Czechoslovak Mathematical Journal

Bohdan Zelinka

Centrally symmetric Hasse diagrams of finite modular lattices

Czechoslovak Mathematical Journal, Vol. 20 (1970), No. 1, 81–83

Persistent URL: http://dml.cz/dmlcz/100945

Czechoslovak Mathematical Journal, 20 (95) 1970, Praha

CENTRALLY SYMMETRIC HASSE DIAGRAMS OF FINITE MODULAR LATTICES

BOHDAN ZELINKA, Liberec

(Received February 5, 1969)

In [3] a centrally symmetric graph, or S-graph, is defined as an undirected graph without loops and multiple edges fulfilling the following conditions:

(1) G contains at least one edge;

(2) for each triplet $\{x, y, z\}$ of its verticex such that $\varrho_G(y, z) = 1$ we have $\varrho_G(x, y) \neq \varrho_G(x, z)$;

(3) for each vertex x of G exactly one vertes $\bar{x}$ exists such that for each vertex w of a neighbourhood of $\bar{x}$ we have $\varrho_G(x, \bar{x}) > \varrho_G(x, w)$.

Here $\varrho_G(a, b)$ denotes the distance of a and b in G. The vertices x and $\bar{x}$ are called opposite to each other.

In [3] the following theorems are proved.

(A) *If for each chosen vertex x of G there exists a Jordan-Dedekind lattice such that its Hasse diagram* (see [2], [4]) *is isomorphic to G and its greatest element is x, then G is an S-graph.*

(B) *If G is an arbitrary S-graph and x is its vertex, then $\bar{\bar{x}} = x$.*

(C) *If G is an arbitrary S-graph and d is its diameter, then arbitrary two opposite vertices and only such two vertices have the distance d.*

Further in [3] A. KOTZIG suggests to study such S-graphs which satisfy the assumption of **(A)**. He conjectures that these graphs are $C^1, K_6, K_8, \ldots$ and Cartesian products of these graphs. This conjecture is expressed also in [1], among the unsolved problems. The symbol C^1 denotes the graph consisting of exactly one edge and its end vertices, the symbol K_n denotes the circuit with n vertices.

In this paper we shall study only S-graphs which satisfy the assumption of **(A)** so that the corresponding lattices are modular and finite.

Theorem. *Let L be a finite modular lattice with n atoms such that its Hasse diagram* *is an S-graph. Then L is a Boolean algebra and its Hasse diagram is the graph of the n-dimensional cube.*

Remark. The assumption of this theorem is more general than that of **(A)**. On the other side, it is evident that the graph of an n-dimensional cube, because of its high degree of symmetry, satisfies not only the assumption of this theorem, but even the assumption of **(A)**.

This result does not contradict to Kotzig's conjecture, because the graph of the n-dimensional cube is the n-th Cartesian power of the graph C^1.

Before proving Theorem we shall state some lemmas.

By $d(x)$ the dimension function on L is denoted.

Lemma 1. *Let L be a finite modular lattice whose Hasse diagram is an S-graph. Then for each $a \in L$ we have $a \wedge \bar{a} = O$, $a \vee \bar{a} = I$.*

Remark. We do not distinguish the elements of L and the vertices of the Hasse diagram of L.

Proof. Assume that $a \wedge \bar{a} = b \succ O$. Then there exists a saturated chain C_1 of the length $d(a) - d(b)$ in L whose least element is b and greatest element is a and a saturated chain C_2 of the length $d(\bar{a}) - d(b)$ in L whose least element is b and greatest element is $\bar{a}$. In the Hasse diagram of L two elementary paths of the same lengths correspond to these chains. The union of these paths is a path joining a and $\bar{a}$ of the length $l = d(a) + d(\bar{a}) - 2d(b)$. As L is modular, we have $d(a) + d(\bar{a}) = d(a \wedge \bar{a}) + d(a \vee \bar{a}) = d(b) + d(a \vee \bar{a})$, so $l = d(a \vee \bar{a}) - d(b)$. As $a \vee \bar{a} \leqq I$, we have $d(a \vee \bar{a}) \leqq d(I) = d(L)$, and as $b \succ O$, we have $d(b) > d(0) = 0$. This implies $l < d(L)$ which is a contradiction because the diameter of the Hasse diagram of L is evidently $d(L)$. So we have proved $a \wedge \bar{a} = O$. The proof of $a \vee \bar{a} = I$ is dual.

