Czechoslovak Mathematical Journal

S. Mrówka

Characterization of classes of functions by Lebesgue sets

Czechoslovak Mathematical Journal, Vol. 19 (1969), No. 4, 738–744

Persistent URL: http://dml.cz/dmlcz/100933

Czechoslovak Mathematical Journal, 19 (94) 1969, Praha

CHARACTERIZATION OF CLASSES OF FUNCTIONS BY LEBESGUE SETS[0])

S. MRÓWKA, Kalamazoo

(Received November 15, 1968)

All functions considered in this paper are real-valued.

By Lebesgue sets of a function f defined on a set X we understand sets of the form $L_a(f) = \{p \in X : f(p) \leqq a\}$ and $L^a(f) = \{p \in X : f(p) \geqq a\}$, where a is an arbitrary real number. Various classes of functions are characterized in terms of their Lebesgue sets. Perhaps the most famous result in this direction is the Lebesgue theorem on the characterization of Baire measurable functions by their Lebesgue sets. In this paper we shall give a few applications of the results of [1] to theorems of the above type.

Let us briefly analyse the proof of the Lebesgue theorem in light of the results of [1]. For simplicity, we shall consider only the 1st Baire class.

Theorem A. (Lebesgue) *A function f on the unit interval $[0,1]$ is of the* 1*st Baire*[1]) *class if, and only if, all the Lebesgue sets of f are G_δ-sets.*

The following is shown in [1] (see Theorem 3.3).

Theorem B. *Let $\mathfrak{F}$ be a uniformly closed ring of functions on a set X that contains all constant functions on X and is closed under inversion. If each of the Lebesgue sets of a function f on X is a Lebesgue set of some function in F, then $f \in \mathfrak{F}$.*

Now, if $\mathfrak{F}$ is the 1st Baire class, then $\mathfrak{F}$ satisfies the assumptions of Theorem B. Consequently, the difficult part of Theorem A (the "if" part) reduces to the following proposition.

Proposition C. *Each G_δ-set $A \subset [0,1]$ is a Lebesgue set of some function of the* 1*st Baire class.*

---

[0]) The preparation of this paper was partially supported by the U.S. National Science Foundation, Grant GP-5286. The author wishes to express his sincere thanks to Professor H. Shapiro who rendered a considerable help in preparing the manuscript.

[1]) We refer here to the inclusive Baire classification; i.e., the 1st Baire class includes all continuous functions.

(Perhaps the simplest way to show Proposition C is to write the complement of A as the union of countably many closed sets, $[0,1] - A = \bigcup_n F_n$, and consider the function $g(p) = 0$ for $p \in A$ and $g(p) = \sum \{1/2^n : p \in F_n\}$ for $p \notin A$. A is a Lebesgue set of g (in fact, $A = \{p : g(p) \leq 0\}$) and g is of the 1st Baire class. Indeed, for each n there is a continuous function $f_n : [0,1] \to [0,1]$ such that $p \in F_n$ iff $f_n(p) = 1$. Setting $g_n(p) = \frac{1}{2}f_1^n(p) + (1/2^2) f_2^n(p) + \ldots + (1/2^n) f_n^n(p)$ we have $g_n(p) \to g(p)$ for every $p \in [0,1]$.)[2])

It is frequently more convenient to apply Theorem B in a somewhat different form. Let $\mathfrak{F}$ be a class of functions on a set X; let $\mathfrak{A}$ denote the class of all the Lebesgue sets of functions in $\mathfrak{F}$; in turn, let $\tilde{\mathfrak{F}}$ be the class of all functions whose Lebesgue sets are in $\mathfrak{A}$. Clearly, $\mathfrak{F} \subset \tilde{\mathfrak{F}}$. Now, Theorem A tells that if $\mathfrak{F}$ is the 1st Baire class, then $\tilde{\mathfrak{F}} = \mathfrak{F}$. On the other hand, Theorem B yields the following general result in this direction.

Theorem B′. *If $\mathfrak{F}$ satisfies the assumptions of Theorem B, then $\tilde{\mathfrak{F}} = \mathfrak{F}$.*

Clearly, Theorem A (strictly speaking, the "if" part of this theorem) is an immediate consequence of Theorem B′ and Proposition C.

In the following sections we shall examine in details conditions leading to the equality $\tilde{\mathfrak{F}} = \mathfrak{F}$. We shall treat both the case of arbitrary functions and that of bounded functions.

