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EXTENSION OF SEQUENTIALLY CONTINUOUS FUNCTIONALS IN INDUCTIVE LIMITS OF BANACH SPACES

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The present note is a continuation of a series of papers [1] [2], [3] devoted to the open mapping theorem in spaces of distributions. In a recent paper, we have introduced the notion of orthogonality for subspaces of inductive limits of sequences of Fréchet spaces which is very useful in formulating sufficient conditions for openness of linear mappings [2]. In a forthcoming note [4] we intend to discuss another important situation in which the notion occurs quite naturally.

In order to obtain conditions which are both necessary and sufficient we shall need a slightly weaker notion which we propose to call semiorthogonality. In the present note we discuss this notion in the case of an inductive limit of a sequence of Banach spaces. This case forms an important step in the treatment of similar questions in inductive limits of sequences of Fréchet spaces. It has the further advantage of being technically considerably simpler although the underlying idea is essentially the same for both Banach and Fréchet spaces. The transition from the *LB*-case to the *LF*-case is then effected using an important idea due to W. SŁOWIKOWSKI [5]. However, because of the technical and notational complications connected with this transition, we postpone this matter to another note.

1. NOTATION, TERMINOLOGY AND PRELIMINARIES

If (E, u) is a locally convex space, we denote by $(E, u)'$ its dual. If more than one locally convex topologies on a vector space are considered, there will be, in general, different dual spaces as well. The usual notation for polarity is not sufficient to distinguish in which duality the polarity is understood. Let us recall a useful convention which we have suggested in [6] and which deals with this ambiguity. If E is a vector space, E^* its algebraic dual and Q a subspace of E^*, we denote, for $A \subset E$, by A^Q the set

$$\{y \in Q; |\langle A, y\rangle| \leqq 1\}.$$

If Y is a subspace of E, we denote by $P(Y)$ the operator which assigns to every $x' \in E'$ its restriction to Y. The term inductive limit of a sequence of Banach spaces is taken in the following restricted sense: we are given a sequence (E_n, u_n) of Banach spaces such that $E_n \subset E_{n+1}$ and the restriction of u_{n+1} to E_n equals u_n. The term sequentially closed is discussed in [1] and [4].

We begin by proving a proposition which describes a situation similar to the notion of semiorthogonality for two subspaces. The reader is referred to [3] for a discussion of semiorthogonality and related material.

(1,1) Proposition. *Let E be a normed space, R, Y and S three closed subspaces of E such that $R \cap Y \subset S \subset R$. Then the following conditions are equivalent*

1° *given an $r^* \in R'$, $|r^*| \leqq 1$ which annihilates S and given $\varepsilon > 0$, there exists an extension x' of r^* with $|x'|_Y < \varepsilon$;*

2° *the annihilator S^0 is contained in the norm closure of $R^0 + Y^0$;*

3° *if A is the canonical mapping of E onto E/S and T the canonical mapping of E onto $E/R \oplus E/Y$ then the pair $[A, T]$ is of type $(\infty, 0)$;*

4° *denote by Z the space $Y + S$; given $z^* \in Z'$ and $r^* \in R'$ such that they coincide on $Z \cap R = S$, there exists, for each $\varepsilon > 0$, an extension x' of r^* such that*

$$|P(Y)\, x' - P(Y)\, z^*| < \varepsilon\ ;$$

5° $\bar{R} \cap \bar{Y} \subset \bar{S}$, *the bar denoting closure in the topology $\sigma(E'', E')$.*

Proof. Let us show first that $R \cap (Y + S) = S$. Indeed, $S \subset R \cap (Y + S)$ immediately. On the other hand, if $r = y + s$, we have $y = r - s \in Y \cap R \subset S$ whence $y + s \in S$ so that $r \in S$.

1° → 2°. If $x' \in S^0$ and $\varepsilon > 0$ is given, consider $P(R)\, x'$ and an extension thereof, z' say, with $|z'|_Y < \varepsilon$. It follows that $x' - z' \in R^0$ and $z' = y^0 + m$ with $y^0 \in Y^0$ and $|m| < \varepsilon$. Hence $x' = (x' - z') + y^0 + m \in R^0 + Y^0 + \varepsilon U^0$.

