [6].

Let $e \in E = E\,Z(S)$, where $S$ is a biregular semigroup. If $S_e = \{x \in S : e(x) \ngeqq e\}$, then in [3] we have shown that $S_e$ is an ideal of $S$. This can be strengthened to

**Lemma 1.2.** *$S_e$ is a prime ideal of the biregular semigroup $S$ for all $e \in E$.*

Proof. Suppose $a, b \in S$ are such that $aSb \subseteq S_e$. Then $S(aSb) \subseteq S_e$ or $SaS\,.\,b \subseteq$ $\subseteq S_e$, therefore $e(a)\,b \in S_e$. Let $x, y \in S$ be such that $xby = e(b)$, then $x\,e(a)\,by \in S_e$, but $x\,e(a)\,by = xby\,e(a) = e(b)\,e(a)$. If $a, b \notin S_e$, then $e(a) \geqq e$ and $e(b) \geqq e$ hence $e(a)\,e(b) \geqq e$ and so $e(a)\,e(b) \notin S_e$, a contradiction.

For each $e \in E$, let $T_e = S - S_e$ and let $E^*$ be the maximal semilattice homomorphic image of $S$. As in [6], a *face* $N$ of a semigroup $S$ will denote the complement of a completely prime ideal of $S$ and $N(x)$ the intersection of all faces of $S$ containing the element $x$ of $S$.

**Lemma 1.3.** *If $\alpha$ is a congruence on the biregular semigroup $S$ then $(x, y) \in \alpha$ implies that $(e(x), e(y)) \in \alpha$ for all $x, y \in S$.*

Proof. Let $(x, y) \in \alpha$ and let $a, b, c, d \in S$ be such that $e(x) = axb$ and $e(y) = cyd$. Then, $(axb, ayb)$ and $(cxd, cyd) \in \alpha$, i.e. $(e(x), ayb)$ and $(cxd, e(y)) \in \alpha$. Now, $(e(x), ayb) \in \alpha$ implies that $(e(x)\, e(y), ayb) \in \alpha$ and so by transitivity $(e(x), e(x)\, e(y)) \in \alpha$. Similarly, $(e(x)\, e(y), e(y)) \in \alpha$; hence $(e(x), e(y)) \in \alpha$.

**Theorem 1.2.** *The following conditions are equivalent for a biregular semigroup $S$.*

(1) *$S$ is a $p$-semigroup.*

(2) *$E = E^*$.*

(3) *$\mathscr{J}$ is a congruence on $S$.*

(4) *$S$ is intraregular.*

Proof. Suppose $S$ is a $p$-semigroup. Since by lemma 1.2 $S_e$ is a prime ideal of $S$, then $S_e$ is a completely prime ideal of $S$ and so $T_e$ is a face of $S$. Let $x \in S$ and let $N$ be a face of $S$ containing $x$, then $x = x\, e(x) \in N$ and so $e(x) \in N$. Let $y \in T_{e(x)}$, then $e(y)\, e(x) = e(x) \in N$ and so $e(y) \in N$ and thus $y \in N$. Therefore $T_{e(x)} \subseteq N$ and since $T_{e(x)}$ is itself a face of $S$, $T_{e(x)} = N(x)$.

Let $N_x = \{y \in S : N(x) = N(y)\}$. By the above $N(x) = N(y)$ if and only if $(x, y) \in \mathscr{J}$ and so $N_x = J_x$ and since $E^* = \{N_x : x \in S\}$ we have that $E = E^*$.

Conversely suppose that $E = E^*$. Then in particular $\mathscr{J}$ is a congruence. If $(x, y) \in \mathscr{J}$, then $(x, y) \in \eta \circ \eta^{-1}$, where $\eta$ denotes the natural homomorphism of $S$ onto $E^* = E$. Let $P$ be a prime ideal of $S$, then by theorem 1.1 $P' = E \cap P$ is a prime ideal of $E$ and so by corollary 3.7 of [6], $\eta^{-1}(P')$ is a completely prime ideal of $S$. But, $P = SP'$ and so $\eta(P) = \eta(SP') = \eta(S)\, \eta(P') \subseteq E^*P' = EP' \subseteq P'$, thus $P \subseteq \eta^{-1}(P')$. Now, let $x \in \eta^{-1}(P')$. Then since $\eta \circ \eta^{-1} = \mathscr{J}\eta(e(x)) \in P'$ and since $\eta$ is the identity on $E$ we have $e(x) \in P'$ and so $x = x\, e(x) \in SP' = P$. Therefore $P = \eta^{-1}(P')$ and so $P$ is a completely prime ideal of $S$, proving the equivalence of (1) and (2).

