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# ON A RESIDUAL SET OF CONTINUOUS FUNCTIONS

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**1. Introduction.** Let $C$ denote the Banach space of real-valued continuous functions $f$ defined on $[0, 1]$ with the uniform norm $\|f\| = \sup_{0 \leq x \leq 1} |f(x)|$.

BANACH [1] and MAZURKIEWICZ [6] proved in 1931 that the non-differentiable functions $f \in C$ form a residual subset of the space (see SAKS [8, p. 211]). JARNÍK investigated further the nature of Dini derivates of a residual set of functions in $C$ and established the following result [5, p. 49, Satz I] in 1933:

I. "*There exists a residual set of functions* $f \in C$ *each of which possesses the following properties*:

(a) *at every* $x \in (0, 1)$, $[D_- f(x), D^- f(x)] \cup [D_+ f(x), D^+ f(x)] = [-\infty, \infty]$,

(b) *at almost all* $x \in (0, 1)$, $D^+ f(x) = D^- f(x) = \infty$ *and* $D_+ f(x) = D_- f(x) = -\infty$,

(c) *at every* $x \in [0, 1)$, $\max \{|D^+ f(x)|, |D_+ f(x)|\} = \infty$ *and at every* $x \in (0, 1]$, $\max \{|D^- f(x)|, |D_- f(x)|\} = \infty$,

(d) *there exist four nonempty perfect sets* $M^+, M_+, M^-, M_-$ *in* $[0, 1]$ *such that at* $x \in M^+[M_+]$, $D^+ f(x) = D_+ f(x) = \infty[-\infty]$, *and at* $x \in M^-[M_-]$, $D^- f(x) = D_- f(x) = \infty[-\infty]$."

As remarked by Jarník, the part I(c) is due to Banach [1] and Mazurkiewicz [6] and I(d) is due to Saks [8].

The result I can be further strengthened by utilizing some of the author's results on non-differentiable functions [4] and some on continuous nowhere monotone functions [3]. The theorem 1 of the present note is devoted to the same. Whereas Jarník referred to a result of Saks [8] for I(d), we shall see that Saks' result as well as I(d) both are already contained in the parts (a) and (b) of Jarník (see remark 1). Jarník's result also yields results on the nature of all but a countable number of level sets of a residual set of functions in $C$ (see theorems 2 and 3). In theorem 4 we establish independently the existence of a dense set of functions in $C$ of which every level set is perfect and of measure zero.

**2. Derivates of a residual set of functions in** $C$. Let, for every function $f \in C$,

$$E_k(f) = \{x : D^+ f(x) = D^- f(x) = \infty\,,\quad D_+ f(x) = D_- f(x) = -\infty\}\,,$$
$$E_1(f)\,[E_2(f)] = \{x : f'_+(x) = \infty[-\infty]\,,\quad D^- f(x) = \infty\,,\quad D_- f(x) = -\infty\}\,,$$
$$E_3(f)\,[E_4(f)] = \{x : f'_-(x) = \infty[-\infty]\,,\quad D^+ f(x) = \infty\,,\quad D_+ f(x) = -\infty\}\,,$$

and, for every $r \in R$ (the set of real numbers), let

$$E_{1r}(f) = \{x : D^+ f(x) = r \geqq D_- f(x)\,,\quad D_+ f(x) = -\infty\,,\quad D^- f(x) = \infty\}\,,$$
$$E_{2r}(f) = \{x : D_- f(x) = r \leqq D^+ f(x)\,,\quad D_+ f(x) = -\infty\,,\quad D^- f(x) = \infty\}\,,$$
$$E_{3r}(f) = \{x : D_+ f(x) = r \leqq D^- f(x)\,,\quad D^+ f(x) = \infty\,,\quad D_- f(x) = -\infty\}\,,$$
$$E_{4r}(f) = \{x : D^- f(x) = r \geqq D_+ f(x)\,,\quad D^+ f(x) = \infty\,,\quad D_- f(x) = -\infty\}\,.$$

The Jarník's result I can then be strengthened to the following form.

**Theorem 1.** *There exists a residual set of functions* $f \in C$ *for each of which the sets* $E_k(f)$, $E_i(f)$ $(i = 1$ *to* $4)$ *and* $E_{ir}(f)$ $(i = 1$ *to* $4$, $r \in R)$ *cover all the points of* $(0, 1)$, *whereas*

i) $E_k(f)$ *is residual in* $(0, 1)$ *with its measure equal to* $1$,

ii) *each of the sets* $E_i(f)$ $(i = 1$ *to* $4)$ *and* $E_{ir}(f)$ $(i = 1$ *to* $4$, $r \in R)$ *is of the first category with its measure equal to zero and has the power of the continuum in every subinterval of* $(0, 1)$, *and*

iii) *for each* $r \in R$, *the sets* $E_{1r}(f) \cap E_{2r}(f)$ *and* $E_{3r}(f) \cap E_{4r}(f)$ *are both everywhere dense in* $(0, 1)$.