2° ↔ 3°. Condition 2° says that $(A^{-1}(0))^0$ is contained in the norm closure of the range of T'.

2° → 1°. Consider an $r^* \in R'$ which annihilates S and such that $|r^*| \leqq 1$. Let z' be an extension of r^* to the whole of E such that $|z'| = |r^*|$. Then $z' \in S^0$; given $\varepsilon > 0$, we may write z' in the form $z' = r^0 + y^0 + m$ with $r^0 \in R^0$, $y^0 \in Y^0$ and $|m| \leqq \varepsilon$. It follows that $x' = z' - r^0 = y^0 + m$ is an extension of r^* with $|x'|_Y = |y^0 + m|_Y \leqq \leqq \varepsilon$.

5° ↔ 2°. If X is a convex space and P, Q two subspaces thereof, the equality $(P + Q)^0 = P^0 \cap Q^0$ holds. Applying this, we obtain

$$\bar{R} \cap \bar{Y} = R^{E'E''} \cap Y^{E'E''} = (R^{E'} + Y^{E'})^{E''}.$$

It follows that $(\bar{R} \cap \bar{Y})^{E'} = (R^{E'} + Y^{E'})^{E''E'}$ = norm closure of $(R^0 + Y^0)$. Condition 5° implies $S^{E'} = \bar{S}^{E'} \subset (\bar{R} \cap \bar{Y})^{E'}$ = norm closure of $(R^0 + Y^0)$ so that 2° is satisfied. On the other hand, if $S^0 \subset$ norm closure of $(R^0 + Y^0)$, we have $\bar{S}^{E'} \subset \subset (\bar{R} \cap \bar{Y})^{E'}$, whence, taking polars in E'', condition 5° follows.

1° → 4°. To see that 1° implies 4° consider extensions z' and r' of z^* and r^*. Since $P(S)(z' - r') = 0$, there exists, by 1°, an element x' such that x' extends $P(R)(z' - r')$ and $|x'|_Y < \varepsilon$. It follows that $z' - r' - x' = r^0 \in R^0$ and $x' = y^0 + m$ where $y^0 \in Y^0$ and $|m| < \varepsilon$. Consider now the difference $z' - x'$. Since $z' - x' = r' + r^0$, the functional $z' - x'$ extends r^*. Further, $P(Y)(z' - x') - P(Y)z' = -P(y)x' = -P(Y)m$. The proof is complete since 1° is a special case of 4°.

2. EXTENSION OF SEQUENTIALLY CONTINUOUS LINEAR FUNCTIONALS

This section is devoted to necessary and sufficient conditions for a sequentially continuous functional to possess an extension to the whole space.

(2,1) Theorem. *Let (E, u) be an inductive limit of a sequence of Banach spaces E_n. Let R be a sequentially closed subspace of E. Denote by v the topology of R considered as the inductive limit of the sequence $R \cap E_n$. Then the following conditions are equivalent.*

1° *each element of $(R, v)'$ has an extension in $(E, u)'$;*

2° *there exists a defining sequence H_n for E which has the following property: given a natural number n, a positive ε and an $r' \in (R \cap H_{n+2})'$ which annihilates $R \cap H_{n+1}$, there exists an extension $x' \in H'_{n+2}$ such that $P(R \cap H_{n+2})\, x' = r'$ and $|P(H_n)\, x'| < \varepsilon$.*

