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Czechoslovak Mathematical Journal, Vol. 20 (1970), No. 1, 98–108

Persistent URL: http://dml.cz/dmlcz/100948

Czechoslovak Mathematical Journal, 20 (95) 1970, Praha

A SEPARATION THEOREM AND APPLICATIONS TO BOREL SETS

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(Received February 24, 1969)

The aim of this paper is to make a further little step in clarifying possible limits of separable Descriptive Theory in the class of all completely regular spaces. A short summary is in [8]. If not otherwise stated, by a space we mean a separated (i.e. Hausdorff) uniformizable (hence completely regular) space.

Whereas the concept of analytic space in general setting is clear because all characterizations and properties of classical analytic sets extend to the general case under appropriate interpretation, the concept of Borel space is not clear because several problems concerning K-Borel sets have not been solved. It is not excluded that Borelian spaces introduced by the author in [3] (for definition and properties see also [6], [7], and [10]) are really the best generalization of classical separable absolute Borel sets (called Lusinian by Bourbaki).

In this paper B-spaces will be introduced and studied. These spaces form another generalization of classical Borel spaces. In section 1 the definitions and basic properties of previous generalizations are given, and two corollaries of the theory of B-spaces to classical theory are described. In section 2 two general separation lemmas are proved, and applied to a separation theorem for a countable collection of analytic sets (Theorem 1). In section 3 B-spaces are introduced and studied. Separation Theorem 1 is essential in Section 3.

1. Introduction. Denote by Σ the set of all infinite sequences in the set (and the discrete space) N of natural numbers, endowed with the topology of coordinate convergence; thus Σ is the product space $\prod\{\mathsf{N} \mid n \in \mathsf{N}\} = \mathsf{N}^{\mathsf{N}}$. In the classical theory, analytic sets are the empty set and continuous images of Σ, and separable absolute Borel sets are just the one-to-one continuous images of closed subspaces of Σ. In the setting of completely regular spaces the basic concepts are defined as images of Σ under certain correspondences (i.e. multi-valued mappings). An upper semi-continuous correspondence (preimages of closed sets are closed) ic called an usco correspondence. An usco-correspondence with compact values is called usco-compact. A space P is called analytic if there exists an usco-compact correspondence f of Σ

onto P. (Observe that the actual domain of f, denoted by $\mathsf{D}f$, is a closed subspace of Σ.)

A dusco-compact correspondence is a disjoint usco-compact correspondence; disjoint means that the values at distinct points are disjoint. Dusco-compact images of Σ are called Borelian spaces (introduced and studied in [3]). A space P is called bi-analytic ([3], Definition 3) if P is analytic and $K - P$ is analytic for some (and then any) compactification K of P. It turns out that Borelian spaces are just the one-to-one continuous images of bi-analytic spaces.

If $\mathcal{M}$ is a collection of sets we denote by $\mathsf{B}(\mathcal{M})$ the smallest collection $\mathcal{N} \supset \mathcal{M}$ of sets that is closed under countable intersections and countable unions. If we restrict the operations to disjoint countable unions and any countable intersections we get $\mathsf{B}_d(\mathcal{M})$. A Baire set in a space P is a set belonging to

$$\mathsf{B}\,(\text{zero sets in } P)\,,$$

where the zero sets are the zero sets of continuous real-valued functions on P. It is important that Baire sets can be defined as elements of

$$\mathsf{B}_d\,(\text{co-zero sets in } P)\,;$$

a co-zero set is the complement of a zero set. Any two disjoint analytic subspaces A_1 and A_2 of a space P admit a separation by a Baire set in P (i.e. there exists a Baire set B in P with $A_1 \subset B \subset P - A_2$). It turns out that bi-analytic spaces are absolute Baire sets in their compactifications. Borelian spaces P are characterized as absolute

$$\mathsf{B}_d\,(\text{closed sets} \cup \text{Baire sets})$$

in completely regular spaces.