Proof. According to Jarník's result there exists a residual subset $C_0$ of $C$ such that for every function $f \in C_0$ the parts (a), (b) and (c) of $I$ hold.

Let $f \in C_0$. It follows from I(a) and I(b) that $f$ is a non-differentiable function of the Weierstrass type [4, p. 135], i.e. the knot-points of $f$, viz. $E_k(f)$, cover almost all the points of $[0, 1]$. It, therefore, follows from author's [4, p. 141, prop. 3] that each of the sets $E_i(f)$ $(i = 1$ to $4)$ has the power of the continuum in every subinterval of $[0, 1]$. Moreover, since every non-differentiable function is a nowhere monotone function of the second species [3, p. 83], it follows from author's [3, p. 87, th. 5] that $E_k(f)$ is a residual subset of $[0, 1]$, and from [3, p. 86, th. 4] that, for every $r \in R$, each of the following four sets

$$E'_{1r}(f) = \{x : D^+ f(x) = r\}\,,\quad E'_{2r}(f) = \{x : D_- f(x) = r\}\,,$$
$$E'_{3r}(f) = \{x : D_+ f(x) = r\}\,,\quad E'_{4r}(f) = \{x : D^- f(x) = r\}$$

has the power of the continuum in every subinterval of $(0, 1)$. The part i) of the theorem thus holds for every $f \in C_0$, and since the complement of $E_k(f)$ is a set of the first category with its measure equal to zero, the same is true for each of its subsets. Moreover, at $x \in E'_{1r}(f)$ it follows from I(c) that $D_+ f(x) = -\infty$ and from I(a) that $D^- f(x) = \infty$ and $D_- f(x) \leqq r$. Thus $E'_{1r}(f) = E_{1r}(f)$, and similarly $E'_{ir}(f) = E_{ir}(f)$ for each of $i = 2$ to 4. The part ii) of the theorem thus also holds for every $f \in C_0$.

It is clear from I(a) and I(c) that for every $f \in C_0$, the sets $E_k(f)$, $E_i(f)$ $(i = 1$ to $4)$ and $E_{ir}(f)$ $(i = 1$ to $4$, $r \in R)$ together cover all the points of $(0, 1)$. The sets $E_{ir}(f)$ $(i = 1$ to $4$, $r \in R)$ are, however, not all mutually disjoint. Since $f$ is a nowhere monotone function of the second species, for every $r \in R$, the function $g(x) = f(x) - rx$ $(0 \leqq x \leqq 1)$ is continuous and nowhere monotone and so has maxima and minima at sets of points $S_1$ and $S_2$ both everywhere dense in $(0, 1)$. At $x \in S_1$ we have $D^+ g(x) \leqq 0 \leqq D_- g(x)$, i.e. $D^+ f(x) \leqq r \leqq D_- f(x)$, whence by I(a) we have $D^+ f(x) = r = D_- f(x)$ and by I(c), $D_+ f(x) = -\infty$, $D^- f(x) = \infty$. so that $x \in$ $\in E_{1r}(f) \cap E_{2r}(f)$. Thus $E_{1r}(f) \cap E_{2r}(f) \supset S_1$, and similarly $E_{3r}(f) \cap E_{4r}(f) \supset S_2$, whence both the sets are everywhere dense in $(0, 1)$. This completes the proof of the theorem.

Remark 1. In the above proof only the first three parts of Jarník's result I are used. The part I(d) was deduced by Jarník from the following result of Saks [8, §2]: "*Every continuous function with the exception of a class of the first category in the space C, has a right-sided derivative* $+\infty$ *at a non-denumerable set of points*". Clearly, this result of Saks as well as I(d) both follow from the above theorem.

Remark 2. It remains to determine exactly one derivate of $f \in C_0$ at the points of $E_{ir}(f)$ $(i = 1$ to $4$, $r \in R)$. As suggested by the derivates of the Weierstrass' non-differentiable function [4, p. 142, th. 2], perhaps the undetermined derivate at each point of $E_{ir}(f)$ is either $r$ or infinite (the sign being uniquely determined by the inequality satisfied by that derivate).