Proof. Define, on $(E, u)'$, the sequence of pseudonorms

$$p_n(x') = \sup \{|\langle x, x' \rangle|;\ x \in E_n,\ |x| \leqq 1\}$$

and, similarly, on $(R, v)'$, the sequence of pseudonorms

$$q_n(r') = \sup \{|\langle r, r' \rangle|;\ r \in R \cap E_n,\ |r| \leqq 1\}$$

so that $p_n \leqq p_{n+1}$ and $q_n \leqq q_{n+1}$ for each $n \in N$. If we denote by p and q the topologies on $(E, u)'$ and $(R, v)'$ defined respectively by the sequences p_n and q_n. It is easy to see that $((E, u)', p)$ and $((R, v)', q)$ are both Fréchet spaces. Suppose now that condition 1° is satisfied. It follows that the mapping $P(R)$ is a continuous linear transformation of $((E, u)', p)$ onto $((R, v)', q)$. It follows from the open mapping theorem that, for each natural number n, there exists an $m(n) > n$ and an $\varepsilon_n > 0$ with the following property: given $z' \in (R, v)'$ such that $q_{m(n)}(z') \leqq \varepsilon_n$, there exists an extension $x' \in \in (E, u)'$ of z' with $p_n(x') \leqq 1$. From this we deduce immediately the following fact:

if $z' \in (R, v)'$ annihilates $R \cap E_{m(n)}$ and if $\varepsilon > 0$ is given, there exists an extension $x' \in (E, u)'$ with $p_n(x') < \varepsilon$. Indeed, $q_{m(n)}(2z'/\varepsilon) \leqq \varepsilon_n$, so that there exists an extension y' of $2z'/\varepsilon$ with $p_n(y') \leqq 1$. Hence $x' = \frac{1}{2}\varepsilon y'$ is an extension of z' and $p_n(x') < \varepsilon$. Define now a sequence of natural numbers $h(n)$ as follows: we set $h(1) = 1$ and $h(n+1) = m(h(n))$. Now it suffices to define H_n as $E_{h(n)}$ and condition 2° is satisfied.

To prove the converse, we assume that we have a defining sequence E_n which satisfies condition 2°. We introduce the following abbreviation: the intersection $R \cap E_n$ will be denoted by R_n. We have thus, for each $n \in N$, the following assertion:

(A_n) given $r' \in R'_{n+2}$ which annihilates R_{n+1}, there exists, for each $\varepsilon > 0$, an extension $x' \in E'_{n+2}$ such that

$$P(R_{n+2})\, x' = r' \quad \text{and} \quad |P(E_n)\, x'| < \varepsilon\,.$$

Consider now a sequentially continuous functional r' on R and let us show that it has an extension to the whole of E. First of all, we introduce the abbreviation r'_n for $P(R_n)\, r'$.

First of all, let x'_2 be an element of E' such that $P(R_2)\, x'_2 = r'_2$. Consider the difference $r'_3 - P(R_3)\, x'_2$; clearly $P(R_2)\,(r'_3 - P(R_3)\, x'_2) = r'_2 - P(R_2)\, x'_2 = 0$.

According to (A_1), there exists an $x'_3 \in E'$ such that

$$P(R_3)\, x'_3 = r'_3 - P(R_3)\, x'_2\,, \quad |P(E_1)\, x'_3| < \tfrac{1}{2}\,.$$

The next step in the construction consists in applying (A_2) to the difference $r'_4 - P(R_4)\,(x'_2 + x'_3)$. Since

$$P(R_3)\,(r'_4 - P(R_4)\,(x'_2 + x'_3)) = r'_3 - P(R_3)\, x'_2 - P(R_3)\, x'_3 = 0\,,$$

there exists an $x'_4 \in E'$ such that

$$P(R_4)\, x'_4 = r'_4 - P(R_4)\,(x'_2 + x'_3)\,, \quad |P(E_2)\, x'_4| < \frac{1}{2^2}\,.$$

Suppose we have already constructed $x'_2, \ldots, x'_n \in E'$ in such a manner that

1° $|P(E_{i-2})\, x'_i| < \dfrac{1}{2^{i-2}}$ for $3 \leqq i \leqq n$,

2° x'_i is an extension of $r'_i - P(R_i)\,(x'_2 + \ldots + x'_{i-1})$ for $3 \leqq i \leqq n$.