A Souslin set in a space P is the image of Σ under a closed-graph correspondence of Σ into P. Analytic spaces are characterized as absolute Souslin sets. Usually the Souslin sets are defined as the sets that can be obtained from the closed sets by the Souslin operation. On the other hand, also in the abstract setting the Souslin operation can be described by means of correspondences defined in Σ.

In this paper the so-called B-spaces are introduced and studied. These can be defined as the images of Σ under busco-compact correspondences; a correspondence defined in Σ is called boxed if the preimages of points are "boxes" in the product space Σ, and "busco-compact" means "boxed usco-compact". The class of all B-spaces contains all Borelian spaces and all σ-compact spaces and countable intersections of σ-compact spaces. Recall that a σ-compact space need not be Borelian ([3], Remark following Theorem 10). No external characterization of B-spaces is known.

Now we are going to describe two corollaries of section 3 to the classical theory.

By the classical Lusin theorem if $f: P \to Q$ is a continuous surjective mapping such that the preimages of points are countable, except for points in a countable subset of Q, then if P is a separable absolute Borel set then so is Q. In the converse direction Purvis [9] proved that if the assumptions of the Lusin's theorem are not satisfied, then the image of some Borel set in P need not be Borel in Q. Our first result says that Q is absolute Borel if P is absolute Borel, under a "geometrical" assumption on the preimages of points.

Theorem 5'. *Let f be an usco-compact correspondence of Σ onto a metrizable space Q (a particular case: f is a continuous mapping onto Q defined on a closed subset of Σ). If the preimages of points are boxes in Σ, then Q is a separable absolute Borel set.* (The proof follows Theorem 5.)

The second result are two characterizations of (Conditions (1) and (3) in the following Theorem 3) separable absolute Borel sets by means of complete sequences of countable coverings. Recall (see, e.g. [3], or [4] or [1]) that a sequence $\mu = \{\mathscr{M}_n\}$ of coverings of a topological space P is said to be complete if every μ-Cauchy filter has a cluster point; a μ-Cauchy filter is a filter $\mathscr{M}$ such that $\mathscr{M} \cap \mathscr{M}_n \neq \emptyset$ for each n.

Theorem 3. *Each of the following four conditions is necessary and sufficient for a metrizable space P to be an absolute Borel separable set:*

(1) *There exists a complete sequence of countable coverings of P such that the elements are analytic sets.*

(2) *Condition* (1) *with disjoint coverings.*

(3) *There exists a B-structure on P* (for definition see Definition 1 below).

(4) *There exists a Borelian structure on P* (see the Remark to Definition 2).

The proof follows Lemma 3. It should be remarked at this point that conditions (2) and (4) characterize Borelian spaces (called descriptive Borel by C. A. Rogers in [10]) introduced in [3].

2. Separation Theorems. Two rather general lemmas will be proved here. Theorem 1 below is an immediate consequence of the First Separation Theorem for analytic sets, and Lemma 2 below.

Definition 1. Given a collection $\mathscr{M}$ of sets, two sets X and Y are said to be $\mathscr{M}$-separated or separated by sets in $\mathscr{M}$ if there exist X_1 and Y_1 in $\mathscr{M}$ with $X \subset X_1$, $Y \subset Y_1$ and $X_1 \cap Y_1 = \emptyset$.

Lemma 1. *Let $\mathscr{M}$ be a finitely additive and finitely multiplicative collection of sets in a set P. Assume that $\mathscr{F}$ is a finite collection of sets in P such that any two disjoint elements of*

$$[\mathscr{F}] \cap [\mathrm{compl}(\mathscr{M})] \, (= \mathsf{E}\{F \cap N \mid F \in \mathscr{F},\ P - N \in \mathscr{M}\})$$

are $\mathscr{M}$-separated.

