Czechoslovak Mathematical Journal

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On a certain class of Λ-structures. I

Czechoslovak Mathematical Journal, Vol. 20 (1970), No. 1, 1–18

Persistent URL: http://dml.cz/dmlcz/100940

CZECHOSLOVAK MATHEMATICAL JOURNAL
Mathematical Institute of Czechoslovak Academy of Sciences at Prague
V. *20* (*95*), PRAHA 26. 3. 1970, No 1

ON A CERTAIN CLASS OF Λ-STRUCTURES. I.

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(Received December 12, 1965, in revised form August 31, 1968)

Introduction. The purpose of this paper is to present a systematic discussion of Λ-structures on uniform spaces.*)

The concept of a Λ-structure was introduced by M. KATĚTOV in [17] and [18]. Further results were obtained by M. KATĚTOV (cf. [19], [20]) and by D. A. RAIKOV (cf. [26]). Some aspects of the algebraic part of Λ-structures have been partially studied in the integration theory (cf. [4], [16]). This point of view was applied for the first time in [25] as a tool of investigation in some questions of compactness in locally convex spaces.

In further considerations we shall be concerned with Λ-structures on uniform spaces. The case of a completely regular space may be included in the preceding one.

The main idea of Λ-structures consists in embedding of any uniform (completely regular) space X into a locally convex space $E(X)$ (cf. [18]). This embedding depends on the continuity structure of X and may, of course, satisfy different sorts of continuity. The space $E(X)$ endowed with an algebraic-topological structure makes it possible to apply the technics of locally convex spaces to diverse investigations of uniform and topological properties of continuity structures.

Let us briefly sketch the contents of this paper. In Section 2 we define the concept of a Λ-structure following [17]. In Section 3 we establish some elementary properties on the dual of a Λ-structure. Some Λ-structures projectively generated by continuous mappings are characterized in the following section. Section 5 is devoted to a norm topology. In Section 6 we present some characterizations of completion of uniform spaces. The Banach-Stone theorem is generalized in the last section.

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*) Revised and augmented. Originally submitted under the titles: *On a certain class of Λ-structures. I., An application of Λ-structures to the compactness, A dual characterization of pseudo-compact spaces.*

The author would like to apologize for a delay in publication which was caused by certain unfavourable and unusual circumstances.

In the second part of this paper we shall study mainly the completion of Λ-structures, the compactness in locally convex spaces and certain generalizations of the extension theorem of V. PTÁK (cf. [24]).

Some results of these articles were communicated at the Seminar of Professor M. KATĚTOV and at the Seminar of Professor D. A. RAIKOV. The present arrangement of the subject coincides with that in [27].

We refer to [2], [3] and to [22] as for the terminology and theorems used throughout this paper.

The author wishes to express his gratitude to Professor M. KATĚTOV for his helpful suggestions and would like to take this opportunity of thanking to Professor D. A. RAIKOV and to Professor V. PTÁK for their valuable remarks and conversations.

1. PRELIMINARY RESULTS AND NOTATION

Let X be an infinite set. The vector space (cf. [17], [18]) of all finite formal linear combinations

$$\sum \lambda_i x_i \tag{1}$$

where $x_i \in X$, λ_i are real numbers, will be denoted by $E(X)$.

For any function f on X the corresponding linear extension $\tilde{f}$ to $E(X)$ is defined by

$$\langle z, \tilde{f} \rangle = \sum \lambda_i \langle x_i, f \rangle \tag{2}$$

where z is of the form (1). We shall use frequently the same letter for the function f as for its linear extension $\tilde{f}$. All functions considered here are assumed to be defined on the whole set X and to be real-valued.

Throughout this paper we suppose, in reality, that X is a uniform (completely regular) space and that all functions are uniformly continuous (continuous). The context will make it clear whether X is taken as a uniform space or as a completely regular space. The uniformity on any completely regular space is defined in a natural way below. For convenience we shall study from now onwards only (infinite) separated uniform spaces.

If $\mathscr{U}$ is a uniformity on X we write $(X, \mathscr{U})$. $P(X)$ denotes the Banach algebra of all uniformly bounded and uniformly continuous functions on X. The norm of an element $f \in P(X)$ is defined by

$$\|f\| = \sup_{x \in X} |\langle x, f \rangle| .$$

The linear space $P_0(X)$ consists of all uniformly continuous functions on X. We write $\mathscr{H} = \mathscr{H}(X)$ for the family of all uniformly bounded and uniformly equicontinuous subsets of $P(X)$. Similarly $\mathscr{H}_0 = \mathscr{H}_0(X)$ represents the collection of all uniformly equicontinuous subsets of $P_0(X)$ bounded in the weak topology (i.e. for arbitrary $x \in X$ and any $H \in \mathscr{H}_0$, $\{\langle x, f \rangle, f \in H\}$ is a bounded subset of real numbers).

The topology on $E(X)$ defined by the system $\mathscr{H}$ or $\mathscr{H}_0$ will be denoted by $t = t(X)$ or by $t_0 = t_0(X)$ respectively. If λ is a topology on $E(X)$ we write $(E(X), \lambda)$.

With respect to the duality of $\langle E(X), P(X) \rangle$ we can define the weak topology $\sigma = \sigma\langle E(X), P(X) \rangle$ on $E(X)$. For the same reasons $\sigma_0 = \sigma_0(E(X), P_0(X))$ is a weak topology on $E(X)$ of the dual pair $\langle E(X), P_0(X) \rangle$. It is evident that $\sigma \leqq t \leqq t_0$, $\sigma_0 \leqq t_0$ (the ordering relation has the same sense as in [2]).

If F is a locally convex space, we write F^* for the space of all continuous linear functions on F. The canonical embedding $\omega : X \to E(X)$ is defined by $\omega : x \to 1 . x$.

Throughout this paper we identify X with $\omega(X)$. Let us recall (cf. [26]) that ω is an isomorphism of X into $(E(X), \lambda)$ where $\lambda = t, t_0$. This implies that any equicontinuous subset on $(E(X), t)$ is of the form $\tilde{H} = \{\tilde{f}; f \in H\}$, $H \in \mathscr{H}(X)$. The same is true for the system $\mathscr{H}_0(X)$. Particularly it holds $(E(X), \sigma)^* = P(X)$ and $(E(X), \sigma_0)^* = P_0(X)$.