1. TERMINOLOGY AND NOTATION

The terminology and notation will follow that of [1]. We shall also use the following:

Suppose that $\mathfrak{F}$ is a class of functions on a set X. (We always assume that $\mathfrak{F}$ contains all constant functions on X.) Let $\mathfrak{F}^*$ be the class of all bounded functions in $\mathfrak{F}$, let $\Lambda(\mathfrak{F})$ denote the class of all Lebesgue sets of functions in $\mathfrak{F}$, and let $\mathfrak{Z}(\mathfrak{F})$ denote the class of all zero-sets of functions in $\mathfrak{F}$. We say that $\mathfrak{F}$ is *lattice-ordered* provided that $f, g \in \mathfrak{F}$ imply that $f \vee g \in \mathfrak{F}$ and $f \wedge g \in \mathfrak{F}$ (where $\vee$ and $\wedge$ stand for maximum and minimum respectively).

Now suppose that $\mathfrak{A}$ is a class of subsets of a set X. We shall always assume that $\emptyset$ and X are members of $\mathfrak{A}$. Let $\lambda(\mathfrak{A})$ (resp. $\lambda^*(\mathfrak{A})$) denote the class of all (resp. all bounded) functions on X whose Lebesgue sets belong to $\mathfrak{A}$. Let $\gamma(\mathfrak{A})$ be the set of all bounded functions on X satisfying the following condition: for every $\varepsilon > 0$ there are $A_1, \ldots, A_n \in \mathfrak{A}$ such that $A_1 \cap \ldots \cap A_n = \emptyset$ and

$$\omega(f, X - A_i) = \sup \{|f(p) - f(q)| : p, q \in X - A_i\} < \varepsilon .$$

[2]) The Lebesgue theorem for functions of a class α, $\alpha < \Omega$, can be derived in the same way from Theorem B and the corresponding Proposition C_α. The proof of Proposition C together with transfinite induction yields Proposition C_α for every $\alpha < \Omega$.

We say that $\mathfrak{A}$ is a *ring* provided that $\mathfrak{A}$ is closed under finite unions and finite intersection. We say that $\mathfrak{A}$ is a *σ-ring* in case $\mathfrak{A}$ is closed under finite unions and countable intersections. We say that $\mathfrak{A}$ is *normal* proved that for every $A, B \in \mathfrak{A}$ with $A \cap B = \emptyset$ there are $C, D \in \mathfrak{A}$ such that $C \cup D = X$, $A \cap C = \emptyset$, and $B \cap D = \emptyset$. Let $c\mathfrak{A}$ denote the class of all complements of members of $\mathfrak{A}$.

2. ELEMENTARY RESULTS

2.1. Proposition. *If $\mathfrak{A}$ is a ring, then $\lambda(\mathfrak{A})$ and $\lambda^*(\mathfrak{A})$ are lattice ordered.*

We omit the obvious proof.

2.2. Proposition. *If $\mathfrak{A}$ is σ-ring, then $\lambda(\mathfrak{A})$ (resp. $\lambda^*(\mathfrak{A})$) is closed under composition with continuous (resp. bounded continuous) functions defined on open subsets of $\mathscr{R}^2$.*

Proof. The proof is the same as that of the implication "(a) *implies* (b)" of Lemma 3.5 in [1].

2.3. Proposition. *If $\mathfrak{A}$ is a σ-ring, then $\lambda(\mathfrak{A})$ and $\lambda^*(\mathfrak{A})$ are both uniformly closed rings. Furthermore, $\lambda(\mathfrak{A})$ is closed under inversion and $\lambda^*(\mathfrak{A})$ is closed under bounded inversion.*

Proof. Suppose that $f_n \to f$ uniformly on X. We may assume that $|f_n - f| < 1/n$. Then clearly $L_a(f) = \bigcap_{n \in N} L_{a+1/n}(f_n)$ and $L^a(f) = \bigcap_{n \in N} L^{a-1/n}(f_n)$ and therefore $\lambda(\mathfrak{A})$ and $\lambda^*(\mathfrak{A})$ are uniformly closed. The result now follows from 2.2 in this paper and 3.5 in [1].