Then for each $M \supset \bigcap \mathscr{F}$, $M \in \mathscr{M}$, *there exists a family* $\mathscr{M}_M = \{M_F \mid F \in \mathscr{F}\}$ *ranging in* $\mathscr{M}$ *such that* $F \subset M_F$, *and*

$$\bigcap \mathscr{M}_M \subset M .$$

In particular, if $\bigcap \mathscr{F} = \emptyset$, *then* $\bigcap \mathscr{M}_\emptyset = \emptyset$.

Corollary a. *If* $\mathscr{F}$ *is a finite collection of compact sets in a separated space* P, *and if* U *is a neighborhood of* $\bigcap \mathscr{F}$, *then there exist neighborhoods* U_F *of* F, $F \in \mathscr{F}$, *such that* $\bigcap \{U_F \mid F \in \mathscr{F}\} \subset U$.

Corollary b. *If* P *is normal then we may replace "compact" by "closed" in Corollary a.*

Lemma 1 (and also the corollaries) may be obviously improved as follows.

Lemma 1'. *Under the assumptions in Lemma* 1, *if for each* $\mathscr{E} \subset \mathscr{F}$ *there is given an* $M_{\mathscr{E}} \supset \bigcap \mathscr{E}$, $M_{\mathscr{E}} \in \mathscr{M}$, *then there exists a family* $\{M_F \mid F \in \mathscr{F}\}$ *ranging in* $\mathscr{M}$ *such that* $F \subset M_F$ *and*

$$\bigcap \{M_F \mid F \in \mathscr{E}\} \subset M_{\mathscr{E}}$$

for each $\mathscr{E} \subset \mathscr{F}$.

Proof of Lemma 1. Evidently it is enough to prove Lemma 1 under the assumption that Lemma 1 is true for each collection of a smaller cardinal than the cardinal of $\mathscr{F}$. Let an M in $\mathscr{M}$ with $\bigcap \mathscr{F} \subset M$ be given. Pick an element K in $\mathscr{F}$, and put $\mathscr{F}_1 = \mathscr{F} - (K)$. The sets $K - M$ and $\bigcap \mathscr{F}_1 - M = \bigcap \{F - M \mid F \in \mathscr{F}_1\}$ are disjoint and therefore, by the assumption in Lemma 1, there exists U, V in $\mathscr{M}$ with

$$U \cap V = \emptyset , \quad \text{and} \quad K - M \subset U , \quad \bigcap \mathscr{F}_1 - M \subset V .$$

Applying Lemma 1 to $\mathscr{F}_1$, by the inductive hypothesis, we get a family $\{M'_F \mid F \in \mathscr{F}_1\}$ in $\mathscr{M}$ such that

$$\{M'_F \mid F \in \mathscr{F}_1\} \subset V, \quad \text{and} \quad M_F \supset F - M .$$

Put

$$M_K = U \cup M , \quad M_F = M \cup M'_F \quad \text{for} \quad F \in \mathscr{F}_1 .$$

Clearly $\bigcap \{M_F \mid F \in \mathscr{F}\} = M$, and $M_F \supset F$ for each F in $\mathscr{F}$.

Lemma 2. *Let* $\mathscr{M}$ *be a finitely additive and* σ*-multiplicative collection of subsets of a set* P, *and let* $P \in \mathscr{M}$. *Assume that* $\mathscr{A}$ *is a countable collection of sets in* P *such that any two disjoint elements in* $[\mathscr{A}] \cap [\text{compl.}\,(\mathscr{M})]$ *are* $\mathscr{M}$*-separated. Finally, assume that a family*

$$\{M_{\mathscr{F}} \mid \mathscr{F} \subset \mathscr{A},\ \mathscr{F} \textit{ finite}\}$$

ranging in $\mathscr{M}$ is given such that $\bigcap\mathscr{F} \subset M_{\mathscr{F}}$ for each $\mathscr{F}$. Then there exists a family

$$\{K_A \mid A \in \mathscr{A}\}$$

ranging in $\mathscr{M}$ such that $K_A \supset A$ for all A in $\mathscr{A}$, and

$$\bigcap\{K_A \mid A \in \mathscr{F}\} \subset M_{\mathscr{F}}$$

for each finite $\mathscr{F} \subset \mathscr{A}$.