Since whenever $E = E^*$, $\eta \circ \eta^{-1} = \mathscr{J}$, (2) implies (3). Now, if $x, y \in S$ are such that $(x, y) \in \mathscr{J}$, then $(\eta(a), \eta(b)) \in \mathscr{J}$ in $E^*$ and since $E^*$ is a semilattice this implies that $\eta(a) = \eta(b)$, therefore $\mathscr{J} \subseteq \eta \circ \eta^{-1}$. If $\mathscr{J}$ is a congruence, since $E^*$ is the maximal semilattice homomorphic image of $S$, we must have $\mathscr{J} = \eta \circ \eta^{-1}$ and thus $E = E^*$. Proving the equivalence of (2) and (3).

By theorem 4.3 of [6], $N_x = J_x$ if and only if for all $x \in S$, $N_x$ is simple and by theorem 4.4 of [2], this is equivalent to intraregularity of $S$. This proves the equivalence of (3) and (4).

From this theorem we can derive a characterization of biregular $p$-semigroups similar to A. H. Clifford's description of inverse semigroups which are union of groups, (see [1]).

**Corollary 1.2.** *Let $E$ be a semilattice. To each element $\alpha \in E$ assign a simple semigroup $S_\alpha$ with identity $e_\alpha$ such that $S_\alpha \cap S_\beta = \Phi$ if $\alpha \neq \beta$. For every $\alpha, \beta \in E$ such that $\alpha \geqq \beta$ assign a homomorphism $\varphi_{\alpha,\beta}$ of $S_\alpha$ into $S_\beta$ such that*

(1) $e_\alpha \varphi_{\alpha,\beta} \leqq e_\beta$,

(2) $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$ *for all* $\alpha \geqq \beta \geqq \gamma$ *in* $E$,

(3) $\varphi_{\alpha,\alpha}$ *is the identity automorphism of* $S_\alpha$.

*Let $S$ be the union of the $S_\alpha$ and define*

(4) $a_\alpha b_\beta = (a_\alpha \varphi_{\alpha,\alpha\beta})(b_\beta \varphi_{\beta,\alpha\beta})$ *for all* $a_\alpha \in S_\alpha$ *and* $b_\beta \in S_\beta$.

*Then, $S$ is a biregular $p$-semigroup and conversely every biregular $p$-semigroup can be obtained in this way.*

Proof. The proof is straightforward and similar to the proof of the corresponding result in [1].

Note. If $S$ is an arbitrary semigroup and $I$ is a semiprime ideal of $S$ (i.e., for all $x$ in $S$, $x^2 \in I$ implies $x \in I$) and $I \neq S$, then the ideal $M$ of $S$, maximal among ideals of $S$ that are disjoint from $\langle b \rangle$ where $b \in S - I$, is a prime ideal of $S$ (not necessarily completely prime) and thus: *every semiprime ideal of $S$ is the intersection of prime ideals of $S$. In particular, if $S$ is a biregular $p$-semigroup then every ideal of $S$ is the intersection of prime ideals of $S$.*

## 2. MAXIMAL AND MINIMAL IDEALS IN BIREGULAR SEMIGROUP

**Theorem 2.1.** *Let $S$ be a biregular semigroup. If $R$ is a 0-minimal right ideal of $S$ then $R = eS$ where $e \in E\,Z(S)$ is a primitive idempotent of $S$. In particular, $R$ is also a 0-minimal left ideal of $S$ and a 0-minimal two-sided ideal of $S$ and thus a group with zero.*

Proof. Let $R$ be a 0-minimal right ideal of $S$. Let $a$ be a non-zero element of $R$ and set $e = e(a)$. Now, since $a \neq 0$, $aS \neq 0$, thus $aS = R$. Let $f \in E$ be such that $f \leqq e$. Since $afS \subseteq aS = R$ either $afS = 0$ or $afS = R$. If $afS = 0$, then $afe = af = 0$ and if $e = sat$ for $s, t \in S$, then $f = ef = satf = saft = 0$. If $afS = aS$, then $a = afu$ for some $u \in S$ and hence $af = fa = a$, therefore $e = sat = satf = ef = f$. Thus, $e$ is primitive in $E$ and so $J(e)$ is 0-minimal.