2.4. Proposition. *If $\mathfrak{A}$ is a ring, then $\gamma(\mathfrak{A})$ is a uniformly closed ring.*

Proof. Let $f, g \in \gamma(\mathfrak{A})$ and let $\varepsilon > 0$ be given. Let $M > 1$ be such that $|f| < M$ and $|g| < M$ and choose $U_1, \ldots, U_n$ and $V_1, \ldots, V_m$ in $c\mathfrak{A}$ so that $U_1 \cup \ldots \cup U_n = V_1 \cup \ldots \cup V_m = X$ and $\omega(f, U_i) < \varepsilon/2M$ and $\omega(g, V_j) < \varepsilon/2M$. It follows that $\omega(f + g, U_i \cap V_j) < \varepsilon$ and $\omega(f \cdot g, U_i \cap V_j) < \varepsilon$. Thus $f + g$ and $f \cdot g$ both belong to $\gamma(\mathfrak{A})$.

Now let $f_n \in \gamma(\mathfrak{A})$ and let $f_n \to f$ uniformly on X. Choose n_0 such that $|f_{n_0} - f| < \varepsilon/3$ and choose $U_1, \ldots, U_n \in c\mathfrak{A}$ so that $U_1 \cup \ldots \cup U_n = X$ and $\omega(f, U_i) < \varepsilon/3$. It follows that $\omega(f, U_i) < \varepsilon/3$ and therefore $\gamma(\mathfrak{A})$ is uniformly closed.

2.5. Proposition. *If $\mathfrak{A}$ is a ring, then $\lambda^*(\mathfrak{A}) \subset \gamma(\mathfrak{A})$.*

Proof. Suppose that $f \in \lambda^*(\mathfrak{A})$ and that $\varepsilon > 0$ is given. Let $I_i = (a_i, b_i)$, $i = 1, \ldots, n$, be a finite number of open intervals covering the range of f and such that $b_i - a_i < \varepsilon$. Let $U_i = X \setminus (L_{a_i}(f) \cup L^{b_i}(f))$. Clearly $U_i \in c\mathfrak{A}$, $U_1 \cup \ldots \cup U_n = X$, and $\omega(f, U_i) < \varepsilon$.

2.6. Proposition. *If $\mathfrak{A}$ is a σ-ring, then* $\lambda^*(\mathfrak{A}) = \gamma(\mathfrak{A})$.

Proof. It suffices to show that if $f \in \gamma(\mathfrak{A})$, then $f \in \lambda^*(\mathfrak{A})$. Let $a \in \mathscr{R}$. For every integer k, choose $U_1, \ldots, U_n \in c\mathfrak{A}$ such that $U_1 \cup \ldots \cup U_n = X$ and $\omega(f, U_i) < 1/k$. Set $G_k = X \setminus \bigcup\{U_i : \inf\{f(p) : p \in U_i\} > a\}$, and note that

$$\text{if} \quad f(p) \leqq a\,, \quad \text{then} \quad p \in G_k \quad \text{for} \quad k = 1, 2, \ldots,$$

$$\text{if} \quad p \in G_k\,, \quad \text{then} \quad f(p) \leqq a + 1/k\,.$$

It follows that $L_a(f) = \bigcap_k G_k$ and hence $L_a(f) \in \mathfrak{A}$. In a similar way we can prove that $L^a(f) \in \mathfrak{A}$. Thus $f \in \lambda^*(\mathfrak{A})$.

2.7. Proposition. *If $\mathfrak{F}$ is a lattice-ordered linear space, then* $A(\mathfrak{F}) = A(\mathfrak{F}^*) = \mathfrak{Z}(\mathfrak{F}) = \mathfrak{Z}(\mathfrak{F}^*)$.

We omit the obvious proof.

2.8. Proposition. *If $\mathfrak{F}$ is a lattice-order linear space, then $A(\mathfrak{F})$ is a ring.*

Again, we omit the proof.

2.9. Proposition. *If $\mathfrak{F}$ is a lattice-ordered linear space and if $\mathfrak{F}$ is closed under bounded inversion, then $A(\mathfrak{F})$ is normal.*

Proof. Let $A, B \in A(\mathfrak{F})$ such that $A \cap B = \emptyset$. By 2.7, there are $f, g \in \mathfrak{F}$ such that $A = Z(f)$ and $B = Z(g)$. Set $h = |f|/(|f| + |g|)$ and note that $h \in \mathfrak{F}$. Let $C = L^{1/3}(h)$ and $D = L_{2/3}(h)$ and observe that $C, D \in A(\mathfrak{F})$, $A \cap C = B \cap D = \emptyset$ and $C \cup D = X$.