Proof. By Lemma 1, for each finite $\mathscr{F} \subset \mathscr{A}$ there exists a family $\{K_F^{\mathscr{F}} \mid F \in \mathscr{F}\}$ in $\mathscr{M}$ such that

$$\bigcap\{K_F^{\mathscr{F}} \mid F \in \mathscr{F}\} \subset M_{\mathscr{F}} .$$

For each A in $\mathscr{A}$ let K_A be the intersection of all $K_F^{\mathscr{F}}$, $A \in \mathscr{F} \subset \mathscr{A}$, $\mathscr{F}$ finite. The family $\{K_A\}$ has the required properties.

Theorem 1. *Let $\mathscr{A}$ be a countable collection of analytic sets in a uniformizable space P. Assume that for each finite $\mathscr{F} \subset \mathscr{A}$ a Baire set $Z_{\mathscr{F}}$ is given such that $\bigcap\mathscr{F} \subset$ $\subset Z_{\mathscr{F}}$. Then there exists a family $\{Z_A \mid A \in \mathscr{A}\}$ of Baire sets in P such that*

$$\bigcap\{Z_A \mid A \in \mathscr{F}\} \subset Z_{\mathscr{F}}$$

for each finite $\mathscr{F} \subset \mathscr{A}$.

Proof. Lemma 2 applies to our $\mathscr{A}$, and the set $\mathscr{M}$ of all Baire sets in P. This follows from the author's separation theorem (see [3], or [6]): Any two disjoint analytic sets admit a separation by a Baire set.

Remark. Theorem 1 is not true for spaces which are not uniformizable. In the case of separated spaces we may replace Baire sets by B (open (P)). More precisely, $Z_{\mathscr{F}}$ are assumed to belong to B (open), and Z_A's are required to belong to B (open). Then the first part of the proof of the author's separation theorem for analytic sets and Souslin sets [6, Theorem 1] gives the existence of $F_A \in$ B (closed) such that $A \subset$ $\subset F_A \subset Z_A$.

3. B-spaces. If not otherwise stated, by a space we mean a uniformizable space.

Definition 2. A B-structure on a space P is a complete sequence $\mu = \{\mathscr{M}_n\}$ of countable coverings of P such that

$$(*) \qquad \bigcap M_n = \bigcap\{\text{cl} \bigcap\{M_k \mid k \leqq n\} \mid n \in N\}$$

for any $M_n \in \mathscr{M}_n$.

By "complete" we mean that any μ-Cauchy filter $\mathscr{M}$ (i.e. $\mathscr{M}_n \cap \mathscr{M} \neq \emptyset$ for each n) has a cluster point in P. A B-space is a space on which there exists a B-structure.

It follows immediately from the definition that

Proposition 1. *If $\mu = \{\mathcal{M}_n\}$ is a B-structure on P, then the restriction of μ to any closed subset of P, and also to any element of any $\mathcal{M}_n$, is a B-structure. Hence: Any closed subspace of a B-space, and also any element of any cover in a B-structure, are B-spaces.*

To clarify the condition (*), observe that closed coverings do have the property (*), and the following is true (in contrast to Theorems below).

Theorem 2. *A regular space P is analytic if and only if there exists a complete sequence $\mu = \{\mathcal{M}_n\}$ of countable coverings of P.*

Proof. Assume that $\mu = \{\mathcal{M}_n\}$ is a complete sequence of countable coverings on P. Let $\{M_n^k\}_k$, $n = 1, 2, \ldots$, be a sequence ranging on $\mathcal{M}_n$. For each $\sigma \in \Sigma$ put

$$f\sigma = \bigcap\{\mathrm{cl} \bigcap\{M_n^{\sigma(n)} \mid n \leqq k\} \mid k \in \mathsf{N}\}$$

where $\sigma(n)$ is the n-th coordinate of σ. It may be proved that $f: \Sigma \to P$ is usco-compact and surjective.