In particular, it follows that $P(R_n)\, x'_n = r'_n - P(R_n)\,(x'_2 + \ldots + x'_{n-1})$ whence

3° $r'_n - P(R_n)\,(x'_2 + \ldots + x'_n) = 0$.

Consider now the difference $p = r'_{n+1} - P(R_{n+1})\,(x'_2 + \ldots + x'_n)$. According to 3°, we have $P(R_n)\, p = 0$ so that, by (A_{n-1}), there exists an $x'_{n+1} \in E'$ such that

$P(R_{n+1})\, x'_{n+1} = p$ and $|P(E_{n-1})\, x'_{n+1}| < 1/2^{n-1}$. This completes the inductive construction. Let us show further that

4° $P(R_n)\, x'_{n+1} = 0$.

This, however, is a simple consequence of 3° since

$$P(R_n)\, x'_{n+1} = P(R_n)\, P(R_{n+1})\, x'_{n+1} = P(R_n)\, p$$

which is zero according to 3°.

Consider now the sum $x'_2 + x'_3 + \dots$ and let us show that it represents a continuous linear functional on E. Take a fixed E_p and consider the series $P(E_p)\, x'_2 + P(E_p)\, x'_3 + \dots$ According to 1°, we have $|P(E_p)\, x'_{p+2}| < 1/2^p$, further $|P(E_p)\, x'_{p+3}| \leqq |P(E_{p+1})\, x'_{p+3}| < 1/2^{p+1}$ and, similarly, $|P(E_p)\, x'_{p+k}| < 1/2^{p+k-2}$ for each $k \geqq 2$. It follows that the series is convergent uniformly on each E_p so that it represents a continuous linear functional $x' \in E'$

$$x' = x'_2 + x'_3 + \dots$$

Let us prove now that x' extends r'. If n is given, we have, according to 4°, the equation $P(R_n)\, x'_{n+1} = 0$ and similarly, $P(R_n)\, x'_{n+k} = P(R_n)\, P(R_{n+k-1})\, x'_{n+k} = 0$ for each $k \geqq 1$. It follows that

$$P(R_n)\, x' = P(R_n)\, (x'_2 + \dots + x'_n)$$

and this equals r'_n by 3°. The proof is complete.

3. A SEQUENTIALLY CONTINUOUS LINEAR FUNCTIONAL WITH NO EXTENSION

This section is devoted to the construction of an example. We intend to construct a pair of convex spaces E and R with the following properties.

1° E is an inductive limit of a sequence of Banach spaces;

2° R is a closed subspace of E;

3° there is a sequentially continuous linear functional f on R which has no extension to an element of E'.

Let C be the Banach space of all bounded real valued functions defined and continuous on the interval $\langle 0, \infty)$ with the norm

$$|x| = \sup \{|x(t)|;\ 0 \leqq t < \infty\}\,.$$

Let C_n be the subspace of C consisting of those $x \in C$ for which $x(0) = 0$ and $x(t) = 0$ for $t \geqq n$. Let E be the inductive limit of the sequence C_n.

First of all, let us introduce an abbreviation. If n is a natural number, $x_0, x_1, \ldots, x_n$ is a sequence of real numbers such that $0 = x_0 < x_1 < \ldots < x_n = 1$ and $y_1, \ldots, y_n$ are arbitrary real numbers, we denote by

$$\begin{pmatrix} x_0, x_1, \ldots, x_n \\ y_0, y_1, \ldots, y_n \end{pmatrix}$$

the function f defined on $\langle 0, 1\rangle$ by the following two postulates:

(1) $f(x_i) = y_i$ for $i = 0, 1, \ldots, n$;

(2) f is linear in each of the intervals $\langle x_{i-1}, x_i\rangle$ for $i = 1, 2, \ldots, n$.

