Ladislav Bican

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# MIXED ABELIAN GROUPS OF TORSION FREE RANK ONE

LADISLAV BICAN, Praha

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In this paper Baer's lemma (see Lemma 46.3 in [1]) is generalized and, by means of this generalization, necessary and sufficient conditions for splittingness of the mixed abelian group of torsion free rank one are given. Further, all the mixed groups of torsion free rank one, any pure subgroup of which splits are characterized. The obtained results are applied to the mixed abelian group $G$ with maximal torsion subgroup $T$ such that $G/T$ is completely decomposable.

By the word "group" we mean always an additively written abelian group. We say that a mixed group $G$ with maximal torsion subgroup $T$ is of torsion free rank one if the factor-group $G/T$ is of rank one.

The definitions of the characteristic and of the type of an element may be extended to arbitrary (mixed) groups, but some properties of these notions do not hold in general. However, those properties which are satisfied in an arbitrary group are applied here to the study of the groups mentioned above. In this paper we shall denote by $h_p^G(g)$, $\tau^G(g)$, $\hat{\tau}^G(g)$ the $p$-height, the characteristic and the type of the element $g$ in the group $G$, respectively. In general, we shall adopt the notation used in [1].

**Lemma 1.** *Let $K$ be a subgroup of $G$ such that $\bar{G} = G/K$ is torsion free and let $\bar{0} \neq \bar{h} \in \bar{G}$ be an arbitrary element. If for some $x \in \bar{h}$ there is $\tau^G(x) = \tau^{\bar{G}}(\bar{h})$, then $\tau^G(\varrho x) = \tau^{\bar{G}}(\varrho\bar{h})$ for any integer $\varrho$. Similarly, if for some $x \in \bar{h}$ there is $\hat{\tau}^G(x) = \hat{\tau}^{\bar{G}}(\bar{h})$, then $\hat{\tau}^G(\varrho x) = \hat{\tau}^{\bar{G}}(\varrho\bar{h})$ for any integer $\varrho$.*

Proof. We shall simply write $\tau(x)$, $\hat{\tau}(x)$, $\hat{\tau}(\bar{x})$ etc. in place of $\tau^G(x)$, $\hat{\tau}^G(x)$, $\hat{\tau}^{\bar{G}}(\bar{x})$ etc. because no confusion can arise.

First, let us show that for any prime $p$ with $(p, \varrho) = 1$ the equation

$$(1) \qquad p^k y = x\,, \quad x \in G$$

has a solution in $G$ if and only if the equation

$$(2) \qquad p^k z = \varrho x\,, \quad x \in G$$

has a solution in $G$. It clearly suffices to prove that the solvability of (2) implies the solvability of (1). However, from $(p, \varrho) = 1$ the existence of integers $s, t$ follows for which $p^k s + \varrho t = 1$. By multiplying (2) by $t$ we get $p^k tz = \varrho tx = x - p^k sx$, hence $p^k(tz + sx) = x$.

Further, from $\varrho x \in \varrho \bar{h}$ it follows

$$\tau(\varrho x) \leqq \tau(\varrho \bar{h}) . \tag{3}$$

If $h_p(\bar{h}) = \infty$, then $h_p(x) = \infty$ by hypothesis, and hence $h_p(\varrho x) = h_p(\varrho \bar{h})$ and the second part of our assertion now easily follows.

Finally, if $k$ is the greatest exponent for which $p^k \mid \varrho$ and if $h_p(\bar{h}) < \infty$ then ($\bar{G}$ is torsion free!) it is $h_p(\varrho \bar{h}) = h_p(\bar{h}) + k$. Now it is easy to see that $h_p(\varrho x) \geqq \geqq h_p(x) + k = h_p(\varrho \bar{h})$, which together with (3) proves our assertion.

**Lemma 2.** *Let $K$ be a subgroup of $G$ such that $\bar{G} = G/K$ is torsion free. If $\bar{h} \neq \bar{0}$ is an element of $\bar{G}$ containing $x$ with $\hat{\tau}^G(x) = \hat{\tau}^{\bar{G}}(\bar{h})$, then there exist non-zero integers $m$, $n$ such that $\tau^G(mnx) = \tau^{\bar{G}}(n\bar{h})$.*

Proof. Let $\mathfrak{M}$ be the set of all pairs $(y, \bar{k})$ where $y \in \bar{k}$, $\bar{k} = s\bar{h}$, $y = sx$ for some non-zero integer $s$, and $\hat{\tau}(y) = \hat{\tau}(\bar{k})$ (the last symbols have the same meaning as in the preceding proof). Then clearly $\tau(y) \leqq \tau(\bar{k})$ and with any $(y, \bar{k}) \in \mathfrak{M}$ we may associate a natural integer $n(y, \bar{k})$ in the following way: If $p_1, p_2, \ldots, p_r$ are all primes with $h_{p_i}(\bar{k}) - h_{p_i}(y) = l_i > 0$, $i = 1, 2, \ldots, r$, then we put $n(y, \bar{k}) = p_1^{l_1} \cdot p_2^{l_2} \cdot \ldots \cdot p_r^{l_r}$. Let us denote $n_0 = n(y_0, \bar{k}_0) = \min \{n(y, \bar{k}); (y, \bar{k}) \in \mathfrak{M}\}$. Lemma 2 will be proved by showing that $\tau(n_0 y_0) = \tau(\bar{k}_0)$. Clearly, only the case $n_0 > 1$ must be discussed. If the equality $\tau(n_0 y_0) = \tau(\bar{k}_0)$ does not hold, then it may be easily shown that $\tau(n_0 \bar{k}_0) \geqq \tau(n_0 y_0) > \tau(\bar{k}_0)$. Now $n(n_0 y_0, n_0 \bar{k}_0) < n_0$ which contradicts the minimality of $n_0$, because $\hat{\tau}(n_0 y_0) = \hat{\tau}(n_0 \bar{k}_0)$ by Lemma 1 and hence $(n_0 y_0, n_0 \bar{k}_0) \in \mathfrak{M}$.