Conversely, assume that f is an usco-compact correspondence of Σ onto P. Let, as usual, S be the set of all finite sequences of natural numbers, S_n the set of all $s \in S$ of the length n. Write σ_n for the restriction of $\sigma \in \Sigma$ to the initial segment of N of the length n. Define an order $s < t$ on S_n to mean that $s \neq t$ and the first coordinate of s distinct from that of t is less than that of t. Put

$$\Sigma s = \mathsf{E}\{\sigma \mid \sigma \in \Sigma,\ s < \sigma\}$$

where $s < \sigma$ means that $s = \sigma_n$ for some n, and define

$$Ms = f[\Sigma s] - \bigcup\{f[\Sigma t] \mid t < s\}\,, \quad \mathcal{M}_n = \mathsf{E}\{Ms \mid s \in S_n\}\,.$$

It is easy to see that $Ms \cap Mt = \emptyset$ for $s \neq t$, $s \in S_n$, $t \in S_n$, and that $Ms \cap Mt \neq \emptyset$ implies that either s is a section of t or t is a section of s. It follows that if $\mathcal{M}$ is a Cauchy filter, and $M_n \in \mathcal{M} \cap \mathcal{M}_n$, then there exists a σ in Σ such that if $M_n = Ms$ then $s < \sigma$. Since f is usco-compact, that implies that $\bigcap\{\mathrm{cl}\, M \mid M \in \mathcal{M}\}$ has a cluster point in $f\sigma$.

Corollary. *Every B-space is an analytic space. Each element of each cover in a B-structure is analytic.*

Remark. A B-structure such that the coverings are disjoint is called a Borelian structure (usually it is assumed that the $(n + 1)$-th covering refines the n-th covering, however this is not important). A space P is Borelian if and only if there exists a Borelian structure on P, see [3], Section 3. Borelian spaces are characterized as absolute

$$([\text{closed}] \cap [\text{Baire}])_{\sigma d \delta}$$

where the subscript d indicates that just disjoint unions are considered. (See [3], section 3.) A similar characterization of B-spaces is not known; however if the space P is quasi-classical at infinity (see the following definition) then P is a B-space if and only if P is absolute

$$([\text{closed}] \cap [\text{Baire}])_{\sigma\delta} .$$

Definition 3. A space R is called quasi-classical if R is the image of a separable metrizable space under an usco-compact correspondence. A space P is called quasi-classical at infinity if $K - P$ is quasi-classical for some, and then any, compactification K of P.

Observe that the class of all quasi-classical spaces is closed under usco-compact correspondences, and hence the class of all spaces quasi-classical at infinity is closed under proper mappings in both directions.

Theorem 4. *Assume that a B-space P is a subspace of a space Q such that $Q - P$ is quasi-classical. Then*

$$(*) \qquad P \in ([\text{closed}\,(Q)] \cap [\text{Baire}\,(Q)])_{\sigma\delta} .$$

It follows that $(*)$ is true for any $Q \supset P$ if P is a B-space quasi-classical at infinity.

The proof follows immediately from the following

Lemma 3. *Let $\alpha = \{\mathscr{A}_n\}$ be a complete sequence of countable coverings of a space P such that the elements of $\mathscr{A} = \bigcup\{\mathscr{A}_n\}$ are analytic subspaces of P. Assume that $Q \supset P$ is a space such that $Q - P$ is quasi-classical. Then there exists a family $\{Z_A \mid A \in \mathscr{A}\}$ of Baire sets in Q such that*

$$P = \bigcap\{\bigcup\{Z_A \cap \text{cl}_Q\, A \mid A \in \mathscr{A}_n\} \mid n \in \mathsf{N}\} ,$$

(and $Z_A \supset A$ for each A).