We define first a sequence of continuous functions on $\langle 0, 1\rangle$ as follows. Set $\sigma_0 = \sigma_1 = 0$ and, for $n \geqq 2$, let $\sigma_n = \sum_{k=1}^{n-1} (1/2^k)$. For each natural number n, we define

$$q_n = \begin{pmatrix} 0 & \sigma_n & \sigma_{n+1} & 1 \\ 0 & 0 & \dfrac{1}{2^n} & 0 \end{pmatrix}$$

and set $q_0 = 0$. It follows that $q_i(\sigma_j) = 1/2^{j-1}$ for $j > i \geqq 1$ and $q_i(\sigma_j) = 0$ for $0 \leqq j \leqq i$. If we denote by d_i the difference $d_i = q_i - q_{i-1}$, $i \in N$, we have

$$(1) \qquad d_n(\sigma_k) = -\delta_{nk} \frac{1}{2^{n-1}} \quad \text{for} \quad n \geqq 2, \quad k \geqq 1\,.$$

$$(2) \qquad d_1(\sigma_1) = 0\,, \quad d_1(\sigma_k) = q_1(\sigma_k) = \frac{1}{2^{k-1}} \quad \text{for} \quad k \geqq 2\,.$$

Further, define

$$p_1 = \begin{pmatrix} 0, & \sigma_2, & \sigma_2 + \frac{1}{4}\sigma_2, & \sigma_3, & 1 \\ 0, & 0, & 1, & 1, & 0 \end{pmatrix}$$

$$p_n = \begin{pmatrix} 0, & \sigma_2 + \frac{1}{4}\sigma_{2n-3}, & \sigma_2 + \frac{1}{4}\sigma_{2n-2}, & \sigma_2 + \frac{1}{4}\sigma_{2n-1}, & \sigma_2 + \frac{1}{4}\sigma_{2n}, & 1 \\ 0, & 0, & -1, & -1, & 0, & 0 \end{pmatrix}$$

for $n \geqq 2$ so that, for $j \geqq 2$ and $k \geqq 1$

$$(3) \qquad p_j(\sigma_2 + \tfrac{1}{4}\sigma_{2k}) = -\delta_{j,k+1} = -\delta_{j-1,k}\,.$$

For $i \in N$ let T_i be the interval $\langle \sigma_i, \sigma_{i+1}\rangle$ and let us denote by K_i the union $T_i \cup \langle i, i+1\rangle$. Let E_i be the subspace of $C(K_i)$ consisting of those $x \in C(K_i)$ which are zero at the four points $\sigma_i, \sigma_{i+1}, i, i+1$. We denote by P_i the restriction operator of E onto $C(K_i)$ and by V_i the injection operator $E_i \to E$ which assigns to each $y \in E_i$ the function x defined by the following two postulates

$$P_i x = y\,, \quad x(t) = 0 \quad \text{for all } t \text{ outside } K_i\,.$$

Denote by E_* the subspace of E consisting of all $x \in E$ such that $x(\sigma_i) = x(i) = 0$ for all $i \in N$. Let as note, that, for each $m \in N$, the superposition $V_m \circ P_m$ is the operator on E_* which keeps the part of the function in K_m and erases the rest. For each $i \in N$, let φ_i be the linear function which takes σ_i into 0 and σ_{i+1} into 1, so that $\varphi_i(t) = 2^i(t - \sigma_i)$. Further, let ψ_i be the linear function which takes i into 0 and $i + 1$ into 1 so that $\psi_i(t) = t - i$.

For each fixed $i \in N$ we define two sequences of functions in $E_i \subset C(K_i)$ as follows: for each pair of natural numbers i and n, we denote by $p_n^{(i)}$ the function defined on K_i as follows

$$\begin{aligned} p_n^{(i)}(t) &= p_n(\varphi_i(t)) \quad \text{for} \quad t \in T_i\,, \\ &= 0 \quad \text{for} \quad t \in K_i \quad \text{outside} \quad T_i\,. \end{aligned}$$

Similarly, $d_n^{(i)}$ is defined by

$$\begin{aligned} d_n^{(i)}(t) &= d_n(\psi_i(t)) \quad \text{for} \quad i \leqq t \leqq i + 1\,, \\ &= 0 \quad \text{for} \quad t \in T_i\,. \end{aligned}$$

Clearly both $p_n^{(i)}$ and $d_n^{(i)}$ belong to E_i for each pair of natural numbers i and n.