**Lemma 3.** *Let $K$ be a subgroup of $G$ such that $\bar{G} = G/K$ is torsion free. If to any $g \in G \dot{-} K$ there exists a non-zero integer $m$ such that $\hat{\tau}^G(mg) = \hat{\tau}^{\bar{G}}(\bar{g})$, then to any $g \in G \dot{-} K$ there exists a non-zero integer $\alpha$ and an element $h \in \bar{g}$ such that $\tau^G(\alpha h) = = \tau^{\bar{G}}(\alpha \bar{g})$.*

Proof. As in the preceding proofs we shall use brief symbols $\tau(x)$, $\tau(\bar{g})$ etc. Let $g \in G \dot{-} K$ be an arbitrary element. By hypothesis there exists a non-zero integer $m$ such that $\hat{\tau}(mg) = \hat{\tau}(\bar{g}) = \hat{\tau}(m\bar{g})$. Lemma 2 guarantees the existence of non-zero integers $\varrho, \sigma$ with $\tau(\varrho \sigma mg) = \tau(\sigma m \bar{g})$. Let us write $\varrho$ in the form $\varrho = \varrho' \cdot \varrho''$ where $\varrho'$ is relatively prime to any prime $p$ with $h_p(\bar{g}) = \infty$ and $\bar{g}$ is $\varrho''$-divisible.

Then it holds

$$\tau(\varrho' \varrho'' \sigma mg) = \tau(\sigma m \bar{g}) = \tau(\varrho'' \sigma m \bar{g}) . \tag{4}$$

Let $\varrho' = p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots \cdot p_r^{k_r}$ be the canonical decomposition of $\varrho'$ and let us put $\bar{\varrho} = = p_1^{l_1} \cdot p_2^{l_2} \cdot \ldots \cdot p_r^{l_r}$ where $l_i = h_{p_i}(\bar{g})$, $i = 1, 2, \ldots, r$. Then there exists an element

$\bar{k} \in \bar{G}$ such that $\bar{\varrho}\bar{k} = \bar{g}$ and $h_{p_i}(\bar{k}) = 0$, $i = 1, 2, \ldots, r$. Let $k \in \bar{k}$ be an arbitrary element. By hypothesis there exists a non-zero integer $n$ with $\hat{\tau}(nk) = \hat{\tau}(\bar{k}) = \hat{\tau}(n\bar{k})$. By Lemma 2 there exist non-zero integers $\mu$, $\nu$ such that

$$(5) \qquad \tau(\mu\nu nk) = \tau(\nu n\bar{k}) .$$

Let $p$ be a prime with $p \mid \varrho'$. Suppose that $u$ is the greatest exponent for which $p^u \mid \nu n$. For $p \mid \mu$ we have $h_p(\mu\nu nk) \geqq u + 1$ while $h_p(\nu n\bar{k}) = u$. This contradiction (with (5)) shows that $(\mu, \varrho') = 1$ so that $\varrho' t + \mu s = 1$ for suitable non-zero integers $t, s$. Let us put

$$(6) \qquad h = \varrho' tg + \mu s\bar{\varrho}k .$$

In view of $\bar{\varrho}\bar{k} = \bar{g}$ there is $\bar{\varrho}k - g \in K$ and $h - g = \varrho' tg + \mu s\bar{\varrho}k - \varrho' tg - \mu sg = \mu s(\bar{\varrho}k - g) \in K$ which implies $h \in \bar{g}$. Finally, if we put $\alpha = \nu n\varrho''\sigma m$ we have by (4), (5), (6) $\tau(\alpha h) \leqq \tau(\alpha\bar{g}) = \tau(\nu n\varrho''\sigma m\bar{g}) = \tau(\nu n\varrho'\varrho''\sigma mg) \cap \tau(\mu\nu\varrho''\sigma mn\bar{\varrho}k \leqq$ $\leqq \tau(\nu n\varrho'\varrho''\sigma mtg) \cap \tau(\mu\nu\varrho''\sigma mn\bar{\varrho}s\bar{k}) \leqq \tau(\nu n\varrho''\sigma m(\varrho' tg + \mu s\bar{\varrho}k)) = \tau(\alpha h)$. Hence $\tau(\alpha h) = \tau(\alpha\bar{g})$ which completes the proof.