Because the system $\mathscr{H}$ is stable with respect to the operation of forming the absolutely convex envelope and to the closure operation in the topology $\sigma(P(X), X) = \sigma(P(X), E(X))$, it follows from the Mackey theorem (cf. [3]) that $(E(X), t)^* = P(X)$. Similarly we have $(E(X), t_0)^* = P_0(X)$ (cf. [26]).

2. THE Λ-STRUCTURE ON X

Now we introduce the concept of the Λ-structure on a uniform space.

Definition 1. The linear space $E(X)$ with a locally convex topology will be called a Λ-structure on X.

For example, the spaces $(E(X), \lambda)$, where $\lambda = t, t_0, \sigma, \sigma_0$, represent some Λ-structures on X.

The Λ-structures $(E(X), t)$ and $(E(X), t_0)$ may be regarded as a certain locally convex extensions of the uniform space X. This implies the possibility of applying the theory of locally convex spaces to some properties of uniform spaces. For example, a subset A of a uniform space X will be called bounded if A is bounded in $(E(X), t_0)$. Evidently this definition coincides with that given in [13]. Some results proved in [13] can be obtained in such manner directly from the theory of locally convex spaces.

If A is a subspace in X, then evidently $E(A)$ is algebraically isomorphic to a linear subspace in $E(X)$. We can of course identify $E(A)$ with its canonical image in $E(X)$. The Λ-structure $E(A)$ with the weak topology $\sigma(E(A), P(A))$ may be regarded as a subspace of $(E(X), \sigma)$. This follows from a result of M. KATĚTOV (cf. [21]). An analogous statement for the topology σ_0 is not true. Evidently, $(E(A), \sigma_0(A))$ is a subspace in $(E(X), \sigma_0(X))$ if and only if any function $f \in P_0(A)$ admits a uniformly continuous extension to X. Such an extension does not, in general, exist (cf. [5], [1]). The canonical embedding $(E(A), \sigma_0(A)) \to (E(X), \sigma_0(X))$ is obviously continuous. For the same reasons $(E(A), t_0(A))$ is not, in general, a subspace of $(E(X), t_0(X))$. For the topology t the question seems to be open. If $\hat{X}$ is the completion of the uniform space X, then $(E(X), t(X))$ is a subspace of $(E(\hat{X}), t(\hat{X}))$. The simple proof of this statement may be left to the reader.

Thus the concept of a Λ-structure indicates another approach to the questions stated above. Further applications will be given in the following sections.

Let $X = (X, \tau)$ be a completely regular space, $C(X)$ the Banach algebra of all bounded and continuous functions on X with the usual norm. We denote by $\mathscr{M}$ the system of all uniformly bounded and equicontinuous subsets M of $C(X)$. Similarly as above we put t_c for the uniformity on X (topology on $E(X)$) defined by the system $\mathscr{M}$. It is easy to see that t_c is compatible with the topology τ. The space of all uniformly bounded and uniformly continuous functions on (X, t_c) is identical with $C(X)$. Obviously the family $\mathscr{M}$ is identical with $\mathscr{H}(X, t_c)$.

Proposition 1. *Let u be a mapping of X into a uniform space Y. Then the following assertions are equivalent:*

(a) *u is a continuous mapping of (X, τ) into Y.*

(b) *u is a uniformly continuous mapping of (X, t_c) into Y.*

Proof. It suffices to prove (a) $\Rightarrow$ (b). This follows from the fact that for any uniformly bounded and uniformly equicontinuous subset $H \subseteq P(Y)$ the system $H \circ u$ is an element of $\mathscr{M}$.

Similarly, by t_{0c} we mean the uniformity on X (topology on $E(X)$) determined by the family $\mathscr{M}_0$ of all equicontinuous and weakly bounded subsets in the vector space $C_0(X)$ of all continuous functions on X.

Under a Λ-structure on a completely regular space X we mean in further discussion the Λ-structure on the uniform space (X, λ), where $\lambda = t_c, t_{0c}$. Therefore the results concerning the completely regular spaces will be formulated only exceptionally.

Added in proof. It is to be noted that by DUGUNDJI's theorem (cf. Pacif. J. Math. 1 (1951), 353—367) any continuous mapping $f\colon A \to E$, A closed in a metric space X and E locally convex, admits a continuous extension $\tilde{f}\colon X \to E$. From here one deduces that $(E(A), t_{0c})$ is in this case a topological subspace of $(E(X), t_{0c})$. If, moreover, f is bounded, then we observe the corresponding extension $\tilde{f}$ presented in the quoted paper satisfy the same property. Consequently the embedding $(E(A), t_c) \to (E(X), t_c)$ is a topological one. These statements answer partially a question raised in Section 2. Furthermore, it may be proved that the mentioned operator of extension $f \to \tilde{f}$ maps $\mathscr{M}_0(A)$ into $\mathscr{M}_0(X)$ and $\mathscr{M}(A)$ into $\mathscr{M}(X)$.

3. THE DUAL SPACE

In this section we shall establish some elementary statements concerning the dual spaces of the above considered Λ-structures. Later we shall obtain two theorems characterizing precompact and pseudocompact spaces.

For a uniform space X the space $P(X)$ is algebraically identical with $(E(X), t)^*$. Now we intend to elucidate the question what kind of topology is induced on $P(X)$ by the strong dual space $(E(X), t)$. Next we shall describe the structure of all bounded subsets in $(E(X), t)$.