If $i \in N$, we denote by M_i the closed linear span in E_i of the sequence $\{p_n^{(i)} + d_n^{(i)};\, n \in N\}$. Let us show now that, for each $x \in M_i$ and each $k = 0, 1, 2, \ldots$

$$x(\varphi_i^{-1}(\sigma_2 + \tfrac{1}{4}\sigma_{2k})) = 2^k x(i + \sigma_{k+1})\,. \tag{4}$$

To see that, we prove first that, for $i \geqq 1$, $j \geqq 1$ and $k \geqq 0$

$$p_j^{(i)}(\varphi_i^{-1}(\sigma_2 + \tfrac{1}{4}\sigma_{2k})) = 2^k d_j^{(i)}(i + \sigma_{k+1})\,. \tag{5}$$

To verify (5), we shall distinguish several cases. If $j = 1$ and $k \geqq 1$, we have

$$\begin{aligned} p_1^{(i)}(\varphi_i^{-1}(\sigma_2 + \tfrac{1}{4}\sigma_{2k})) &= p_1(\sigma_2 + \tfrac{1}{4}\sigma_{2k}) = 1\,, \\ d_1^{(i)}(i + \sigma_{k+1}) &= d_1(\sigma_{k+1}) = \frac{1}{2^k} \end{aligned}$$

by (2) and this proves (5). If $j = 1$ and $k = 0$, we have

$$\begin{aligned} p_1^{(i)}(\varphi_i^{-1}(\sigma_2)) &= p_1(\sigma_2) = 0\,, \\ d_1^{(i)}(i + \sigma_1) &= d_1(\sigma_1) = 0\,. \end{aligned}$$

If $j \geqq 2$ and $k \geqq 1$, we have, using (3) and (1)

$$\begin{aligned} p_j^{(i)}(\varphi_i^{-1}(\sigma_2 + \tfrac{1}{4}\sigma_{2k})) &= p_j(\sigma_2 + \tfrac{1}{4}\sigma_{2k}) = -\delta_{j,k+1} \\ d_j^{(i)}(i + \sigma_{k+1}) &= d_j(\sigma_{k+1}) = -\delta_{j,k+1}\,\frac{1}{2^k}\,. \end{aligned}$$

Finally, for $j \geqq 2$ and $k = 0$, we have

$$p_j^{(i)}(\varphi_i^{-1}(\sigma_2)) = p_j(\sigma_2) = 0\,, \quad d_j^{(i)}(i + \sigma_1) = d_j(\sigma_1) = 0\,.$$

In this manner, the relation (5) is established. Since the $p_j^{(i)}$ are zero on $\langle i, i+1\rangle$ and the $d_j^{(i)}$ are zero on $\langle \sigma_i, \sigma_{i+1}\rangle$, equation (5) implies

$$p_j^{(i)}(\varphi_i^{-1}(\sigma_2 + \tfrac{1}{4}\sigma_{2k})) + d_j^{(i)}(\varphi_i^{-1}(\sigma_2 + \tfrac{1}{4}\sigma_{2k})) = \\ = 2^k(p_j^{(i)}(i + \sigma_{k+1}) + d_j^{(i)}(i + \sigma_{k+1}))$$

for each $k \geqq 0$. This proves (4).

Further, each function $x \in M_i$ is constant on each interval of the form

$$\langle \varphi_i^{-1}(\sigma_2 + \tfrac{1}{4}\sigma_{2k}),\ \varphi_i^{-1}(\sigma_2 + \tfrac{1}{4}\sigma_{2k+1})\rangle .$$

Let us prove now the following implication

(6) if $r \in M_i$ and $r(t) = 0$ for $i \leqq t \leqq i + 1$, then $r = 0$.