**Theorem 1.** *Let $K$ be a subgroup of $G$ such that $\bar{G} = G/K$ is a torsion free group of rank one. If*

1) *to any $g \in G \dot{-} K$ there exists a non-zero integer $m$ such that $\hat{\tau}^G(mg) = \hat{\tau}^{\bar{G}}(\bar{g})$ and*
2) *to any $g \in G \dot{-} K$ there exists a non-zero integer $m'$ such that for any prime $p$ with $h_p^{\bar{G}}(\bar{g}) = \infty$ there exist elements $h_0^{(p)} = m'g, h_1^{(p)}, h_2^{(p)}, \ldots$ such that $ph_{n+1}^{(p)} = h_n^{(p)}$, $n = 0, 1, 2, \ldots$*

*then $K$ is a direct summand of $G$.*

Proof. By the condition 1) and Lemma 3 there exists a coset $\bar{h} \neq \bar{0}$ containing an element $h$ such that $\tau(h) = \tau(\bar{h})$ (the notation is the same as in the preceding proofs). In view of Lemma 1 the elements $h$ and $\bar{h}$ can be chosen so that 1 can be taken as the integer $m'$ belonging to $h$ which appears in the condition 2).

For the sake of simplicity let us denote $h_p(h) = n(p)$ for all primes $p$ such that $n(p)$ is either natural integer or zero or the symbol $\infty$. For any prime $p$ let us define the elements $h_k^{(p)}$, $0 \leqq k < n(p) + 1$ (where $\infty + 1 = \infty$) in the following way: $h_0^{(p)} = h$ for all primes $p$. If $n(p)$ is a natural integer, then let $h_{n(p)}^{(p)}$ be some solution of the equation $p^{n(p)}x = h$ lying in $G$ and for $1 \leqq k < n(p)$ we put $h_k^{(p)} = p^{n(p)-k}h_{n(p)}^{(p)}$. If $n(p) = \infty$, then let $h_k^{(p)}$ be those elements from $G$ whose existence is guaranteed by 2).

Now we shall define the subgroup $H$ of $G$ in the following way:

$$(7) \quad H = \{h_k^{(p)};\ p \text{ runs over all primes and } 0 \leqq k < n(p) + 1\}.$$

First, we shall prove that

$$(8) \qquad K \cap H = 0 .$$

Note that if $h_{i_1}^{(p)}, h_{i_2}^{(p)}, \ldots, h_{i_k}^{(p)}$, $i_1 < i_2 < \ldots < i_k$ are generators of $H$ belonging to the prime $p$, then in view of the definition of these elements it holds $\sum_{j=1}^{k} \lambda_j h_{i_j}^{(p)} = (\sum_{j=1}^{k} \lambda_j p^{i_k - i_j})\, h_{i_k}^{(p)}$.

Now if $g$ is an element in $K \cap H$, then $g$ can be written in the form $g = \sum_{i=1}^{n} \lambda_i h_{k_i}^{(p_i)}$ where $p_1, p_2, \ldots, p_n$ are mutually different primes. If we put $r = p_1^{k_1} . p_2^{k_2} . \ldots . p_n^{k_n}$ we have $rg = (\sum_{i=1}^{n} \lambda_i r/p_i^{k_i})\, h \in K \cap H$. Thus we have $\bar{h} \neq 0$, $(\sum_{i=1}^{n} \lambda_i r/p_i^{k_i})\, \bar{h} = \bar{0}$ (in $\bar{G} = G/K$), so that from the torsion freeness of $\bar{G}$ it follows $\sum_{i=1}^{n} \lambda_i r/p_i^{k_i} = 0$ which implies $p_i^{k_i} \mid \lambda_i$, i.e. $\lambda_i = p_i^{k_i} \lambda_i'$, $i = 1, 2, \ldots, n$. Hence $g = (\sum_{i=1}^{n} \lambda_i')\, h \in K \cap H$ and the same argument as above leads to $\sum_{i=1}^{n} \lambda_i' = 0$ and then $g = 0$.

Now we are going to prove that

$$G = \{K, H\}\,. \tag{9}$$

Let $g \in G$ be an arbitrary element. For $g \in K$ there is nothing to prove. In the opposite case there exist non-zero integers $r, s$ such that $r\bar{g} = s\bar{h}$ (in view of the fact that $\bar{G}$ is a torsion free group of rank one) and we may assume that $r, s$ are relatively prime and $r > 0$. Hence we have

$$rg = sh + k\,, \quad k \in K\,. \tag{10}$$

Let $r = p_1^{k_1} . p_2^{k_2} . \ldots . p_l^{k_l}$ be the canonical decomposition of $r$. From $(r, s) = 1$ it easily follows that $h_{p_i}(s\bar{h}) = h_{p_i}(\bar{h}) = h_{p_i}(h)$ so that $k_i \leqq n(p_i)$, $i = 1, 2, \ldots, l$. The numbers $p_1^{k_1}, p_2^{k_2}, \ldots, k_l^{k_l}$ are relatively prime which implies the existence of integers $u_1, u_2, \ldots, u_l$ such that $\sum_{i=1}^{l} ru_i/p_i^{k_i} = 1$. If we put $a = \sum_{i=1}^{l} u_i h_{k_i}^{(p_i)}$ we have $a \in H$ and $ra = \sum_{i=1}^{l} u_i h_{k_i}^{(p_i)} = (\sum_{i=1}^{l} ru_i/p_i^{k_i})\; h = h$, hence $rsa = sh$. In view of (10) we get $r(g - sa) = k \in K$. The torsion free character of $\bar{G}$ now implies that $g - sa \in K$. Hence (9) is true which together with (8) completes the proof of our Theorem.