Evidently X is a bounded subset of $(E(X), t)$, therefore any scalar multiple of the absolutely convex envelope ΓX of X is bounded in $(E(X), t)$. Conversely it holds

Proposition 2. *Let B be a bounded subset of $(E(X), t)$. Then for some suitable integer n the subset B is contained in $n\,\Gamma X$.*

Proof. Suppose that B is bounded and that B is not contained in any set of the form $n\,\Gamma X$, $n = 1, 2, \ldots$ Then for any $n = 1, 2, \ldots$ we can choose an element $z_n \in B$, $z_n \notin n\,\Gamma X$, $z_n = \sum \lambda_i^n x_i^n$; obviously $\sum |\lambda_i^n| > n$. Let f_n be a function on $P(X)$, $-1 \leqq f_n \leqq 1$, $n = 1, 2, \ldots$, with $f_n(x_i^n) = 1$ for $\lambda_i^n > 0$ and $f_n(x_i^n) = -1$ for $\lambda_i^n < 0$.

Evidently the subset H of all $(n)^{-1/2} f_n$, $n = 1, 2, \ldots$, is an element of $\mathscr{H}$, hence $\{\langle z, f \rangle;\ z \in B, f \in H\}$ is a bounded subset of the real line. This contradicts the choice of H, because it holds $(n)^{-1/2} f_n(z_n) > (n)^{1/2}$.

The following theorem is a generalization of the above mentioned statement of D. A. Raikov.

Theorem 1. *The canonical mapping $f \leftrightarrow \tilde{f}$ is a topological isomorphism of the locally convex structure of $P(X)$ onto the dual space $(E(X), t)^*$ with the strong topology.*

Proof. It suffices to note that the canonical image of the unit ball in $P(X)$ is identical with the polar set of X in $(E(X), t)^*$. The rest of the proof follows immediately from Proposition 2.

Corollary. *Let X be a uniform space, then $(E(X), t)^*$ is a complete space in the strong topology. Particularly, if X is precompact, then $(E(X), t)^*$ is a complete space in the topology of precompact convergence.*

Remark. It should be noticed that the last statement may be obtained directly by an elementary reasoning. In the same way it can be proved that for a precompact space X the dual space $P(X)$ is complete in the Mackey topology $\tau(P(X), \hat{E})$ where $\hat{E}$ is the completion of $(E(X), t)$. This follows in another way from some theorems on completion of Λ-structures (see below).

The following theorem was suggested by a theorem of A. Grothendieck (cf. [12]).

Theorem 2. *Let U be the absolutely convex envelope of X in $E(X)$. The uniformity of the Banach space $P(X)$ will be denoted by ϱ^*. Then the following assertions are equivalent*:

(a) *X is precompact.*

(b) *The uniformities t and σ are identical on U.*

(c) *Any $H \in \mathscr{H}$ is relatively compact in $P(X)$.*

(d) *The uniformities ϱ^* and σ are identical on each $H \in \mathscr{H}$.*

Proof. (a) $\Rightarrow$ (d). If $H \in \mathscr{H}$ then the uniformity $\sigma = \sigma(P(X), E(X))$ coincides on H with the uniformity p defined by the family of all precompact subsets of $(E(X), t)$. Evidently we have $\sigma \leqq \varrho^* \leqq p$. This implies (d).

(d) $\Rightarrow$ (c). If (d) holds, then each $H \in \mathscr{H}$ is relatively compact in $P(X)$.

(c) $\Rightarrow$ (b). It suffices to note that U is equicontinuous on the space $(P(X), \varrho^*)$.

(b) $\Rightarrow$ (a). Immediately.

Theorem 3. *Let X be a completely regular space, U the set defined in Theorem 2, ϱ^* the uniformity of the Banach space $C(X)$. Then the following properties are equivalent*:

(a) *X is pseudocompact.*

(b) *The uniformities t_c and σ coincide on X.*

(c) *The uniformities t_c and σ coincide on U.*

(d) *Any $M \in \mathscr{M}$ is relatively compact in $C(X)$.*

(e) *The uniformities σ and ϱ^* coincide on each $M \in \mathscr{M}$.*

Proof. First we recall that X is pseudocompact if and only if any uniformity on X compatible with the topology on X is precompact[1]). From Theorem 2 it follows (a) $\Rightarrow$ (e) $\Rightarrow$ (d) $\Rightarrow$ (c). Evidently (c) $\Rightarrow$ (b). To prove (b) $\Rightarrow$ (a) we consider a uniformity r on X compatible with the topology. Since $r \leqq t_c$, it follows that (X, r) is precompact. This completes the proof.

Remark. Let us note that the equivalence of properties (a) and (d) has been established in [10] (cf. Theorem 2, (f)). The above stated proof is of topological character and eliminates therefore the integration theory applied in [10].

Now we turn our attention to the case of the unbounded topology t_0. To be more accurate we consider only the topology t_{0c} and a locally compact (and paracompact if necessary) space X.

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[1]) This theorem is due to T. Ishiwata. For an elementary proof we refer to [6] (see Appendix; exercises 5.8, 4.4).

Proposition 3. *Let X be a locally compact and paracompact space. A subset $B \subseteq (E(X), t_{0c})$ is bounded if and only if for some compact $K \subseteq X$ and an integer n it holds $B \subseteq n\,\Gamma K$.*

Proof. Let X be a locally compact and paracompact space. For any $x \in X$, we choose an open and relatively compact neighbourhood $U(x)$. The family $\{U(x); x \in X\}$ forms an open covering of X and we take a continuous pseudometric (cf. [7], p. 538) ϱ with the property that for any $x \in X$ there exists some $y \in X$ such that $\varrho(x, z) \leqq 1$ implies $z \in U(y)$. Evidently $\{z; \varrho(x, z) < 1\}$ is relatively compact for any $x \in X$.

Let B be a bounded subset in $(E(X), t_{0c})$. For any $z = \sum_{i=1}^{n} \lambda_i x_i \in E(X)$, $\lambda_i \neq 0$ we put $[z] = \{x_i; 1 \leqq i \leqq n\}$ and

$$B_0 = \bigcup_{z \in B} [z] . \tag{3}$$

It suffices to prove that B_0 is contained in a compact subset $K \subseteq X$. Indeed, since $t \leqq t_0$, the subset B is bounded in $(E(X), t)$. According to Proposition 2 we have $B \subseteq n\,\Gamma X$ and the required inclusion $B \subseteq n\,\Gamma K$ follows from $B_0 \subseteq K$.