Lemma 2. *Let G be the Hasse diagram of a finite modular lattice L. Let a, b be two of its vertices (and at the same time elements of L) and let P_0 be an elementary path of the minimal length l joining $a \in L$ and $b \in L$. Then there exists a path P' of the length l joining a and b so that $P' = P_1' \cup P_2'$ where P_1' and P_2' are Hasse diagrams of two chains C_1 and C_2 in L, the least element of C_1 or C_2 is a or b respectively; the chains C_1 and C_2 have a common greatest element which is their only common element.*

Proof. As L is a modular lattice, there exists a dimension function $d(x)$ on L such that $d(x) + d(y) = d(x \wedge y) + d(x \vee y)$ for arbitrary x and y of L. If two elements x and y of L are joined by an edge in G, then either $d(y) = d(x) + 1$ or $d(y) = d(x) - 1$. If P is an elementary path in the Hasse diagram of L, we denote by $D(P)$ the sum of $d(x)$ for all vertices x of the path P. Now let P_0 be an elementary path of the length l joining a and b in the Hasse diagram of L. Let P_0 contain three elements x, y, z such that xy and yz are the edges of P_0 and $d(x) = d(z)$, $d(y) = d(x) - 1$. Then x and z

cover y, so $y = x \wedge z$ and, as L is modular, $x \vee z$ covers x and z. Denote $x \vee z = t$. Omit the vertex y and the edges xy and yz from P_0 and substitute them by the vertex t and the edges xt and tz. We obtain a path P_1 again of the length l joining a and b. We have $D(P_1) = D(P_0) + 2$. We continue this process and obtain a sequence $P_0, P_1, P_2, \ldots$ of the paths between a and b having all the same length l such that $D(P_i)$ increases. As $D(P_i)$ increases, no path can occur in the sequence more than once. As L is finite, such a sequence can have only a finite number of elements. Thus the last path P' in the sequence is a path in which no element (vertex) is covered by two other vertices of the path. Such a path must be a path described in the assertion of the lemma. As all paths of the sequence have the length l, also P' has this length.

Lemma 3. *Let L be a finite modular lattice whose Hasse diagram is an S-graph. If $a \in L$, $b \in L$, $b \neq \bar{a}$, then either $a \vee b \prec I$ or $a \wedge b \succ O$.*

Proof. As $b \neq \bar{a}$, the distance between a and b in the Hasse diagram of L is less than $d(L)$. Let P be a shortest elementary path between a and b, let l be its length. According to Lemma 2 there exists a path P' of the length l between a and b such that $P' = P'_1 \cup P'_2$ where P'_1 and P'_2 are Hasse diagrams of two saturated chains C_1 and C_2 in L, the least element of C_1 or C_2 is a or b respectively, the chains C_1 and C_2 have a common greatest element which is their only common element. Let this element be denoted by c. The length of C_1 is $d(c) - d(a)$, the length of C_2 is $d(c) - d(b)$. Thus the length of P' (and also of P) is $l = 2\,d(c) - d(a) - d(b)$. As $l < d(L)$, we have $2\,d(c) - d(a) - d(b) < d(L)$. Assume that $a \wedge b = O$. Then $d(a) + d(b) = d(a \wedge b) + d(a \vee b) = d(O) + d(a \vee b) = d(a \vee b)$. So we have $2\,d(c) < d(L) + d(a \vee b)$. The element c is greater than both a and b, so $c \geqq a \vee b$ and $d(c) \geqq d(a \vee b)$. Thus $2\,d(a \vee b) \leqq 2\,d(c) < d(L) + d(a \vee b)$, from which $d(a \vee b) < d(L)$ follows and, as $d(L) = d(I)$, we have $a \vee b \prec I$. So $a \wedge b = O$ implies $a \vee b \prec I$, q.e.d.

Proof of Theorem. If $a \in L$, then according to Lemma 1 there exists at least one complement of a, the opposite element $\bar{a}$, and according to Lemma 3 no other complement of a exists. So L is a uniquely complementary modular lattice. According to [4], p. 125, the lattice L is distributive. As L is distributive and uniquely complementary, it is a Boolean algebra. As it is well-known, the Hasse diagram of the finite Boolean algebra with n atoms is the graph of the n-dimensional cube.

References

[1] Beiträge zur Graphentheorie. Leipzig 1968.
[2] *G. Birkhoff:* Lattice Theory. New York 1948.
[3] *A. Kotzig:* О центрально симметрических графах. Czech. Math. J. *18* (1968), 606—615.
[4] *G. Szász:* Einführung in die Verbandstheorie. Leipzig 1962.

Author's address: Liberec, Studentská 5, ČSSR (Vysoká škola strojní a textilní).