Remark. An immediate consequence of Theorem 1 is the so called Baer's lemma (see Lemma 46.3 in [1]).

In the sequel we shall deal with the mixed group $G$ with the maximal torsion subgroup $T$ and $\bar{G}$ will denote the factor-group $G/T$. The bar over the elements will denote the elements from $\bar{G}$. For the sake of simplicity we shall write briefly $\tau(g)$, $\tau(\bar{g})$, $\hat{\tau}(g)$, $\hat{\tau}(\bar{g})$ etc. in place of $\tau^G(g)$, $\tau^{\bar{G}}(\bar{g})$, $\hat{\tau}^G(g)$, $\hat{\tau}^{\bar{G}}(\bar{g})$ etc. $T_p$ will denote the $p$-primary component of $T$.

First of all we shall formulate three conditions.

**Condition** $(\alpha)$. We say that a mixed group $G$ with the maximal torsion subgroup $T$ satisfies Condition $(\alpha)$ if to any $g \in G \dot{-} T$ there exists integer $m \neq 0$ (depending on $g$, of course) such that $\hat{\tau}(mg) = \hat{\tau}(\bar{g})$.

**Condition** $(\beta)$. We say that a mixed group $G$ with the maximal torsion subgroup $T$ satisfies Condition $(\beta)$ if to any $g \in G \dot{-} T$ there exists an integer $m \neq 0$ such that for any prime $p$ with $h_p(\bar{g}) = \infty$ there exist elements $h_0^{(p)} = mg, h_1^{(p)}, h_2^{(p)}, \ldots$ such that $ph_{n+1}^{(p)} = h_n^{(p)}$, $n = 0, 1, 2, \ldots$.

**Condition** $(\gamma)$. We say that a mixed group $G$ satisfies Condition $(\gamma)$ if it holds: if $\bar{G} = G/T$ contains a non-zero element of infinite $p$-height, then $T_p$ is a direct sum of a divisible and a bounded groups.

**Lemma 4.** *Let $G = T \dot{+} A$ be a splitting mixed group where $T$ is torsion and $A$ torsion free. Then $G$ satisfies Conditions $(\alpha)$ and $(\beta)$.*

Proof. Let $g = t + a$, $t \neq 0$ be an arbitrary element from $G \dot{-} T$ and let $m$ be the order of $t$. Then clearly $\hat{\tau}(mg) = \hat{\tau}(ma) = \hat{\tau}^A(ma) = \hat{\tau}^A(a) = \hat{\tau}(\bar{a}) = \hat{\tau}(\bar{g})$ and Condition $(\beta)$ is satisfied in view of the fact that $A$ is a direct summand of $G$.

**Theorem 2.** *Let $G$ be a mixed group of torsion free rank one. Then $G$ splits if and only if $G$ satisfies Conditions $(\alpha)$ and $(\beta)$.*

Proof. Conditions $(\alpha)$ and $(\beta)$ are sufficient according to Theorem 1 and necessary according to Lemma 4.

**Lemma 5.** *Let $G$ be a mixed group satisfying Conditions $(\alpha)$ and $(\gamma)$. Then $G$ satisfies Condition $(\beta)$.*

Proof. In view of the torsion freeness of $\bar{G}$ and of Condition $(\alpha)$ we may restrict ourselves to such elements $g \in G \dot{-} T$ for which $\hat{\tau}(g) = \hat{\tau}(\bar{g})$. Let $p$ be an arbitrary prime with $h_p(g) = h_p(\bar{g}) = \infty$ and let $T_p = D \dot{+} U$ where $D$ is divisible and $p^k U = 0$. Using the method of induction we shall construct a sequence of elements $h_0 = g, h_1, h_2, \ldots$ such that $ph_{n+1} = h_n$, $n = 0, 1, 2, \ldots$. Put $h_0 = g$ and assume that we have constructed the elements $h_0, h_1, \ldots, h_n$ such that

$$h_0, h_1, \ldots, h_n \quad \text{are of infinite } p\text{-height in } G\,, \tag{11}$$

$$ph_{l+1} = h_l\,, \quad l = 0, 1, \ldots, n-1\,. \tag{12}$$

Since $h_n$ is of infinite $p$-height in $G$, there exists an element $h^{(1)}$ such that $p^{k+1}h^{(1)} = h_n$. Put

$$h_{n+1} = p^k h^{(1)}\,. \tag{13}$$

First, $ph_{n+1} = p^{k+1}h^{(1)} = h_n$, hence $h_{n+1}$ satisfies (12). It suffices to prove that $h_{n+1}$ is of infinite $p$-height in $G$. Due to (11) there exist in $G$ elements $h^{(s)}$, $s = 1, 2, \ldots$ such that $p^{k+s}h^{(s)} = h_n$. Then $p^{k+s}h^{(s)} - p^{k+1}h^{(1)} = 0$, i.e. $p^{k+1}(p^{s-1}h^{(s)} - h^{(1)}) = 0$ so that $p^{s-1}h^{(s)} - h^{(1)} \in G[p^{k+1}] \subseteq T_p$. Hence we can write

$$(14) \qquad p^{s-1}h^{(s)} - h^{(1)} = d^{(s)} + u^{(s)} \quad \text{where} \quad d^{(s)} \in D\,, \quad u^{(s)} \in U\,.$$