Lemma. *Let B be a bounded subset of $(E(X), t_{0c})$, B_0 defined by (3). Then for a suitable compact subset $K \subseteq X$ it holds $B_0 \subseteq K$.*

Proof. Suppose that the assertion is not true. For any $x \in X$ we denote by $V(x)$ the relatively compact subset of all $z \in X$, $\varrho(x, z) < 1$. If x_0 is an arbitrary point of X, then we choose $x_1 \in B_0$, $x_1 \notin V(x_0)$. Evidently, the subset $K_1 = V(x_0) \cup V(x_1)$ is relatively compact and therefore there exists $x_2 \in B_0$, $x_2 \notin K_1$. Successively we obtain $x_0, x_1, \ldots, x_{n-1}$, $x_i \in X$, $0 \leqq i \leqq n-1$ and the relatively compact subsets $K_1, \ldots$ $\ldots, K_{n-1}$, $K_r = \bigcup_{i=0}^{r} V(x_i)$, $x_r \notin K_{r-1}$, $2 \leqq r \leqq n-1$. Similarly as above for some $x_n \in B_0$ we have $x_n \notin K_{n-1}$. Evidently the sequence $\{x_n; n = 1, 2, \ldots\}$ has the property $\varrho(x_n, x_m) \geqq 1$ for $n \neq m$. Let $W(x)$ be the subset of all $z \in X$ with $\varrho(x, z) < \frac{1}{4}$ for each $x \in X$. For any integer n we choose an element $z_n \in B$ such that $x_n \in [z_n]$. Let $\lambda_n \neq 0$ be the scalar corresponding to x_n in the representation z_n. We now take a continuous function f_n on X, $f_n(x_n) = n/\lambda_n$, $f_n(y) = 0$ for any $y \in [z_n]$, $y \neq x_n$ and $f_n(y) = 0$ for $y \notin W(x_n)$. The family $\{W(x_n); n = 1, 2, \ldots\}$ is locally finite, hence $\{f_n; n = 1, 2, \ldots\}$ is an element of $\mathscr{M}_0(X)$. Since $f_n(z_n) = n$, the subset B is not bounded in the topology t_{0c}. This contradicts the assumption.

An immediate consequence is

Theorem 4. *Let X be a locally compact and paracompact space, $C_0(X)$ locally convex space of all continuous functions on X under the topology of compact convergence. Then the correspondence $f \leftrightarrow \tilde{f}$ is a topological isomorphism of $C_0(X)$ onto the strong dual $(E(X), t_{0c})^*$.*

Remark. Let X be a locally compact space. Then by the same argument as in Theorem 2 we obtain

(a) Any $M \in \mathcal{M}_0$ is relatively compact in $C_0(X)$.

(b) The uniformity of compact convergence coincides with the uniformity of pointwise convergence on each $M \in \mathcal{M}_0$.

(c) Any relatively compact subset in $C_0(X)$ is an element of $\mathcal{M}_0(X)$.

Problem. Let X be a uniform space; to describe the family of all bounded subsets of $(E(X), t_0)$.

4. PROJECTIVELY GENERATED TOPOLOGIES ON $E(X)$

Now we shall characterize the topology t and the weak topologies σ and σ_0 on the vector space $E(X)$. The reader is referred to compare these results with those obtained in [26].

Definition 2. Let X be a uniform space, F a locally convex space. A mapping u of X into F is said to be bounded if $u(X)$ is a bounded subset of F.

Every uniformly continuous mapping of a precompact space in a locally convex space is bounded. The canonical embedding of X into $(E(X), t)$ (into $(E(X), \sigma)$) is bounded in view of the preceding definition.

If u is a mapping of a uniform space X into a vector space F, then the linear extension $\tilde{u} : E(X) \to F$ is defined by the formula $\tilde{u} : z \to \sum \lambda_i \langle u, x_i \rangle$ for any $z = \sum \lambda_i x_i \in E(X)$.

We turn now to the following theorem which shall play an important role in the sequel.

Theorem 5. *Let X be a uniform space. Then the topology t on $E(X)$ is characterized as the unique locally convex topology on $E(X)$ with the following properties:*

1° *The canonical embedding ω of X into $(E(X), t)$ is bounded and uniformly continuous.*

2° *For any locally convex space F and for any bounded and uniformly continuous mapping u of X into F the linear extension $\tilde{u} : E(X) \to F$ is continuous on $(E(X), t)$.*

Proof. Let u be a bounded and uniformly continuous mapping of X into a locally convex space F. For any equicontinuous subset $N \subseteq F^*$ the family $N \circ u$ is bounded and uniformly equicontinuous on X. This implies the continuity of $\tilde{u}$. The uniqueness of such a topology is clear.

Remark. The topology t on $E(X)$ may be characterized by the property 1° of the preceding theorem and by

2°′ For any Banach space B and for any bounded and uniformly continuous mapping $u : X \to B$ the corresponding linear extension $\tilde{u}$ is continuous on $(E(X), t)$.

To prove this statement, it suffices to note that any locally convex space may be regarded as a subspace of the Cartesian product of Banach spaces.

Corollary. (a) *If u is a uniformly continuous mapping of X into Y, then the linear extension of u to $(E(X), t)$ is a continuous mapping into $(E(Y), t)$.*

(b) *Let F be a normed space, S a bounded open subset of F with the topology induced by the norm. Then F is isomorphic to a quotient space of $(E(S), t)$.*

Proof. The proof of (a) is evident. To prove (b), we denote by j the identical mapping of S into F, J the corresponding linear extension to $E(S)$. It is clear that J is a continuous mapping of $E(S)$ onto F. Putting

$$h(x) = 1 \,.\, x - 1 \,.\, x_0$$

for an $x_0 \in S$, we see that h is a continuous mapping of S into $(E(S), t)$. For an open neighbourhood U of the origin in $(E(X), t)$ the inverse image $h^{-1}(U)$ is open in S, hence open in F. For an $x_0 \in h^{-1}(U)$ the set $h^{-1}(U) - x_0$ is an open neighbourhood of zero element in F and the assertion (b) follows from $J(U) \supseteq h^{-1}(U) - x_0$.