Proof of Lemma 3. Let $\mathscr{A}$ be the union of all $\mathscr{A}_n$, $n \in \mathsf{N}$. By our assumption there exists an usco-compact correspondence k of a separable metrizable space R onto $Q - P$. Choose a countable base $\mathscr{B}$ of R. For each finite $\mathscr{F} \subset \mathscr{A}$ and each B in $\mathscr{B}$ let $Z(\mathscr{F}, B)$ be a Baire set in Q such that $\bigcap\mathscr{F} \subset Z(\mathscr{F}, B)$, and if there exists a Baire set Z with $\bigcap\mathscr{F} \subset Z \subset Q - k[B]$, then $Z(\mathscr{F}, B) \subset Q - k[B]$. Let, for each finite $\mathscr{F} \subset \mathscr{A}$, the set $Z_{\mathscr{F}}$ be the intersection of all $Z(\mathscr{F}, B)$, $B \in \mathscr{B}$.

Apply Theorem 1 to Q, the collection $\mathscr{A}$ and the family $\{Z_{\mathscr{F}}\}$; we get a family $\{Z_A \mid A \in \mathscr{A}\}$ of Baire sets such that $A \subset Z_A$, and $\bigcap\{Z_A \mid A \in \mathscr{F}\} \subset Z_{\mathscr{F}}$ for each finite $\mathscr{F} \subset \mathscr{A}$. We shall prove that

$$P = \bigcap\{\bigcup\{Z_A \cap \text{cl}_Q\, A \mid A \in \mathscr{A}_n\} \mid n \in \mathsf{N}\} .$$

The inclusion $\subset$ being evident, assume that a point $x \in Q - P$ is in the set on the right-hand side. Thus there exist A_n in $\mathscr{A}_n$ such that

$$(**) \qquad x \in \bigcap\{Z_{A_n} \cap \mathrm{cl}_Q A_n\} .$$

Consider the set

$$K = \bigcap\{\mathrm{cl}_P \bigcap\{A_m \mid m \leqq n\} \mid n \varepsilon \mathsf{N}\} .$$

Since α is complete, the set K is compact (it may be empty!), and for each neighborhood U of K there exists an $n_U \in \mathsf{N}$ such that

$$\mathrm{cl}_P \bigcap\{A_m \mid m \leqq n_U\} \subset U .$$

Choose a y in R with $x \in ky$. The set ky is compact, and if V is a neighborhood of ky in Q then there exists a B_V in $\mathscr{B}$ such that

$$y \in B_V , \quad \text{and} \quad k[B_V] \subset V .$$

The sets K and ky are disjoint, because $K \subset P$ and $ky \cap P = \emptyset$. The space Q is uniformizable and therefore there exists a neighborhood Z of K, which is a zero set in Q such that $Z \cap ky = \emptyset$.

Put $\mathscr{F} = \mathsf{E}\{A_m \mid m \leqq n_Z\}$, aud consider the set $k[B_{Q-Z}]$ $(\subset Q - Z)$. Therefore

$$Z_{\mathscr{F}} \subset Q - k[B_{Q-Z}] .$$

Hence

$$\bigcap\{Z_{A_n} \mid n \leqq n_Z\} \cap ky = \emptyset ,$$

thus

$$x \notin \bigcap\{Z_{A_n} \mid n \leqq n_Z\} .$$

This contradicts our assumption (**) above, and establishes the converse inclusion.