From the divisibility of $D$ the existence of elements $d^{(s)}_{s-1}$, follows for which

$$(15) \qquad p^{s-1}d^{(s)}_{s-1} = d^{(s)}\,.$$

If we put $h^{(s)}_* = h^{(s)} - d^{(s)}_{s-1}$, then by using (15), (14), (13) and the relation $p^kU = 0$ we get $p^{k+s-1}h^{(s)}_* = p^{k+s-1}(h^{(s)} - d^{(s)}_{s-1}) = p^k(p^{s-1}h^{(s)} - d^{(s)}) = p^k(h^{(1)} + u^{(s)}) = h_{n+1}$ and $h_{n+1}$ satisfies (11), which completes the proof of the lemma.

**Theorem 3.** *Let $G$ be a mixed group of torsion free rank one. If $G$ satisfies Conditions $(\alpha)$ and $(\gamma)$, then $G$ splits.*

Proof. It suffices to use Lemma 5 and Theorem 2.

Recall that WANG [3] has called a subgroup $H$ of a torsion free group $G$ regular in $G$ if any element of $H$ has in $H$ the same type as in $G$.

**Lemma 6.** *Let $G$ be a mixed group satisfying Condition $(\alpha)$. Then any subgroup $S$ pure in $G$ satisfies $(\alpha)$, too, and $\bar{S} = S/S \cap T$ is isomorphic to some regular subgroup of $\bar{G}$.*

Proof. Let $s \in S \dot{-} S \cap T$ be an arbitrary element. By Condition $(\alpha)$ there exists a non-zero integer $m$ such that

$$(16) \qquad \hat{\tau}^G(ms) = \hat{\tau}^{\bar{G}}(\bar{s})\,.$$

Now by the isomorphism theorem it is

$$(17) \qquad S/S \cap T \cong \{S, T\}/T \subseteq G/T\,.$$

Using (16), (17), the well-known properties of the types of elements in a torsion free group and the purity of $S$ in $G$ we get $\hat{\tau}^G(ms) = \hat{\tau}^{\bar{G}}(\bar{s}) \geqq \hat{\tau}^{\bar{S}}(s + S \cap T) = \hat{\tau}^{\bar{S}}(ms + S \cap T) \geqq \hat{\tau}^S(ms) = \hat{\tau}^G(ms)$ which completes the proof of the lemma.

**Lemma 7.** *Let $G$ be a mixed group with the maximal torsion subgroup $T$. If $S$ is a pure subgroup of $G$, then $(S \cap T)_p$ is pure in $T_p$.*

Proof. Let the equation $p^kx = s$, $s \in (S \cap T)_p$ be solvable in $T_p$. From the purity of $S$ in $G$ the existence of $y \in S$ follows for which $p^ky = s$. If $p^l$ is the order of $s$ then $p^{k+l}y = 0$ so that $y \in (S \cap T)_p$.

**Lemma 8.** *Let $T$ be a p-primary group of the form $T = D \dotplus U$ where $D$ is divisible and $U$ bounded. If $S$ is a pure subgroup of $T$, then $S \cap D$ is divisible.*

Proof. Suppose that

$$(18) \qquad p^k U = 0$$

and let $s \in S \cap D$ be an arbitrary element. From the divisibility of $D$ the existence of elements $d_l \in D$ follows such that $p^{k+l} d_l = s$, $l = 1, 2, \ldots$. Now in view of the purity of $S$ in $T$ there exist elements $s_l \in S$ such that

$$(19) \qquad p^{k+l} s_l = s\,.$$

Then

$$(20) \qquad d_l - s_l = d'_l + u_l\,, \quad d'_l \in D\,, \quad u_l \in U\,.$$

If we put

$$(21) \qquad s'_l = p^k s_l\,,$$

it holds on the one hand by (19) $p^l s'_l = p^{k+l} s_l = s$ and on the other hand by (21), (20) and (18) $s'_l = p^k s_l = p^k(d_l - d'_l - u_l) = p^k(d_l - d'_l) \in D \cap S$, so that $s$ has in $S \cap D$ an infinite $p$-height. Because $s$ was an arbitrary element from $S \cap D$, our Lemma is proved.

**Lemma 9.** *Let $T$ be a p-primary group of the form $T = D \dotplus U$ where $D$ is divisible and $U$ bounded. Then any subgroup $S$ pure in $T$ is a direct sum of divisible and bounded groups again.*

Proof. $S \cap D$ is divisible by Lemma 8, hence

$$(22) \qquad S = (S \cap D) \dotplus S'\,.$$

It suffices to prove that $S'$ is bounded. First of all let us note that

$$(23) \qquad \{S, D\} = D \dotplus (\{S, D\} \cap U)\,.$$

Using the well-known isomorphism theorems and the relations (22) and (23) we get $S' \cong S/S \cap D \cong \{S, D\}/D = D \dotplus (\{S, D\} \cap U)/D \cong \{S, D\} \cap U \subseteq U$, which completes the proof.