**3. Level sets of a residual set of functions in** $C$. If a function $f \in C$ is non-differentiable, all of its level sets $f^{-1}(c) = \{x : f(x) = c\}$ $(c \in R)$ are non-dense and except for a set of values of $c$ that is of the first category [2, p. 60, th. 1] and of measure zero [7, pp. 31, 33], all the level sets $f^{-1}(c)$ are perfect sets of measure zero. The same is, therefore, true of the level sets of a residual set of functions in C. The Jarník's result I, however, yields even further

**Theorem 2.** *There exists a residual set of functions $f \in C$ of which every level set is non-dense and except for at most a countable number of them, each level set is a perfect set of measure zero.*

Proof. Let $C_1$ be the residual set of functions $f \in C$ for which I(a) holds, and let $f \in C_1$. Then a point $x \in (0, 1)$ is an isolated point of the level set $f^{-1}(c)$, where $f(x) =$

$= c$, if and only if $f$ has a strict maximum or minimum at $x$. Hence if $G$ denotes the set of strict extremum values of $f$ together with $f(0)$ and $f(1)$, $G$ is at most countable and for every $c \notin G$, each point of $f^{-1}(c)$ is a limit point of the set. As $f$ is continuous, each of its level sets is closed, and so $f^{-1}(c)$ is perfect (possibly void) for every $c \notin G$. Also since $f$ cannot have a line of invariability, $f^{-1}(c)$ is non-dense for every $c \in R$.

The function $f$ is obviously measurable and so is each of its level sets. Let $H$ denote the set of values of $c$ for which $f^{-1}(c)$ is of positive measure and let, for every natural number $n$, $H_n = \{c \in R : \text{mes} f^{-1}(c) > 1/n\}$. Since the level sets of $f$ are mutually disjoint and their union has its measure equal to 1, each of the sets $H_n$ is finite, whence $H = \bigcup_{n=1}^{\infty} H_n$ is at most countable. Clearly, $G \cup H$ is also at most countable and for every $c \notin G \cup H$, $f^{-1}(c)$ is a perfect set of measure zero. Hence the theorem.

We next observe that if $f \in C_1$ and $\lambda \in R$, then the function $g(x) = f(x) - \lambda x$ $(0 \leqq x \leqq 1)$ is again a function belonging to $C_1$. The above Theorem 2 can therefore be strengthened to the following form:

**Theorem 2′.** *There exists a residual set of functions $f \in C$ such that for every real value of $\lambda$, the graph of $f$ intersects with the line $y = \lambda x + c$ in a perfect set of linear measure zero except possibly for a countable set of real values of $c$ (depending on $f$ and $\lambda$).*

Let, again, $f \in C_1$, $\lambda \in R$, $g(x) = f(x) - \lambda x$ $(0 \leqq x \leqq 1)$ and $m_\lambda = \inf_{0 \leqq x \leqq 1} g(x)$, $M_\lambda = \sup_{0 \leqq x \leqq 1} g(x)$. As $g$ is nowhere monotone, we have $m_\lambda \neq M_\lambda$, and as $g$ is continuous, there exist two distinct points $x_1, x_2$ in $[0, 1]$ such that $g(x_1) = m_\lambda$ and $g(x_2) = M_\lambda$. For any $c \in (m_\lambda, M_\lambda)$, the function $h(x) = g(x) - c$ $(0 \leqq x \leqq 1)$ still belongs to $C_1$ and we have $h(x_1) < 0 < h(x_2)$. Using a result due to ZAHORSKI [11, p. 43, II] it then follows that the set $h^{-1}(0) = \{x : g(x) = c\} = \{x : f(x) = \lambda x + c\}$ has the power of the continuum. Thus we further have the following

**Theorem 3.** *There exists a residual set of functions $f \in C$ such that its graph intersects with the line $y = \lambda x + c$, for every $\lambda \in R$ and for all values of $c$ in between its two extreme values for which they intersect (not necessarily inclusive), in a set which has the power of the continuum.*

**4. Level sets of a dense set of functions in** $C$. A natural question that arises from Theorem 2 is whether the exception to a countable number of level sets in it is necessary or not. The existence of functions $f \in C$ of which every level set is a perfect set of measure zero is known. A. N. SINGH [9] has constructed a class of such non-differentiable functions. In the following theorem we answer the question partially by establishing that such functions are at least everywhere dense in $C$.

**Theorem 4.** *There exists a set of functions everywhere dense in $C$ of which every level set is a perfect set of measure zero.*

Proof. Let $U$ be an arbitrary open subset of $C$. There exists a nowhere monotone function $g \in U$ since such functions are everywhere dense in $C$. Moreover, $U$ being open, there exists $\varepsilon > 0$ such that $f \in U$ whenever $f \in C$ and $\|f - g\| < \varepsilon$.