Indeed, if $r(t) = 0$ for all $t \in \langle i, i+1\rangle$, it follows from (4) that $r(\varphi_i^{-1}(\sigma_2 + \frac{1}{4}\sigma_{2k})) = 0$ for $k \geqq 0$; the function x being constant on $\langle \varphi_i^{-1}(\sigma_2 + \frac{1}{4}\sigma_{2k}), \varphi_i^{-1}(\sigma_2 + \frac{1}{4}\sigma_{2k+1})\rangle$, we have $r(\varphi_i^{-1}(\sigma_2 + \frac{1}{4}\sigma_{2k+1})) = 0$ as well. It follows from the piecewise linearity of r that $r = 0$.

Now set $R_i = V_i M_i$ and let R be the linear hull of the sequence R_i. Let us show first that R is closed in E.

Denote by E_0 the subspace of E consisting of all $x \in E_*$ which satisfy the following conditions.

1° for each $k \geqq 0$ and each $i \geqq 1$ the function x is constant on the intervals $\langle \varphi_i^{-1}(\sigma_2 + \frac{1}{4}\sigma_{2k}), \varphi_i^{-1}(\sigma_2 + \frac{1}{4}\sigma_{2k+1})\rangle$ and $\langle \sigma_i, \varphi_i^{-1}(\sigma_2)\rangle$;

2° for each $k \geqq 0$ and each $i \geqq 1$ the function x is linear on the interval $\langle \varphi_i^{-1}(\sigma_2 + \frac{1}{4}\sigma_{2k+1}), \varphi_i^{-1}(\sigma_2 + \frac{1}{4}\sigma_{2k+2})\rangle$;

3° equation (4) is satisfied for each $i \geqq 1$ and each $k \geqq 0$.

Clearly E_0 is closed in E and $R \subset E_0$. We observe further that every $x \in E_0$ may be written in the form $x = \sum_{m=1}^{\infty} V_m P_m x$, only a finite number of the terms $V_m P_m x$ being different from zero. Indeed x has a compact support and condition 3° implies that x can be different from zero on a finite number of intervals $\langle \sigma_i, \sigma_{i+1}\rangle$ only. Suppose now that $y \in E$ lies outside R. If y is outside E_0 as well, y does not belong to the closure of R since $R \subset E_0$ and E_0 is closed in E. We may thus suppose that $y \in E_0$ so that $y = \sum_{m=1}^{s} V_m P_m y$ for a suitable s. If $V_m P_m y \in R_m$ for each $m = 1, 2, \ldots, s$ we have

$$y = \sum_{m=1}^{s} V_m P_m y \in R_1 + \ldots + R_s \subset R$$

which is a contradiction. If follows that there exists an m, $1 \leqq m \leqq s$ such that $V_m P_m y$ non $\in R_m$ whence $P_m y$ non $\in M_m$. Now M_m is closed in E_m and E_m is closed in $C(K_m)$; it follows that there exists a linear functional $x'_m \in C(K_m)'$ such that $\langle P_m y, x'_m \rangle = 1$ and $\langle M_m, x'_m \rangle = 0$. Clearly the linear form on E defined by

$$x \to \langle P_m x, x'_m \rangle$$

is an element of E'; let us denote it by x'. We have thus

$$\langle x, x' \rangle = \langle P_m x, x'_m \rangle$$

so that

$$\langle y, x' \rangle = \langle P_m y, x'_m \rangle = 1$$

and

$$\langle R, x' \rangle = \langle P_m R, x'_m \rangle = \langle P_m R_m, x'_m \rangle = \langle M_m, x'_m \rangle = 0 .$$

This proves that R is closed in E.

For $x \in R$ set

$$f(x) = \sum 2^i \int_{T_i} x(t) \, dt .$$

Observe that the sum is finite since every $x \in R$, being a finite linear combination $x = r_1 + r_2 + \ldots + r_p$, is zero on all T_i with the exception of a finite number. Let us show first that f is continuous on R. Indeed, if $x_n \in R$ and $x_n \to 0$, all x_n are contained in a fixed space $R_1 + \ldots + R_p$; on this space, f is expressed as a finite linear combination of continuous functionals. Hence f is continuous on R. Suppose now that $x' \in E'$ is an extension of f.