Remark. From Lemmas 7 and 9 and from Theorem 50.3 from [1], the following assertion follows immediately: If a torsion group $T$ is a direct sum of a divisible and a bounded groups, then any pure subgroup of an arbitrary mixed group $G$ containing $T$ as a maximal torsion subgroup splits.

**Lemma 10.** *Let $G$ be a mixed group satisfying Condition $(\gamma)$. Then any subgroup $S$ pure in $G$ satisfies Condition $(\gamma)$.*

Proof. Let $S$ be a pure subgroup of $G$. If $\bar{S} = S/S \cap T$ contains an element of infinite $p$-height, then by Lemma 6 $\bar{G}$ contains an element of infinite $p$-height, too. Hence by hypothesis $T_p$ is a direct sum of a divisible and a bounded groups and Lemmas 7 and 9 complete the proof.

**Lemma 11.** *Let $T$ be a reduced $p$-primary group containing elements of arbitrary great orders. Then an arbitrary basic subgroup $B$ of $T$ contains elements of arbitrary great orders, too.*

Proof. In view of the fact that two basic subgroups of $T$ are isomorphic it suffices to observe one basic subgroup $B$. Suppose that $B$ is bounded. From the purity of $B$ in $T$ it follows by Theorem 24.5 from [1] that $B$ is a direct summand of $T$, $T = B \dotplus D$ where $D \cong T/B$ is divisible. Due to reducedness of $T$ it is $D = 0$, hence $T = B$ which contradicts our hypothesis.

**Lemma 12.** *Let $G$ be a mixed group of the form $G = \sum_{i=1}^{\infty} \{b_i\} \dotplus A$ where $\{b_i\}$ is a cyclic group of order $p^{l_i}$, $l_i < l_{i+1}$, $i = 1, 2, \ldots$ and $A$ is a $p$-divisible rank one torsion free group. Then $G$ contains a non-splitting pure subgroup.*

Proof. We shall prove this Lemma in several steps.

a) Let $a \in A$ be an arbitrary non-zero element and let $a_i \in A$ be such elements that $p^{l_i}a_i = a$, $i = 1, 2, \ldots$. Put $s_i = a_i + b_i$, $i = 1, 2, \ldots$, and $U = \{s_i, i = 1, 2, \ldots\}$. Now let $S$ be a subgroup of $G$ generated by $U$ and by all solutions (if they exist) of the equations of the form $q^k x = u$ where $u \in U$, $q \neq p$ and $k$ is a natural integer.

We show now that $S$ is precisely the set of those elements $s \in G$ for which $ms \in U$ for a suitable non-zero integer $m$ relatively prime to $p$. It is easy to see that any element from $S$ has the property just mentioned. Conversely, let $s$ be an element of $G$ such that $ms \in U$ for a suitable non-zero integer $m$ relatively prime to $p$. Let $m = p_1^{k_1} . p_2^{k_2} \ldots p_r^{k_r}$ be the canonical decomposition of $m$ ($m$ can be assumed positive). In view of $(m/p_1^{k_1}, m/p_2^{k_2}, \ldots, m/p_r^{k_r}) = 1$ there exist integers $\eta_i$, $i = 1, 2, \ldots, r$ such that $\sum_{i=1}^{r} m\eta_i/p_i^{k_i} = 1$. Further, $p_i^{k_i}(ms/p_i^{k_i}) = ms \in U$ so that $ms/p_i^{k_i} \in S$, $i = 1, 2, \ldots, r$ in view of the definition of $S$. Then $s = \sum_{i=1}^{r} \eta_i ms/p_i^{k_i} \in S$, which was to prove.

b) From the preceding part, the $q$-purity of $S$ in $G$ immediately follows for all primes $q \neq p$. Now we shall deal with the $p$-purity of $S$ in $G$.
Suppose, at first, that the equation

$$p^k x = u\,, \quad u \in U \tag{24}$$

has the solution $x$ in $G$. We are going to show that this equation has a solution in $U$. Let

$$x = \sum_{i=1}^{n} \mu_i b_i + a'\,, \quad a' \in A \tag{25}$$

and

$$(26) \qquad u = \sum_{i=1}^{m} \lambda_i s_i \, .$$

It is easy to see that there is no loss of generality in assuming that $m = n$. If we substitute (25) and (26) into (24), we obtain $\sum_{i=1}^{n} p^k \mu_i b_i + p^k a' = \sum_{i=1}^{n} \lambda_i a_i + \sum_{i=1}^{n} \lambda_i b_i$. In view of the form of $G$ this equality holds if and only if $p^k a' = \sum_{i=1}^{n} \lambda_i a_i$ and $p^{l_i} \mid \lambda_i - p^k \mu_i$, $i = 1, 2, \ldots, n$. The last relation implies the existence of integers $v_i$, $i = 1, 2, \ldots n$ such that

$$(27) \qquad \lambda_i = p^k \mu_i + p^{l_i} v_i \, , \quad i = 1, 2, \ldots, n \, .$$