Now we characterize the weak topology σ on $E(X)$.

Theorem 6. *Let X be a uniform space. Then $\sigma = \sigma(E(X), P(X))$ is the unique weak locally convex topology on $E(X)$ with the following properties:*

1° *The canonical embedding w of X into $(E(X), \sigma)$ is bounded and uniformly continuous.*

2° *For any bounded and continuous mapping u of X into a locally convex space F with the topology $\sigma(F, F^*)$ the linear extension of u to $E(X)$ is continuous on $(E(X), \sigma)$.*

The proof is clear.

Remark 1. A similar result holds for the topology σ_0 on $E(X)$.

Remark 2. The assertion of Theorem 4 is true if we suppose in 2° that F is an arbitrary Banach space.

Proof. Any locally convex space F is isomorphic to a subspace of the Cartesian product ΠB_α where B_α are Banach spaces. Let x' be an element of F^*. It suffices to prove that $x \to \langle u(x), x' \rangle$ is continuous on X. But any $x' \in F^*$ (exactly the extension of x' to ΠB_α) is of the form

$$x' : y \to \sum_{\alpha \in A} \langle pr_\alpha y, x'_\alpha \rangle$$

where A is a finite set, $x'_\alpha \in B^*_\alpha$ for $\alpha \in A$.

According to Theorem 5 the embedding $\omega : X \to (E(X), t_0)$ admits a continuous extension to $(E(X), t)$ for any precompact space X. Hence we have proved that $t_0(X) = t(X)$ on $E(X)$. But the families $\mathscr{H}(X)$ and $\mathscr{H}_0(X)$ are saturated which implies $\mathscr{H}(X) = \mathscr{H}_0(X)$ (cf. [22]).

Corollary 1. *If X is a precompact space, then*

$$\mathscr{H}(X) = \mathscr{H}_0(X) \quad \text{and} \quad t(X) = t_0(X).$$

Corollary 2. *If X is a pseudocompact space, then*

$$\mathscr{M}(X) = \mathscr{M}_0(X) \quad \text{and} \quad t_c(X) = t_{0c}(X).$$

Remark. On the other hand, from the equality $\mathscr{M}(X) = \mathscr{M}_0(X)$ it follows immediately that X is pseudocompact. Similarly, if the topologies $t_c(X)$ and $t_{0c}(X)$ coincide on $E(X)$, then X is pseudocompact. Let us note that the uniformities $t_c(X)$ and $t_{0c}(X)$ coincide on X for every completely regular space X. This fact buttresses the guiding motive of introducing the Λ-structure as an adequate and finer superstructure over the given continuity space X.

5. THE NORM ON $E(X)$

A norm ϱ is defined on $E(X)$ in this section and some properties of $(E(X), t)$ are investigated.

We denote by $U = \Gamma X$ the absolutely convex envelope of X in $E(X)$. It holds

Lemma. *The set U defined above is a bounded barrel in $E(X)$ with every topology compatible with the duality of the dual pair $\langle E(X), P(X) \rangle$.*

Proof. It suffices to prove that U is closed in $(E(X), \sigma)$. Let $z \in E(X)$ be an element of the form (1), $z \notin U$. This implies $\sum |\lambda_i| > 1$. We choose a function $f \in P(X)$, $\|f\| \leqq 1$ such that $f(x_i) = 1$ for $\lambda_i > 0$, $f(x_i) = -1$ for $\lambda_i < 0$. We have $|\langle y, f \rangle| \leqq 1$ for any $y \in U$ and

$$\langle z, f \rangle = \sum \lambda_i \langle x_i, f \rangle = \sum |\lambda_i| > 1.$$

This concludes the proof.

Now we define a norm ϱ on $E(X)$: the unit ball for ϱ is U. Evidently $\sigma \leqq t \leqq \varrho$. We denote by $(\hat{E}(X), \varrho)$ the completion of $E(X)$ by ϱ. The space $l^1(X)$ is defined as the set of all $\lambda = \{\lambda_x; x \in X\}$ satisfying $\|\lambda\| = \sum|\lambda_x| < \infty$. A linear function f on $(E(X), \varrho)$ is continuous if and only if the restriction of f to X is a bounded function. This fact enables us to identify the dual space $(E(X), \varrho)^*$ with the space $l^\infty(X)$ of all bounded functions on X. Now we are going to describe the completion $(\hat{E}(X), \varrho)$.

Theorem 7. *The vector space $(\hat{E}(X), \varrho)$ is linearly isometric to $l^1(X)$.*

Proof. For any $z = \sum\lambda_i x_i$ it holds $\varrho(z) \leqq \sum|\lambda_i|$. The polar of U in $l^\infty(X)$ is evidently the unit ball which implies

$$\varrho(z) = \sup_{\|f\| \leqq 1, f \in l^\infty(X)} |\langle z, f\rangle| .$$

Let f be a function in $l^\infty(X)$ with $f(x_i) = 1$ for $\lambda_i > 0$ and $f(x_i) = -1$ for $\lambda_i < 0$. Then $\varrho(z) \geqq |\langle z, f\rangle| = \sum|\lambda_i|$. Thus we have proved the equality $\varrho(z) = \sum|\lambda_i|$ for any $z = \sum\lambda_i x_i \in E(X)$. Now it is easy to see that $(\hat{E}(X), \varrho)$ is linearly isometric to $l^1(X)$.

In view of the relation $t \leqq \varrho$ a natural question arises under what conditions the topology t will be equal to the greatest lower bound of all norm topologies σ greater than t. The answer is contained in

Proposition 4. *Suppose that Σ is a family of all norm topologies σ on $E(X)$, $\sigma \geqq t$. Then the following properties are equivalent:*

(a) *It holds $t = \inf\{\sigma; \sigma \in \Sigma\}$.*

(b) *The space $(E(X), t)$ is the inductive limit of all spaces $\{E(X), \sigma\}$, $\sigma \in \Sigma$.*

(c) *The space $(E(X), t)$ is bornological.*

(d) *It holds $t = \varrho$.*

(e) *The uniformity of X is discrete.*

Proof. The following statements

$$(d) \Rightarrow (a) \Rightarrow (b) \Rightarrow (c)$$

are clear.