Remark. In the case of a Borelian structure the proof is much more simpler without any assumption on $Q - P$. By the first separation theorem we can choose Baire sets Z_A, $A \in \mathscr{A}$, in Q such that $Z_A \supset A$ for each A in $\mathscr{A}$, $Z_A \subset Z_B$ if $A \subset B$, and the collections

$$\{Z_A \mid A \in \mathscr{A}_n\}$$

are disjoint. Then we may interchange $\bigcup$ and $\bigcap$ in

$$\bigcap\{\bigcup\{Z_A \cap \mathrm{cl}_Q A \mid A \in \mathscr{A}_n\} \mid n \in \mathsf{N}\} ,$$

and the resulting set is easy to prove to be P (the proof is the same as in the proof of Theorem 2). For details see [3], Theorem 11 or [6], Theorem 7. This method can be used to prove that if there exists a B-structure on a space P such that the elements of the coverings are Baire sets in P, then P is Borelian.

Proof of Theorem 4. Any element of an B-structure is analytic by Corollary to Theorem 2, and hence Lemma 3 applies to the first part of Theorem 4. To prove the second part it is enough to verify it for compactifications Q of P, and this follows from the first part.

Proof of Theorem 3. Let P be any space. Clearly condition (4) implies conditions (1), (2), and (3); condition (3) implies condition (1); and condition (2) implies condition (1). Assume now that P is metrizable. It remains to prove that condition (1) is sufficient, and condition (4) is necessary. First assume (1). Every analytic space is Lindelöf as an usco-compact image of a Lindelöf space, namely Σ. Any metrizable Lindelöf space is separable, and hence P is separable. Take a metrizable compactification K of P, and apply Lemma 3 to get that P is a Borel set in K, which established the sufficiency of condition (1).

The proof of necessity and sufficiency of condition (4) without any use of classical results is given in [3]. The proof is easy if we know that each separable absolute Borel set is the image of a closed subspace F of Σ under a one-to-one continuous mapping f. Then a Borelian structure is constructed as follows. For each finite sequence s of natural numbers denote by Σs the set of all σ in Σ that extend s. The n-th covering of a required Borelian structure consists of all $f[\Sigma s]$ with s of length n. The verification is routine.

We note the following

Corollary to Theorem 3. *A metrizable space is a B-space if and only if it is a separable absolute Borel set.*

Definition 4. A box in $\Sigma = \mathsf{N}^{\mathsf{N}} = \prod\{\mathsf{N} \mid n \in \mathsf{N}\}$ is a set of the form $\prod\{X_n \mid n \in \mathsf{N}\}$. A correspondence of Σ into something is said to be boxed if the preimages of points are boxes. For example, any dusco-compact correspondence is boxed.

Theorem 5. *A space P is a B-space if and only if there exists a boxed usco-compact correspondence of Σ onto P.*

Proof. Let $\mu = \{\mathscr{M}_n\}$ be a B-structure on P and let arrange $\mathscr{M}_n$ in a sequence $\{M_n^k \mid k \in \mathsf{N}\}$. Define f by

$$f\sigma = \bigcap\{\operatorname{cl} \bigcap\{M_n^{k_n} \mid n \leqq m\} \mid m \in \mathsf{N}\} = \bigcap\{M_n^{k_n}\}$$

where $\sigma = \{k_n\}$. Then

$$f[\Sigma_j^i] = M_j^i,$$

where Σ_j^i is the set of all σ where j-th coordinate is i. It is easy to see that f is boxed and usco-compact.

Assume that f is a boxed usco-compact mapping of Σ onto P, $M_j^i = f[\Sigma_j^i]$, and let $\mathcal{M}_n$ be the set of all M_n^i, $i \in \mathsf{N}$. We shall prove that $\mu = \{\mathcal{M}_n\}$ is a B-structure on P. Let $\mathcal{M}$ be a Cauchy filter, $M_n^{i_n} \in \mathcal{M}$, $n \in \mathsf{N}$. Consider the point $\sigma = \{i_n\} \in \Sigma$. Since f is a boxed correspondence, we have

$$f[\Sigma\{i_n \mid n \leqq k\}] = \bigcap\{f[\Sigma_n^{i_n}] \mid n \leqq k\}$$