Now if $l_j$ is some of integers $l_i$, $i = 1, 2, \ldots$ with $l_j \geqq k$, then we put

$$(28) \qquad v = \sum_{i=1}^{n} v_i$$

and $u' = \sum_{i=1}^{n} \mu_i s_i + v p^{l_j - k} s_j$ so that $u' \in U$. Then by (27), by the definitions of $s_i$ and $b_i$, and further by (26) and (28), there is $p^k u' = \sum_{i=1}^{n} p^k \mu_i s_i + v p^{l_j} s_j = \sum_{i=1}^{n} \lambda_i s_i - \sum_{i=1}^{n} p^{l_i} v_i s_i + va = u - \sum_{i=1}^{n} v_i a + va = u$. Now let the equation $p^k y = s$, $s \in S$ have a solution in $G$. Then for a suitable non-zero integer $m$ with $(m, p) = 1$ there is $ms = u \in U$ so that the equation (24) has a solution in $G$. Hence the equation (24) has a solution $u'$ in $U \subseteq S$. The integers $m$, $p$ are relatively prime so that $m\varrho + p^k \sigma = 1$ for suitable integers $\varrho$, $\sigma$. Then $p^k \varrho u' = \varrho u = m \varrho s = s - p^k \sigma s$, hence $p^k(\varrho u' + \sigma s) = s$ with $\varrho u' + \sigma s \in S$ which proves the purity of $S$ in $G$.

c) Now we shall prove that

$$(29) \qquad b_1 \notin S \, .$$

There exists an integer $m$ relatively prime to $p$ (see part a) of this proof) such that $mb_1 \in U$, i.e. $mb_1 = \sum_{i=1}^{n} \lambda_i s_i = \sum_{i=1}^{n} \lambda_i a_i + \sum_{i=1}^{n} \lambda_i b_i$. In view of the form of $G$ this equality holds if and only if

$$(30) \qquad p^{l_i}(m - \lambda_1) \, , \quad p^{l_i} \mid \lambda_i \, , \quad i - 2, 3, \ldots, n$$

$$(31) \qquad \sum_{i=1}^{n} \lambda_i a_i = 0 \, .$$

From $(m, p) = 1$ it follows

$$(32) \qquad (p, \lambda_1) = 1 \, .$$

Further, (30) implies the existence of integers $\lambda_i'$, $i = 2, 3, \ldots, n$ such that $\lambda_i = p^{l_i}\lambda_i'$, $i = 2, 3, \ldots, n$ so that $\sum_{i=1}^{n} \lambda_i a_i = \lambda_1 a_1 + \sum_{i=2}^{n} \lambda_i' p^{l_i} a_i = \lambda_1 a_1 + (\sum_{i=2}^{n} \lambda_i') a = 0$. Multiplying this equality by $p^{l_1}$ we obtain $(\lambda_1 + p^{l_1} \sum_{i=2}^{n} \lambda_i') a = 0$. $a$ is an element of infinite order, hence $\lambda_1 + p^{l_1} \sum_{i=2}^{n} \lambda_i' = 0$ which implies $p/\lambda_1$, a contradiction to (32).

d) Suppose now that the group $S$ splits,

$$S = P \dotplus B \tag{33}$$

where $P$ is torsion and $B$ torsion free. Then $a$ can be written in the form $a = t + b$, $t \in P$, $b \in B$ (because $a = p^{l_1} s_1 \in S$). $a$ is of infinite $p$-height in $G$, hence in $S$ as well, so that by (33) $t$ is of infinite $p$-height, too. In view of the inclusion $P \subseteq \sum_{d\, i=1}^{\infty} \{b_i\}$ this may happen only if $t = 0$. Hence $a \in B$ and the purity of $B$ in $G$ guarantees the existence of $c_j \in B$ such that $p^{l_j} c_j = a$. All $c_j$, $j = 1, 2, \ldots$ are clearly of infinite $p$-height in $G$, so that $a_j - c_j$, $j = 1, 2, \ldots$ are of infinite $p$-height in $G$, too. But $p^{l_j}(a_j - c_j) = 0$ i.e. $a_j - c_j \in G[p^{l_j}] \subseteq \sum_{i=1}^{\infty} \{b_i\}$, so that it is necessarily $a_j = c_j$. Particularly it is $a_1 = c_1 \in B \subset S$ and hence $b_1 = s_1 - a_1 \in S$ which contradicts (29). This contradiction completes the proof of Lemma 12.

**Theorem 4.** *Let $G$ be a mixed group of torsion free rank one. Then any pure subgroup of $G$ splits if and only if $G$ satisfies Conditions $(\alpha)$ and $(\gamma)$.*

Proof. Let $S$ be a pure subgroup of $G$. By Lemmas 6 and 10 and by Theorem 3 $S$ splits and the proof of the sufficiency is obvious.