2.10. Proposition. *If $\mathfrak{F}$ is a lattice-ordered uniformly closed linear space, then $A(\mathfrak{F})$ is a σ-ring.*

Proof. It suffices to show that $A(\mathfrak{F})$ is closed under countable intersections. By 2.7, let $A_n = Z(f_n)$, where $f_n \in F$. Set $f = \sum_n (1/2^n)(1 \wedge |f_n|)$. Clearly $\bigcap_n A_n = Z(f)$ and $f \in \mathfrak{F}$.

We conclude this section with an abstract version of the Urysohn lemma.

2.11. Proposition. *Suppose that $\mathfrak{A}$ is a ring. Then $\mathfrak{A}$ is normal if and only if for every two disjoint members A and B of $\mathfrak{A}$, there exists an $f \in \gamma(\mathfrak{A})$ such that $f(p) = 0$ for $p \in A$, $f(p) = 1$ for $p \in B$, and $0 \leqq f \leqq 1$.*

Proof. The existence of f is proved by repetition of the Urysohn procedure. To prove the converse, assume that A and B are both non-empty (otherwise there is nothing to prove) and choose $A_1, \ldots, A_n$ in $\mathfrak{A}$ such that $A_1 \cap \ldots \cap A_n = \emptyset$ and $\omega(f, X - A_i) < \frac{1}{3}$. Set $C = \bigcap\{A_i : (X - A_i) \cap A \neq \emptyset\}$ and $D = \bigcap\{A_i : (X - A_i) \cap B \neq \emptyset\}$. Then $C, D \in \mathfrak{A}$, $A \cap C = B \cap D = \emptyset$, and $C \cup D = X$.

3.1. Theorem. *If $\mathfrak{F} \subset F^*(X)$ is a uniformly closed ring, then $\lambda^*(A(\mathfrak{F}))$ is the smallest uniformly closed ring that contains $\mathfrak{F}$ and that is closed under bounded inversion.*

Proof. Let $\tilde{\mathfrak{F}}$ be the smallest uniformly closed ring that contains $\mathfrak{F}$ and that is closed under bounded inversion (the existence of $\tilde{\mathfrak{F}}$ is obvious). It is known that $\tilde{\mathfrak{F}}$ is a linear lattice (see, for instance, 2.1 in [1]) and therefore, by 2.10, $A(\tilde{\mathfrak{F}})$ is a σ-ring. It follows by 2.3 that $\tilde{\mathfrak{F}} \subset \lambda^*(A(F))$. Conversely, suppose that $f \in \lambda^*(A(\mathfrak{F}))$ and that $a, b \in \mathscr{R}$ with $a < b$. Let $g_1, g_2 \in \mathfrak{F}$ be such that $L_a(f)$ and $L^b(f)$ are Lebesgue sets of g_1 and g_2 respectively. It follows that $\mathfrak{F}$, and therefore $\tilde{\mathfrak{F}}$, S_3-separates $L_a(f)$ and $L^b(f)$ and hence by Theorem 2.9 in [1], $f \in \tilde{\mathfrak{F}}$. Consequently, $\tilde{\mathfrak{F}} = \lambda^*(A(\mathfrak{F}))$.

3.2. Corollary. *Suppose that $\mathfrak{F} \subset F^*(X)$ is a uniformly closed ring. Then $\lambda^*(A(\mathfrak{F})) = \mathfrak{F}$ (or equivalently $\gamma(A(\mathfrak{F})) = \mathfrak{F}$) if and only if $\mathfrak{F}$ is closed under bounded inversion.*

3.3. Theorem. *If $\mathfrak{F} \subset F^*(X)$ is a lattice-ordered ring that is closed under bounded inversion, then $\gamma(A(\mathfrak{F})) = \bar{\mathfrak{F}}$ (the uniform closure of $\mathfrak{F}$).*

Before proving Theorem 3.3 we need the following lemma.