for each n, and therefore the sets

$$\mathsf{D}f \cap \Sigma s\,, \quad s < \sigma$$

must be non-void, and $\sigma \in \mathsf{D}f$. If cl $M \cap f\sigma$ were empty for some M in $\mathcal{M}$, then we would choose an $s < \sigma$ such that $f[\Sigma s] \cap \text{cl}\, M = \emptyset$, which would contradict the fact that $\mathcal{M}$ is a filter. Therefore the collection of all $f\sigma \cap \text{cl}\, M$, $M \in \mathcal{M}$, is a filter base on $f\sigma$, and $f\sigma$ being compact, $\mathcal{M}$ has a cluster point in $f\sigma$. The proof is finished.

Proof of Theorem 5′. Apply Theorem 5 and Corollary to Theorem 3 (following the proof of Theorem 3).

Using the correspondence technique it is easy to prove that the collection of B-spaces $P \subset Q$ is closed under B_d.

Theorem 6. *The set of all B-spaces $P \subset Q$, where Q is a space, is closed under countable intersections, and countable disjoint unions.*

Proof. Let f^n, $n \in \mathsf{N}$, be boxed usco-compact correspondences of Σ into Q. Define

$$f : \Sigma^{\mathsf{N}} \to Q$$

by

$$f\{\sigma(n)\} = \bigcap\{f^n_{\sigma(n)}\}\,.$$

It is easy to see that f is boxed, and it is easy to show that f is usco-compact. Clearly

$$\mathsf{E}f = \bigcap\{\mathsf{E}f^n\}\,.$$

As concerns the disjoint unions, if $\{\mathsf{E}f^n\}$ is disjoint then

$$g : \sum\{\Sigma \mid n \in \mathsf{N}\} \to Q$$

defined by

$$g\langle n, \sigma\rangle = f^n\sigma$$

is boxed, and of course, usco-compact. Clearly $\mathsf{E}g = \bigcup\{\mathsf{E}f^n\}$. The spaces Σ^{N} and $\sum\{\Sigma \mid n \in \mathsf{N}\}$ are homeomorphic to Σ, which proves Theorem 6.

Theorem 7. *If P is the image under a one-to-one continuous mapping of a space R which is $K_{\sigma\delta}$ in some $Q \supset R$, then P is a B-space.*

This follows immediately from the following two propositions; the proof of the first one is left to the reader. As usual, $K_{\sigma\delta}$ means a countable intersection of σ-compact sets.

Proposition 2. *The image of a B-structure under a one-to-one continuous mapping is a B-structure.*

Proposition 3. *If R is a $K_{\sigma\delta}$ in Q, then R is a B-space.*

Proof. Let $R = \bigcap_n \bigcup_m \{K(n, m)\}$ with $K(n, m)$ compact. Put $\mathcal{M}_n = \mathsf{E}\{R \cap K(n, m) \mid m \in \mathsf{N}\}$, and prove that $\{\mathcal{M}_n\}$ is a B-structure on R.

Remark. In [4] the term B-space was used in another meaning; B-spaces in the sense of [4] coincide with analytic spaces by Theorem 2 above.

The main unsolved question concerning B-spaces is to decide if there exists an external characterization of B-spaces in the form "absolute something". The second problem is to find how much larger is the class of all B-spaces than the class of all Borelian spaces. We know that each $K_{\sigma\delta}$ is a B-space (Theorem 7), while a K_σ need not be Borelian ([3], Remark following Theorem 10). The third problem: is the class of all B-spaces $P \subset Q$ closed under countable unions? One more: Is any B-space the image of a $K_{\sigma\delta}$ under a one-to-one continuous mapping?

Call a space P a K-Borel space if P belongs to B (compact (Q)) for some $Q \supset P$. There are many unsolved problems concerning K-Borel spaces, e.g., is a metrizable K-Borel space an absolute Borel set?

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