Condition $(\alpha)$ is necessary by Theorem 2. To prove the necessity of $(\gamma)$ let us suppose that $G$ does not satisfy $(\gamma)$. Hence there exists a prime $p$ such that $G/T$ is $p$-divisible and $T_p$ is not a direct sum of a divisible and a bounded groups. If $G$ does not split, the proof is complete. In the opposite case let be $G = T \dotplus A$ where $T$ is torsion and $A$ a $p$-divisible torsion free group of rank one. We can write $T_p$ in the form $T_p = T_p' \dotplus D$ where $D$ is divisible and $T_p'$ reduced. The orders of elements are not bounded (by our hypothesis), hence by Lemma 11 the basic subgroup $B$ of $T_p'$ contains elements of arbitrary great orders. Hence there exists a direct summand $Q$ of $B$ which is of the form $Q = \sum_{i=1}^{\infty} \{b_i\}$ where $\{b_i\}$ is a cyclic group of order $p^{l_i}$, $l_i < l_{i+1}$, $i = 1, 2, \ldots$. It is clear that $Q$ is pure in $T$ from where it is easy to derive that $Q \dotplus A$ is pure in $G = T \dotplus A$. The application of Lemma 12 completes the proof of necessity of Condition $(\gamma)$.

**Theorem 5.** *Let $G$ be a mixed group with the maximal torsion subgroup $T$. If $G/T$ is a completely decomposable torsion free group, then Conditions $(\alpha)$ and $(\beta)$ are necessary and sufficient for the splittingness of $G$.*

Proof. Conditions $(\alpha)$ and $(\beta)$ are necessary by Lemma 4.

To prove the sufficiency suppose that $G$ satisfies Conditions $(\alpha)$ and $(\beta)$ and that $G/T = \bar{G} = \sum_{\lambda\in\Lambda}{}_{\mathrm{d}}\, \bar{G}_\lambda = \sum_{\lambda\in\Lambda}{}_{\mathrm{d}}\, G_\lambda/T$ where $r(\bar{G}_\lambda) = 1$, $\lambda \in \Lambda$. For an arbitrary $g \in G_\lambda \dot{-} T$ there exists a non-zero integer $m$ such that $\hat{\tau}^G(mg) = \hat{\tau}^{\bar{G}}(\bar{g})$. In view of the purity of $G_\lambda$ in $G$ there is $\hat{\tau}^{G_\lambda}(mg) = \hat{\tau}^G(mg) = \hat{\tau}^{\bar{G}}(\bar{g}) = \hat{\tau}^{\bar{G}_\lambda}(\bar{g})$ so that $G_\lambda$ satisfies Condition $(\alpha)$ for any $\lambda \in \Lambda$.

Further, to any $g \in G_\lambda \dot{+} T$ there exists an integer $m$ such that for any prime $p$ with $h_p^{\bar{G}}(\bar{g}) = \infty$ there exist in $G$ elements $h_0^{(p)} = mg, h_1^{(p)}, \ldots$ such that $ph_{n+1}^{(p)} = h_n^{(p)}$, $n = 0, 1, 2, \ldots$. In view of the purity of $G_\lambda$ in $G$ there exist elements $g_n^{(p)} \in G_\lambda$ with $p^n g_n^{(p)} = mg$, $n = 1, 2, \ldots$ Then $g_n^{(p)} - h_n^{(p)} \in G[p^n] \subseteq T \subseteq G_\lambda$, which implies $h_n^{(p)} \in G_\lambda$, $n = 0, 1, 2, \ldots$ so that $G_\lambda$ satisfies Condition $(\beta)$ for any $\lambda \in \Lambda$. Hence $G_\lambda$ splits by Theorem 2 for any $\lambda \in \Lambda$ and Lemma 2.2 from [1] completes the proof.

**Theorem 6.** *Let $G$ be a mixed group with the maximal torsion subgroup $T$. If $G/T$ is a completely decomposable torsion free group whose type set is ordered and satisfies the maximum condition, then any pure subgroup of $G$ splits if and only if $G$ satisfies Conditions $(\alpha)$ and $(\gamma)$.*

Proof. Condition $(\alpha)$ is necessary by Lemma 4. Let $G/T = \bar{G} = \sum_{\lambda\in\Lambda}{}_{\mathrm{d}}\, \bar{G}_\lambda = \sum_{\lambda\in\Lambda}{}_{\mathrm{d}}\, G_\lambda/T$ be a complete decompositition of $G/T$. Suppose that $G_{\lambda_0}/T$ is a rank one direct summand of this complete decomposition, the type of which is maximal in the type set of $\bar{G}$. If $\bar{G}$ contains an element of infinite $p$-height, then clearly $G_{\lambda_0}/T$is $p$-divisible. Now $G_{\lambda_0}$ is pure in $G$, hence Condition $(\gamma)$ is necessary by Theorem 4.

Conversely, let Conditions $(\alpha)$ and $(\gamma)$ be satisfied and let $S$ be a pure subgroup of $G$. We are going to prove that $S$ splits. By Lemma 6 $S$ satisfies Condition $(\alpha)$ and $\bar{S} = S/S \cap T$ is isomorphic to a regular subgroup of $\bar{G}$. Hence $\bar{S}$ is completely decomposable by Theorem 1 from [3]. By Lemma 10, $S$ satisfies Condition $(\gamma)$. Hence $S$ satisfies Condition $(\beta)$ by Lemma 5 and Theorem 5 completes the proof.

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*Author's address:* Praha 8 - Karlín, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta UK).