As $g$ is as well uniformly continuous on $[0, 1]$, there exists a $\delta > 0$ such that

$$|g(x) - g(x')| < \tfrac{1}{3}\varepsilon \quad \text{whenever} \quad x, x' \in [0, 1] \quad \text{and} \quad |x - x'| < \delta .$$

Let

$$0 = a_0 < a_1 < \ldots < a_i < \ldots < a_n = 1$$

be a finite partition of $[0, 1]$ such that $a_i - a_{i-1} < \delta$ for every $i = 1, 2, \ldots, n$. We shall further assume that this partition is so chosen that $g(a_i) \neq g(a_{i-1})$ for every $i = 1, \ldots, n$, the existence of such a partition following from the fact that $g$ has no lines of invariability. Thus for each $i$ for which $g(a_i) = g(a_{i-1})$, there exists a point $a_i' \in (a_{i-1}, a_i)$ such that $g(a_{i-1}) \neq g(a_i') \neq g(a_i)$, adding such points to the original partition we obtain a new finite partition with the desired properties.

We shall employ the Singh's non-differentiable function $\Phi_{3,1,3}$ [9, p. 3] of which every level set is a perfect set of measure zero [9, p. 16] (or see [10], pp. 53, 89), to construct a function $f \in U$ with the same property. Let every point $x \in [0, 1]$ be expressed as a radix fraction with base 3, viz.

$$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \ldots + \frac{a_n}{3^n} + \ldots ,$$

where each $a_n = 0, 1$ or $2$. Let for every $a = 0, 1, 2$ and for every $s = 0, 1, 2, \ldots,$ $K^s(a) = a$ or $2\text{-}a$ according as $s$ is even or odd. The Singh's function is then defined at $x$ as

$$\Phi(x) = \frac{b_1}{3} + \frac{b_2}{3^2} + \ldots + \frac{b_n}{3^n} + \ldots ,$$

where $b_1 = a_1$ and for every natural number $n$,

$$b_{n+1} = K^{a_2 + a_3 + a_5 + a_6 + \ldots + a_{3n-1} + a_{3n}}(a_{3n+1}) .$$

Clearly, $\Phi(0) = 0$, $\Phi(1) = 1$ and $0 \leqq \Phi(x) \leqq 1$ for every $x \in [0, 1]$.

Let a function $f$ be defined in each interval of the partition $\{a_i : 0 \leqq i \leqq n\}$ separately as follows:

$$f(x) = g(a_{i-1}) + \{g(a_i) - g(a_{i-1})\}\, \Phi\left(\frac{x - a_{i-1}}{a_i - a_{i-1}}\right), \quad a_{i-1} \leqq x \leqq a_i ,$$

$$(i = 1, 2, \ldots, n) .$$

As $f$ agrees with $g$ at the points $a_i$ $(0 \leqq i \leqq n)$, $f$ is uniquely defined. Also $\Phi$ being continuous, so is $f$, i.e. $f \in C$. For every $x \in [0, 1]$, there exists an $i$ such that $a_{i-1} \leqq \leqq x \leqq a_i$, and since then $|x - a_{i-1}| \leqq a_i - a_{i-1} < \delta$, we have

$$|f(x) - g(x)| \leqq |f(x) - g(a_{i-1})| + |g(x) - g(a_{i-1})| \leqq$$
$$\leqq |g(a_i) - g(a_{i-1})| \,.\, \|\Phi\| + |g(x) - g(a_{i-1})| < \tfrac{1}{3}\varepsilon \,.\, 1 + \tfrac{1}{3}\varepsilon = \tfrac{2}{3}\varepsilon \,.$$

Thus $\|f - g\| = \sup\limits_{0 \leqq x \leqq 1} |f(x) - g(x)| \leqq \frac{2}{3}\varepsilon < \varepsilon$, and hence $f \in U$.

Let $c \in R$. For each of $i = 1, 2, \ldots, n$, as $g(a_i) \neq g(a_{i-1})$, we have $x \in f^{-1}(c) \cap \cap [a_{i-1}, a_i]$ if and only if

$$\Phi\left(\frac{x - a_{i-1}}{a_i - a_{i-1}}\right) = \frac{c - g(a_{i-1})}{g(a_i) - g(a_{i-1})},$$

or, equivalently, if and only if

$$x = a_{i-1} + (a_i - a_{i-1})\,\xi\,, \quad \text{where} \quad \xi \in \Phi^{-1}\left\{\frac{c - g(a_{i-1})}{g(a_i) - g(a_{i-1})}\right\}.$$

As the level sets of $\Phi$ are perfect sets of measure zero, it follows that $f^{-1}(c) \cap \cap [a_{i-1}, a_i]$ is such a set for every $i$, and the set $f^{-1}(c)$ being in turn a finite union of such sets is again a perfect set of measure zero. This completes the proof of the theorem.