(c) $\Rightarrow$ (d): follows from Proposition 2.

(e) $\Rightarrow$ (d): is evident.

(d) $\Rightarrow$ (e): if $t = \varrho$, then $(E(X), t)^* = l^\infty(X)$. For any $x \in X$ we put $f_x(y) = 1$ for $y = x$, $y \in X$ and $f_x(y) = 0$ for $y \neq x$, $x \in X$. The family $F = \{f_x; x \in X\}$ is contained as a subset in the unit ball of $l^\infty(X)$. The subset V of all (y, z) in the Cartesian product $X \times X$ satisfying $|f_x(y) - f_x(z)| < \frac{1}{2}$ for all $x \in X$ is a uniform neighbourhood in X. But due to the choice of the family F the subset V is equal to the diagonal in $X \times X$.

Remark 1. The space $(E(X), t)$ is not barrelled. Indeed, if $(E(X), t)$ were barrelled, then U would be a neighbourhood in $(E(X), t)$ which implies $t = \varrho$. But it is well known that $(E(X), \varrho)$ is not a barrelled space.

Remark 2. With regard to the completeness we note only that $(E(X), t)$ does not posses any property of this kind. It is easy to show that $(E(X), t)$ is not sequentially complete. The completion of $(E(X), t)$ will be investigated in Section 8.

Remark 3. The topologies t_0 and ϱ are, in general, not comparable. If X is a discrete uniform space, then $(E(X), t_0)^*$ is algebraically isomorphic with the KÖTHE space $\omega(X)$ (cf. [22]). In this case t_0 is the finest locally convex topology on $E(X)$.

Some results of this section were indicated in [17].

6. THE COMPLETION OF UNIFORM SPACES

The aim of this section is to present a characterization of the completion of uniform spaces. Let X be a uniform space. The embedding $x \to \hat{x}$ of X into $P^*(X)$ defined by the formula

$$\langle \hat{x}, f \rangle = \langle f, x \rangle$$

admits an extension to $E(X)$. In this way we identify any $x \in E(X)$ with the corresponding $\hat{x} \in P^*(X)$. Similarly we denote by the same letter t the extended topology of $\mathscr{H}$-convergence on $P^*(X)$.

Putting $\hat{E} = (\hat{E}(X), t)$ for the completion of $(E(X), t)$, we denote by $\bar{X}$ the closure of X in $\hat{E}$ with the topology $\sigma(\hat{E}, P(X))$ and by $\hat{X}$ the closure of X in $(\hat{E}(X), t)$. Evidently $\hat{X}$ is the completion of X and $\hat{X} \subseteq \bar{X}$. We start our discussion with

Theorem 8. *Let X be a uniform space. Then it holds*

$$\bar{X} = \hat{X}\,.$$

Proof. If $y \in (\hat{E}(X), t)$, $y \in \bar{X}$, then there exists a net $\{x_\alpha;\ \alpha \in A\}$ in X, $\lim x_\alpha = y$ in the topology $\sigma(\hat{E}, P(X))$. Next we prove

Lemma. *For any neighbourhood V of the origin in $(\hat{E}(X), t)$ there exists an index $\alpha_V \in A$ such that*

$$x_\beta \in x_{\alpha_V} + V$$

for some confinal subset $B \subseteq A$ of indices β.

Proof. On the contrary, suppose that V_0 is an absolutely convex and closed neighbourhood of the origin in $(\hat{E}(X), t)$ not having the required property. Let $\bar{f}$ be a uniformly continuous function on $(\hat{E}(X), t)$ with

$$\bar{f}(0) = 1\,, \quad \bar{f}(x) = 0 \quad \text{for all} \quad x \in \hat{E}(X) \setminus V_0\,, \quad 0 \leqq \bar{f} \leqq 1\,.$$

If we put $\bar{f}_x(z) = f(x - z)$, then $\bar{f}_x(x) = 1$, $\bar{f}_x(y) = 0$ for all $y \notin x + V_0$. We denote by f_x the restriction of $\bar{f}_x$ to X. It is easy to see that the family $\{f_x;\ x \in X\}$ is an element of $\mathscr{H}$ (see also [9]). The same is true for the set H of all $\max(f_{x_1}, \ldots, f_{x_n})$, $x_i \in X$, $1 \leqq i \leqq n$, n an arbitrary integer. Let ε be a real number, $0 < \varepsilon < \frac{1}{2}$. From $y \in \bar{X}$ it follows that for some convex mean $\sum_{i=1}^{n} \lambda_i x_{\alpha_i}$, $\lambda_i \geqq 0$, $\sum_{i=1}^{n} \lambda_i = 1$ it holds

$$\left| \sum_{i=1}^{n} \lambda_i \langle x_{\alpha_i}, f \rangle - \langle y, f \rangle \right| < \varepsilon$$

for any $f \in H$.

In view of the choice of V we can find to each α_i, $1 \leqq i \leqq n$, an index $\beta_i \in A$, $1 \leqq i \leqq n$, such that $x_\beta \notin x_{\alpha_i} + V_0$ for all $\beta \geqq \beta_i$, $1 \leqq i \leqq n$. If $\beta_0 \in A$, $\beta_0 \geqq \beta_i$, $1 \leqq i \leqq n$, then for some convex mean $\sum_{j=1}^{m} \mu_j x_{\beta'_j}$, $\mu_j \geqq 0$, $\sum_{j=1}^{m} \mu_j = 1$, $\beta'_j \geqq \beta_0$ we have

$$\left| \sum_{j=1}^{m} \mu_j \langle x_{\beta'_j}, f \rangle - \langle y, f \rangle \right| < \varepsilon$$

for all $f \in H$. Hence

$$\left| \sum \lambda_i \langle x_{\alpha_i}, f \rangle - \sum \mu_j \langle x_{\beta'_j}, f \rangle \right| < 2\varepsilon < 1$$

for all $f \in H$.