3.4. Lemma. *Suppose that $\mathfrak{A}$ is a normal ring. If $f \in \gamma(\mathfrak{A})$ with $0 \leqq f(p) \leqq 1$ for every $p \in X$, then for every integer n there exist $A_1, \ldots, A_n \in \mathfrak{A}$ such that*

(i) $$A_1 \cup \ldots \cup A_n = X\,.$$

(ii) $$\frac{3i-4}{3n} \leqq f(p) \leqq \frac{3i+1}{3n} \quad \text{for every} \quad p \in A_i\,, \quad i = 1, \ldots, n\,.$$

(iii) $$A_i \cap A_j = \emptyset \quad \text{for} \quad i + 2 \leqq j\,, \quad i = 1, \ldots, n-2\,.$$

Proof. Let n be given and let $V_1, \ldots, V_k$ be elements of $c\mathfrak{A}$ such that $V_1 \cup \ldots \ldots \cup V_k = X$ and $\omega(f, V_j) < 1/3n$, $j = 1, 2, \ldots, k$. For $i = 1, \ldots, n$, set

$$U_i = \bigcup_j \left\{ V_j : f(V_j) \cap \left[\frac{i-1}{n}, \frac{i}{n}\right] \neq \emptyset \right\}.$$

Clearly $U_i \in c\mathfrak{A}$, $U_1 \cup \ldots \cup U_n = X$, and $(3i - 4)/3n \leqq f(p) \leqq (3_i + 1)/3n$ for $p \in U_i$. Since $\mathfrak{A}$ is normal, there are A'_i and A''_i in $\mathfrak{A}$ such that $A'_i \subset U_1 \cup \ldots \cup U_i$, $A''_i \subset U_{i+1} \cup \ldots \cup U_n$, and $A'_i \cup A''_i = X$ for $i = 1, \ldots, n-1$. Then set $A_1 = A'_1$, $A_i = A'_i \cap A''_{i-1}$ for $i = 2, 3, \ldots, n-1$, and $A_n = A''_{n-1}$ and note that $A_1, \ldots, A_n$ satisfy the requirements.

Proof of Theorem 3.3. By 2.8 and 2.9, $\Lambda(\mathfrak{F})$ is a normal ring and hence by 2.4, $\gamma(\Lambda(\mathfrak{F}))$ is a uniformly closed ring. Clearly $\mathfrak{F} \subset \gamma(\Lambda(\mathfrak{F}))$ and therefore $\overline{\mathfrak{F}} \subset \gamma(\Lambda(\mathfrak{F}))$. Conversely, suppose that $f \in \gamma(\Lambda(\mathfrak{F}))$ and assume that $0 \leqq f(p) \leqq 1$ for every $p \in X$. Let $a, b \in \mathcal{R}$ with $0 \leqq a < b \leqq 1$ and choose an integer n sufficiently large so that $a < (3i_0 - 4)/3n < (3i_0 + 1)/3n < b$ for some i_0, $1 \leqq i_0 \leqq n$. Let $A_1, \ldots, A_n$ be the sets described in Lemma 3.4. Clearly $L_a(f) \subset A_1 \cup \ldots \cup A_{i_0-1}$ and $L^b(f) \subset \subset A_{i_0+1} \cup \ldots \cup A_n$. Moreover $A_1 \cup \ldots \cup A_{i_0-1}$ and $A_{i_0+1} \cup \ldots \cup A_n$ are elements of $\Lambda(\mathfrak{F}) = \mathfrak{Z}(\mathfrak{F})$ and furthermore by (iii) of 3.4, $A_1 \cup \ldots \cup A_{i_0-1}$ and $A_{i_0+1} \cup \ldots \cup A_n$ are disjoint. Let g_1 and g_2 be functions in $\mathfrak{F}$ with $A_1 \cup \ldots \cup A_{i_0-1} = Z(g_1)$ and $A_{i_0+1} \cup \ldots \cup A_n = Z(g_2)$ and set $g = |g_1|/(|g_1| + |g_2|)$. Since $\mathfrak{F}$ is closed under bounded inversion, $g \in \mathfrak{F}$. Clearly, $g(p) = 0$ for $p \in L_a(f)$, $g(p) = 1$ for $p \in L^b(f)$, and $0 \leqq g(p) \leqq 1$ for $p \in X$. Thus $\mathfrak{F}$, and hence $\overline{\mathfrak{F}}$, S_1-separates the Lebesgue sets of f. By the Tietze Approximation Theorem (Theorem 2.7 in [1]), $f \in \overline{\mathfrak{F}}$.