Now, $R^2 \neq 0$ since $e \in R$, thus there is a $b \in R$ such that $bR \neq 0$, hence $bR = R = bS$. Let $e = e(b)$ and $x, y \in S$ be such that $e = xby$. If $f = byx$, then $f^2 = f \in bS = R$ and since $e = xbyxby = xfby, f \neq 0$, thus $fS = R$. Suppose $g \in E(S)$ is such that $g \leqq f$, then by an argument similar to above $g = 0$ or $g = f$ and so $f$ is primitive and since $f \in J(e)$ which is a 0-minimal two-sided ideal of $S$ and thus a 0-simple subsemigroup of $S$, $J(e)$ is a group with zero and so $e = f \in E$. Since $R = J(e)$ is a group with zero it is also a 0-minimal left ideal of $S$.

**Corollary 2.1.** *A biregular semigroup $S$ has minimal one-sided ideals if and only if it has primitive idempotents in the center.*

Proof. The necessity follows from the theorem. If $e \in E$ is a primitive idempotent of $S$, then for $a \in J(e)$, $a \neq 0$, if $e(a) = xay$, then $f = ayx$ is an idempotent in $aS$ and thus $f \leqq e(a) = e$. Hence $f = 0$ or $f = e$. If $f = 0$, then $e = xfay = 0$ thus $f = e$ and so $aS = eS = J(e)$ is a 0-minimal right ideal of $S$.

Let $S^*$ denote the *right socle* of $S$, i.e. the union of the 0-minimal right ideals of $S$. Then by theorem 2.1 $S^*$ is also the left and two-sided socle of $S$ and is a 0-direct union of groups.

In section one we saw that $S_e$ is a prime ideal of $S$ for all $e \in E$. If $e$ is primitive we can state moreover:

**Theorem 2.2.** *If $e$ is a non-zero idempotent in the socle $S^*$ of $S$, then $S_e$ is a completely prime ideal of $S$.*

Proof. Let $x \in S - S_e = T_e$. If $xe = 0$, then since $x \neq 0$ $e(x) = sxt$ for some $s, t \in S$ and hence $e = e\, e(x) = esxt = sxet = 0$, a contradiction. Thus for all $x \in T_e$ $xe \neq 0$.

Let $x, y \in T_e$ be such that $xy = 0$, then $xey = 0$, but since $xe \neq 0$, $xe \in H_e$ and so there is a $z \in H_e$ such that $z \,.\, xe = e$. Thus $z \,.\, xey = ey = 0$, a contradiction. Now, $xy \neq 0$ implies that $e(xy) \neq 0$. But, $e(xy) \leqq e(x)\, e(y)$ and so $e \,.\, e(xy) \leqq e \,.\, e(x) \,.\, e(y) = e$ and since $e(xy) \,.\, e \neq 0$ [for otherwise $xey = 0$], we must have $e \leqq e(xy)$ and so $xy \in T_e$.

We can easily derive from this theorem the following

**Corollary 2.2.** *The left annihilator $S^*A$ of $S^*$ in $S$ is also the right and two sided annihilator of $S^*$ in $S$ and is the intersection of completely prime ideals of $S$.*

Note that $S^*A$ need not be trivial. E.g. take $S$ to be the 0-direct union of a group $G$ and the real interval $I = [0, 1]$, where multiplication in $I$ is given by $x \,.\, y = \min(x, y)$. Then, $S$ is a biregular semigroup with $S^* = G \cup 0$ and $S^*A = 1$.

**Theorem 2.3.**[1]) *Let $S$ be a biregular semigroup. A proper ideal $M$ of $S$ is a maximal ideal of $S$ if and only if $M = S - J_e$, where $e$ is a maximal idempotent of $S$ in the center of $S$.*

Proof. Suppose $M$ is a proper maximal ideal of $S$; then $S/M$ is a 0-simple biregular semigroup with identity and the center $Z(S/M)$ is therefore a group with zero. Hence, $S - M$ contains a unique central idempotent $e$ (central in $S - M$). If $e \notin Z(S)$, then there is an $f \in E$ such that $J(f) = J(e)$ and so $e = fe$. If $f \in M$, then $e \in M$ a contradiction; thus $f \in S - M$ and therefore $f = e$. Now, if $g \in E$ is such that $e \leqq g$, then $e = eg$ and thus $g \in S - M$ and so $g = e$. Therefore $e$ is maximal in $E$. Let $x \in S - M$, then $SxS \subseteq SeS$ and by maximality of $M$, $M \cup SxS = S$ and hence $e \in SxS$; therefore $x \in J_e$. On the other hand if $x \in J_e$, then $e = sxt$ for some $s, t \in S$ and thus $x \in S - M$ [for otherwise $e \in M$]. Thus, $S - M = J_e$.