In particular, for the function $f_0 = \max(f_{x_{\alpha_1}}, \ldots, f_{x_{\alpha_n}})$ we obtain the relation

$$1 - \sum \mu_j \langle x_{\beta_j'}, f_0 \rangle < 1 .$$

The function f_0 is equal to zero for every element which belongs to the complement of $\bigcup_{i=1}^{n} (x_{\alpha_i} + V_0)$. From $x_{\beta'_j} \notin \bigcup_{i=1}^{n} (x_{\alpha_i} + V_0)$, $1 \leqq j \leqq m$ it follows $\langle x_{\beta'_j}, f_0 \rangle = 0$, $1 \leqq j \leqq m$. This contradiction establishes the result.

Making use of the preceding lemma we finish now the proof of Theorem 8.

Let V be an absolutely convex and closed neighbourhood of the origin in $(\hat{E}(X), t)$. The set $x_{\alpha_V} + V$ being convex and closed for the topology t, we conclude by Mackey theorem that the same holds for the weak topology. But $y = \lim \{x_\alpha;\ \alpha \in A\}$, hence $y \in x_{\alpha_V} + V$. From the symmetry of V we derive $x_{\alpha_V} \in y + V$. Thus we have proved that y is a closure point of X in the space $(\hat{E}(X), t)$, hence $y \in \hat{X}$. This concludes the proof of Theorem 8.

Let $\bar{X}^\sigma$ be the closure of X in $P^*(X)$ with the weak topology $\sigma(P^*(X), P(X))$. Evidently

$$\bar{X} = \bar{X}^\sigma \cap (\hat{E}(X), t) .$$

According to Theorem 8 we can write

$$\hat{X} = \bar{X}^\sigma \cap (E(X), t) . \tag{4}$$

The last equality is basic for future considerations.

Remark 1. The assertion of Theorem 8 can be modified in the following manner.

(a) The topology on $\bar{X}$ induced by the uniformity t coincides with the topology induced on X by the weak uniformity σ.

(b) Any $H \in \mathscr{H}$ is equicontinuous on $\bar{X}$ in the weak topology.

The proof of this statements does not present any difficulty.

Remark 2. If X is a uniform space, then $\alpha X = \bar{X}^\sigma$ with the weak topology possesses analogous properties as the Čech-Stone compactification βX. To this point, we state only the relation between βX and αX. Any function $f \in P(X)$ can be extended to a continuous function $\bar{f}$ on βX. If we define on βX the equivalence relation R by: $x, y \in \beta X$, $x \mathrel{R} y$ if and only if $\bar{f}(x) = \bar{f}(y)$, then the quotient space with the uniformity induced by the collection $\{\bar{f}; f \in P(X)\}$ is uniformly isomorphic to αX.

Remark 3. The construction of the family H in the proof of Theorem 8 was applied for the first time in [9]. In fact, the idea of the present proof is based – roughly speaking – on the method of the sliding hump.

In the rest of this section we show how the complete envelope of any uniform space can be described as a collection of extremal points of the unit ball in $P^*(X)$ satisfying some complementary conditions. This result will generalize the well-known dual characterization of the Čech-Stone compactification βX.

The collection of all points z in $P^*(X)$ with

$$z(e) = 1\,, \quad z \geqq 0 \tag{5}$$

will be called the polar set in the unit ball of $P^*(X)$ (briefly the polar set) and will be denoted by K. Here e means the function on X identical to the unit, $z \geqq 0$ means $\langle z, f\rangle \geqq 0$ for all $f \in P(X)$, $f \geqq 0$. Evidently K is convex and weakly compact in $P^*(X)$.

Lemma. *The family of all extremal points of the polar set K coincides with the subset $\bar{X}^\sigma$.*

Proof. If $x \in \bar{X}^\sigma$ then x is an extremal point of the unit ball in $P^*(X)$, hence x is also an extremal point of K. On the contrary, the weakly closed and convex envelope $\overline{\text{co}\, X^\sigma}$ of X is equal to K. Indeed, for any $z \in K$, $z \notin \overline{\text{co}\, X^\sigma}$ we find a function $f \in P(X)$, $\langle z, f\rangle > 1$, $\langle y, f\rangle < 1$ for all $y \in \overline{\text{co}\, X^\sigma}$. Hence $f \leqq e$ which implies $\langle z, f\rangle \leqq \leqq \langle z, e\rangle = 1$. To prove the rest of the lemma, we note that according to Krein-Milman theorem any extremal point of $\overline{\text{co}\, X^\sigma}$ is contained in $\bar{X}^\sigma$.

By a theorem of Grothendieck (cf. [11]) the completion $(\hat{E}(X), t)$ is identical to the collection of all linear functions on $P(X)$ continuous on each $H \in \mathscr{H}$ with the pointwise topology $\sigma(P(X), X)$. From (4) we obtain

Theorem 9. *The completion $\hat{X}$ of the uniform space X coincides with the family of all extremal points of the polar set K continuous on each $H \in \mathscr{H}$ with the pointwise topology $\sigma(P(X), X)$.*

Remark 1. Particularly, if we consider a uniform space X with the weak uniformity σ, then the completion $\bar{X}^\sigma$ of (X, σ) coincides with the space of all maximal ideals of the Banach algebra $P(X)$. This enables us to identify the space $\bar{X}^\sigma$ with the family of all (isotone) homomorphisms of the Banach algebra onto the real line. The same can be applied to a rather general case, namely, if the uniformity $\mathscr{U}$ on X is induced by a linear algebra of functions separating points on X in the strong sense (cf. [14]).

Remark 2. By an analogous procedure as in [23] it can be shown that the family of all extremal points of the polar set K is identical with the collection of all elementary normed linear functions on $P(X)$ (i.e. satisfying (5) and having the property: $0 \leqq h \leqq \leqq z$, $h \in P(X)$ implies $h = \alpha z$ for a suitable scalar α). In this terms the characterization of the complete envelope of a uniform space may be formulated in the same way as in [9].

7. THE GENERALIZATION OF THE BANACH-STONE THEOREM

We end this paper by applying the preceding results to prove a generalization of the Banach-Stone theorem.