A slightly general version of Theorem 4 follows as a

**Corollary.** *For every real number $\lambda$, there exists a set of functions $f$ everywhere dense in $C$ such that the graph of $f$ intersects with every straight line of which the slope is $\lambda$ in a perfect set with linear measure zero.*

Proof. Let $\lambda \in R$, $f_\lambda(x) = \lambda x$ $(0 \leqq x \leqq 1)$ and $U$ be an arbitrary open subset of $C$. Then $V = \{f - f_\lambda : f \in U\}$ is also an open subset of $C$. According to Theorem 4, there exists a function $h \in V$ such that each of its level sets is a perfect set of measure zero. Then $f = h + f_\lambda \in U$ and for every real number $c$, $h^{-1}(c) = \{x : f(x) = \lambda x + c\}$ is a perfect set of measure zero. This being the projection of the set in which $y = f(x)$ intersects with $y = \lambda x + c$, the later set is also a perfect subset of the line $y = = \lambda x + c$ with linear measure zero.

As for the perfectness of all the sets in which non-vertical straight lines intersect with the graph of a function $f \in C$, we have the following

**Theorem 5.** *There exists a set of functions $f$ everywhere dense in $C$ such that for every real value of $\lambda$, the graph of $f$ intersects with the line $y = \lambda x + c$ in a perfect set for all but a finite number of real values of $c$.*

Proof. As in the proof of Theorem 4, let $g$ be a nowhere monotone function belonging to a given open subset $U$ of $C$, and let $\{a_i : 0 \leqq i \leqq n\}$ be a partition of $[0, 1]$ as defined there. Let $\{A_i : 0 \leqq i \leqq n\}$ be the points on the graph of $g$ corresponding to the points of the partition. Then none of the segments $A_{i-1}A_i$ $(i = 1, \ldots, n)$ is parallel to any of the coordinate axes. Using a geometrical method of construction due to J. GILLIS [12], starting with the segment $A_{i-1}A_i$ we can define a continuous function $f$ on $[a_{i-1}, a_i]$ such that $f$ agrees with $g$ at the end points of the interval,

$$\min\{g(a_{i-1}), g(a_i)\} \leqq f(x) \leqq \max\{g(a_{i-1}), g(a_i)\} \quad \text{for} \quad a_{i-1} \leqq x \leqq a_i,$$

and the graph of $f$ intersects with every non-vertical straight line in a perfect set, except possibly the ones passing through $A_{i-1}$ and $A_i$. Defining the function $f$ likewise in every interval of the partition, $f$ has the requisite properties and, as before, $\|f - g\| \leqq \frac{2}{3}\varepsilon < \varepsilon$, so that $f \in U$.

### *References*


[1] *S. Banach:* Über die Baire'sche Kategorie gewisser Funktionenmengen, Studia Math. *3* (1931), 174—179.

[2] *K. M. Garg:* On level sets of a continuous nowhere monotone function, Fund. Math. *52* (1963), 59—68.

[3] *K. M. Garg:* On nowhere monotone functions III: Functions of first and second species, Rev. Math. Pures Appl. *8* (1963), 83—90.

[4] *K. M. Garg:* On asymmetrical derivates of non-differentiable functions, Can. Jour. Math. *20* (1968), 135—143.

[5] *V. Jarník:* Über die Differenzierbarkeit stetiger Funktionen, Fund. Math. *21* (1933), 48—58.

[6] *S. Mazurkiewicz:* Sur les fonctions non dérivables, Studia Math. *3* (1931), 92—94.

[7] *S. Minakshisundaram:* On the roots of a continuous non-differentiable function, Jour. Indian Math. Soc. *4* (1940), 31—33.

[8] *S. Saks:* On the functions of Besicovitch in the space of continuous functions, Fund. Math. *19* (1932), 211—219.

[9] *A. N. Singh:* On the method of construction and some properties of a class of non-differentiable functions, Proc. Benares Math. Soc. *13* (1931), 1—17.

[10] *A. N. Singh:* The Theory and Construction of Non-differentiable Functions, Lucknow University Studies *1*, 1935.

[11] *Z. Zahorski:* Sur l'ensemble des racines de l'équation $W(x) = f(x)$, C. R. Soc. Sci. Lett. Varsovie (3) *41* (1948), 43—45.

[12] *J. Gillis:* Note on a conjecture of Erdös, Quart. Jour. Math. (Oxford) *10* (1939), 151—154.



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