Conversely, suppose that $e$ is a maximal idempotent in $E$ and let $M = S - J_e$; then it is readily verifiable that $M$ is a maximal ideal of $S$.

## 3. PRIMITIVE BIREGULAR SEMIGROUPS WITHOUT ZERO

We use here the term *primitive* in the sense of Hoehnke [4], i.e. a semigroup admitting a faithful transitive representation as a semigroup of mappings.

Let $\mathcal{S}$ denote the class of simple semigroups with identity. If $S$ is a semigroup, $\mathcal{S} - radS$ be the intersection of all congruences $\sigma$ on $S$ such that $S/\sigma \in \mathcal{S}$.

**Lemma 3.1.** *Let $S$ be a biregular semigroup without zero. Then $\mathcal{S} - radS = \{(x, y) \in SxS : ex = ey$ for some $e \in E\}$. In particular, $\mathcal{S} - radS$ is the identity congruence on $S$ if and only if $S$ is simple with an identity.*

Proof. If $\sigma$ is a congruence on $S$ such that $S/\sigma \in \mathcal{S}$ then since the center of $S$ is mapped into the center of $S/\sigma$ by the natural homomorphism and $Z(S/\sigma)$ is a group. We have for all $e, f \in E$ $(e, f) \in \sigma$. Thus, $(e, f) \in \mathcal{S} - radS$ for all $e, f \in E$. Let $\varrho = \{(x, y) \in SxS : ex = ey$ for some $e \in E\}$. Clearly $\varrho$ is a congruence and for any $e, f \in E$, $(e, f) \in \varrho$. Thus, $S/\varrho \in \mathcal{S}$ and so $\mathcal{S} - radS \subseteq \varrho$. Now, let $\sigma$ be a congruence on $S$ such that $S/\sigma \in \mathcal{S}$; then if $(x, y) \in \varrho$, $ex = ey$ for some $e \in E$ and since $(e, e(x))$ and $(e, e(y)) \in \sigma$, $(x, y) \in \sigma$ so $\varrho \subseteq \sigma$ and hence $\varrho = \mathcal{S} - radS$. Note, that $\varrho$ is the least congruence on $S$ such that $(e, f) \in \varrho$ for all $e, f \in E$. Now, if $\mathcal{S} - radS$ is the identity equivalence on $S$ then $e = f$ for all $e, f \in E$ and thus $S$ is simple with identity. The converse is clear.

[1]) It was brought to the author's attention that P. A. Grillet in his paper *Intersections of Maximal Ideals in Semigroups*, Amer. Math. Monthly 76 (1969), 503—509 has proved a more general statement.

**Theorem 3.1.** *Let $S$ be a biregular semigroup without zero. If $S$ is primitive, then $S$ is simple with an identity.*

Proof. If $S$ is primitive then there is a modular right congruence $\varrho$ on $S$ such that $S/\varrho$ is a transitive $S$-operand, i.e. every element of $S$ is a right unit of $S$ modulo $\varrho$. Since $\varrho$ is modular there is an $i \in S$ such that $(ia, a) \in \varrho$ for all $a \in S$; thus $(iae(i), ae(i)) \in \varrho$ or $(ia, e(i)\, a) \in \varrho$, i.e. $(a, e(i)\, a) \in \varrho$. If $c$ is a right unit of $S$ modulo $\varrho$ then for all $a \in S$ there is an $s \in S$ such that $(cs, a) \in \varrho$. In particular for all $e \in E$ there is an $s \in S$ such that $(es, e(i)) \in \varrho$, therefore $(ese, e(i)\, e) \in \varrho$ and by the above, $(e(i)\, e, e) \in \varrho$ so $(es, e) \in \varrho$ and hence $(e, e(i)) \in \varrho$.

Now, let $\varrho L = \{(a, b) \in SxS : (sa, sb) \in \varrho$ for all $s \in S^1\}$. If $e, f \in E$, then for all $s \in S$, $(se, e(i)\, s) \in \varrho$ and $(e, e(i)) \in \varrho$, so since $(e(i)\, s, s) \in \varrho$ we have $(se, s) \in \varrho$. Similarly $(sf, s) \in \varrho$ thus $(e, f) \in \varrho L$ and so by lemma 3.1 $\varrho L \supseteq \mathcal{S} - radS$. Now, since $S/\varrho$ is faithful, $\varrho L$ is the identity equivalence and thus $S$ is simple.

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