3.5. Theorem. *If $\mathfrak{F} \subset F(X)$ is a lattice-ordered uniformly closed linear space, then $\lambda(\Lambda(\mathfrak{F}))$ is the smallest uniformly closed ring that contains $\mathfrak{F}$ and is closed under inversion.*

Proof. The proof is identical with that of Theorem 3.1 except that now we use Theorem 3.3 in [1].

3.6. Corollary. *Suppose that $\mathfrak{G} \subset F^*(X)$ is a uniformly closed ring that is closed under bounded inversion and let $\tilde{\mathfrak{G}}$ be the smallest uniformly closed ring that contains $\mathfrak{G}$ and is closed under inversion. Then the set of all bounded function in $\tilde{\mathfrak{G}}$ coincides with $\mathfrak{G}$. Furthermore, $\tilde{\mathfrak{G}}$ can be defined as the set of all $f \in F(X)$ such that each of the truncations $f^{(i)}$, $i = 1, 2, \ldots$, belong to $\mathfrak{G}$.*

(The truncation $f^{(i)}$ is defined by $f^{(i)}(p) = -i \vee (f(p) \wedge i)$ for every $p \in X$.)

Proof. By Corollary 3.2 we have $\mathfrak{G} = \lambda^*(\Lambda(\mathfrak{G}))$ and by Theorem 3.5, $\tilde{\mathfrak{G}} = \lambda(\Lambda(\mathfrak{G}))$. But $\lambda^*(\Lambda(\mathfrak{G}))$ is obviously the set of all bounded functions in $\lambda(\Lambda(\mathfrak{G}))$, so the first part of the corollary is shown. The second part follows the first and Corollary 3.4 in [1].

Let $\Phi^*(X)$ denote the class of all uniformly closed ring $\mathfrak{G} \subset F^*(X)$ that are closed under bounded inversion and let $\Phi(X)$ denote the class of uniformly closed rings $\mathfrak{F} \subset F(X)$ that are closed under inversion. Corollary 3.6 actually shows the existence of a natural one-to-one correspondence between $\Phi^*(X)$ and $\Phi(X)$; stating this formally we obtain:

3.7. Corollary. *The map $\varkappa : \Phi(X) \to \Phi^*(X)$ defined by*

$$\varkappa(\mathfrak{F}) = \mathfrak{F}^* = \text{the set of all bounded functions in } \mathfrak{F} \ (\mathfrak{F} \in \Phi(X))$$

is one-to-one and onto. Furthermore the inverse map $\varkappa^{-1}$ can be defined by each of the following formulas

$\varkappa^{-1}(\mathbf{G})$ = *the smallest member of* $\Phi(X)$ *that contains* $\mathfrak{G}$

$= \lambda(\Lambda(\mathbf{G}))$

= *the set of all* $f \in F(X)$ *such that all truncations* $f^{(1)}, f^{(2)}, \ldots$, *belong to* $\mathfrak{G}$.

Remark. Actually, a more general form of this corollary is true. For if $\mathfrak{C} \subset F(X)$ is a fixed uniformly closed ring that is closed under inversion and if $\Phi_{\mathfrak{C}}(X)$ (resp. $\Phi^*_{\mathfrak{C}}(X)$) denotes the set of all members of $\Phi(X)$ (resp. $\Phi^*(X)$) that are contained in $\mathfrak{C}$, then the restriction of the map $\varkappa$ to $\Phi_{\mathfrak{C}}(X)$, $\varkappa_{\mathfrak{C}}$, is still a one-to-one map of $\Phi_{\mathfrak{C}}(X)$ onto $\Phi^*_{\mathfrak{C}}(X)$. Furthermore, the inverse map $\varkappa_{\mathfrak{C}}^{-1}$ can still be defined by each of the formulas in Corollary 3.7.

This more general form of Corollary 3.7 can be applied when, for instance, X is a topological space and $\mathfrak{C}$ is the set of all continuous functions on X; or if X is the unit interval and $\mathfrak{C}$ is the set of all Lebesgue measurable functions on X.

References

[1] *S. Mrówka:* On some approximation theorems, Nieuw Archief for Wiskunde (*3*), (1968), 94—111.

Author's address: Western Michigan University, Department of Mathematics, Kalamazoo, Michigan 49001, U.S.A.