If X and Y are two completely regular spaces then any linear isometry between $C(X)$ and $C(Y)$ induces a topological homeomorphism between the Čech-Stone compactifications βX and βY. The basic idea of this assertion consists – roughly speaking – in the fact that a morphism (with additional conditions if necessary) between "dual" objects induces a morphism between original objects.

A question arises for which "dual" objects and under what conditions the above – mentioned statement admits a generalization for uniform spaces. First we state.

Example. Let X be a uniform non-precompact space. Denoting by Y the set X with the coarsest uniformity defined by the system $P(X)$, we see that $P(X) = P(Y)$, but X is not isomorphic to Y ($\hat{X}$ is not homeomorphic to $\hat{Y}$).

The situation is clear; the space $C(X)$ determines the topology on X, similarly $P(X)$ determines the topology on the collection of all maximal ideals of the Banach algebra $P(X)$ but not the uniformity on X.

In the following theorem we shall show that the question is solved if we replace the space $P(X)$ by $(E(X), t)$.

Let X and Y be two uniform spaces, u an isomorphism of the locally convex space $(E(X), t)$ onto $(E(Y), t)$. From Theorem 1 it follows that the adjoint mapping ${}^t u$ is an isomorphism of the locally convex space $P(Y)$ onto $P(X)$. Similarly ${}^{tt}u$ is an isomorphism of the locally convex structure of $P^*(X)$ onto $P^*(Y)$. If V is the unit ball in $P(Y)$, then ${}^t u(V)$ need not be, in general, the unit ball in $P(X)$.

Theorem 10. *Let X and Y be two uniform spaces, u an isomorphism of $(E(X), t)$ onto $(E(Y), t)$. Then the following statements are equivalent:*

(a) *${}^t u$ is a linear isometry.*

(b) *There exists a function $\alpha(x)$, $|\alpha(x)| = 1$ for all $x \in X$ such that $v(x) = \alpha(x)\, u(x)$ is a uniform isomorphism of X onto Y.*

Proof. (a) $\Rightarrow$ (b). If ${}^t u$ is a linear isometry then ${}^{tt}u$ is a linear isometry of $P^*(X)$ onto $P^*(Y)$. Hence, if z is an extremal point of the unit ball U^0 in $P^*(X)$, then ${}^{tt}u(z)$ is an extremal point of the unit ball V^0 in $P^*(Y)$. But the collection of all extremal points of V^0 is identical to $\bar{Y}^\sigma \cup \overline{(-Y)}^\sigma$ (cf. [8]) where $\bar{Y}^\sigma$ is the closure of Y in $P^*(Y)$ under the weak topology. Now we put

$$A = \{x \in \bar{X}^\sigma;\ {}^{tt}u(x) \in \bar{Y}^\sigma\}\,, \quad B = \{x \in \bar{X}^\sigma;\ {}^{tt}u(x) \in \overline{(-Y)}^\sigma\}\,.$$

Evidently A and B are disjoint and weakly compact in $P^*(X)$. For some neighbourhood V of the origin in $P^*(X)$ it holds

$$(A + V) \cap B = \emptyset\,. \tag{6}$$

Let α be a function defined as follows: $\alpha(x) = 1$ for all $x \in A$, $\alpha(x) = -1$ for all $x \in B$.

The continuous extension $\tilde{u}$ of u to $(\hat{E}(X), t)$ coincides on $(\hat{E}(X), t)$ with ${}^{tt}u$. The function v is now defined by

$$v = \alpha \circ {}^{tt}u\,.$$

According to (4) we obtain

$$v(\hat{X}) = \hat{Y}\,. \tag{7}$$

From (6) it follows that v is a uniformly continuous mapping of $\hat{X}$ onto $\hat{Y}$. A symmetric consideration leads to the conclusion that u is a uniform isomorphism.

In [26] it was proved that X is a closed subset in $(E(X), t)$. Hence, $X = \hat{X} \cap \cap (E(X), t)$; from (7) we now obtain the desired equality $v(X) = Y$.

(b) $\Rightarrow$ (a). Conversely, if $v = \alpha \,.\, u$ is a uniform isomorphism of X onto Y, then for any $f \in P(Y)$ we have

$$\|\langle {}^t u, f\rangle\| = \sup_{x\in X} |\langle f, u(x)\rangle| = \sup_{x\in X} |\langle f, v(x)\rangle| = \|f\|\,.$$

Remark. Let u be an isomorphism (a weak isomorphism) from $(\hat{E}(X), t)$ onto $(\hat{E}(Y), t)$. If the condition (a) of Theorem 10 is satisfied then the spaces $\hat{X}$ and $\hat{Y}$ are uniformly isomorphic (homeomorphic). To prove this statement it suffices to repeat the same procedure as in the proof of Theorem 10. Particularly, if X and Y are two compact spaces, u a linear isometry of $C(Y)$ onto $C(X)$, then ${}^t u$ is a weak isomorphism of $C^*(X)$ onto $C^*(Y)$. In view of the fact that for a compact space X it holds $C^*(X) = (\hat{E}(X), t_c)$ (cf. [19]; see section 8) we obtain the above mentioned Banach-Stone theorem.

The proof of Theorem 10 is based on topological terms. If we take into consideration the order structure of $P(X)$, we obtain

Theorem 11. *Let u be an isomorphism of $(E(X), t)$ onto $(E(Y), t)$. The restriction of ^{tt}u on $\hat{X}$ (on X) is a uniformly isomorphic mapping onto $\hat{Y}$ (onto Y), if and only if ^{t}u possesses the following properties:*

(a) *^{t}u is an order preserving mapping of the spaces $P(Y)$ and $P(X)$ (i.e. $f \in P(Y)$, $g \in P(Y), f \leqq g$ implies $^{t}u(f) \leqq {}^{t}u(g)$ and conversely).*

(b) *The image $^{t}u(e)$ of the unit element of the Banach algebra $P(Y)$ is the unit element in $P(X)$.*

The proof follows from the equalities

$$^{tt}u(\hat{E}(X), t) = (\hat{E}(Y), t)\,, \quad ^{tt}u(\bar{X}^{\sigma}) = \bar{Y}^{\sigma}$$

and from the above mentioned theorem of FIODOROVA (cf. [9]).

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