Take now a fixed i and let us prove that

$$\lim_n \langle V_i(p_1^{(i)} + \ldots + p_n^{(i)}), x' \rangle = \tfrac{1}{8} .$$

To see that, we rewrite $p_1^{(i)} + \ldots + p_n^{(i)}$ in the form

$$\sum_{k=1}^{n} p_k^{(i)} = \Big(\sum_{k=1}^{n} p_k^{(i)} + q_n^{(i)}\Big) - q_n^{(i)} = \sum_{k=1}^{n} (p_k^{(i)} + d_k^{(i)}) - q_n^{(i)} = r - q_n^{(i)}$$

where $r \in M_i$. It follows that

$$\begin{aligned}
\langle V_i(\sum_{k=1}^{n} p_k^{(i)}), x' \rangle &= \langle V_i r, x' \rangle - \langle V_i q_n^{(i)}, x' \rangle = \\
&= f(V_i r) - \langle V_i q_n^{(i)}, x' \rangle = \\
&= 2^i \int_{T_i} r(t) \, dt - \langle V_i q_n^{(i)}, x' \rangle = 2^i \int_{T_i} (p_1^{(i)}(t) + \ldots + p_n^{(i)}(t)) \, dt - \langle V_i q_n^{(i)}, x' \rangle = \\
&= 2^i \int_{\sigma_i}^{\sigma_{i+1}} \sum_{k=1}^{n} p_k(2^i(t - \sigma_i)) \, dt - \langle V_i q_n^{(i)}, x' \rangle = \\
&= \int_0^1 \sum_{k=1}^{n} p_k(w) \, dw - \langle V_i q_n^{(i)}, x' \rangle .
\end{aligned}$$

Clearly $\lim_n V_i q_n^{(i)} = 0$ for each i so that the limit $\lim_n \langle V_i(\sum_{k=1}^{n} p_k^{(i)}), x' \rangle$ equals the limit $\lim_n \int_0^1 \sum_{k=1}^{n} p_k(w)\,\mathrm{d}w$ if the latter exists. We have first $\int_0^1 p_1(w)\,\mathrm{d}w = \frac{5}{16}$. Further, it is easy to verify that

$$p_2 + \ldots + p_n = \begin{pmatrix} 0, & \sigma_2, & \sigma_2 + \frac{1}{4}\sigma_2, & \sigma_2 + \frac{1}{4}\sigma_{2n-1}, & \sigma_2 + \frac{1}{4}\sigma_{2n}, & 1 \\ 0, & 0, & -1, & -1, & 0, & 0 \end{pmatrix}$$

whence

$$\int_0^1 (p_2(w) + \ldots + p_n(w))\,\mathrm{d}w = -\left(\frac{1}{16} + \left(\frac{1}{4}\sigma_{2n-1} - \frac{1}{8}\right) + \frac{1}{8}\frac{1}{2^{2n-1}}\right) =$$

$$= \frac{1}{16} - \frac{1}{4}\sigma_{2n-1} - \frac{1}{8}\frac{1}{2^{2n-1}} \to -\frac{3}{16}.$$

This completes the proof of our assertion.

Accordingly, there exists, for each i, a natural number $n(i)$ such that

$$\langle V_i(\sum_{k=1}^{n(i)} p_k^{(i)}), x' \rangle > \tfrac{1}{9}.$$

The sequence $s_n = \sum_{i=1}^{n} V_i(\sum_{k=1}^{n(i)} p_k^{(i)})$ is clearly bounded in E; however, $\langle s_n, x' \rangle > \frac{1}{9}n$. It follows that f cannot have a continuous extension.

Denote by u the topology of E and let v be the topology of R taken as the inductive limit of the sequence $R_1 + \ldots + R_n$. Clearly v is finer than u_R and convergent sequences are the same for v and u_R. However, the preceding example shows that v and u_R are not even equivalent.

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