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A CONTRIBUTION TO THE FOUNDATIONS OF NETWORK THEORY USING THE DISTRIBUTION THEORY

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In recent years there have appeared many publications concerning the formation of the foundations of Network Theory. In the papers by R. W. NEWCOMB [1], that by A. H. ZEMANIAN [2] and M. R. WOHLERS, E. J. BELTRAMI [3], the fundamental concepts of Network Theory were formulated by means of the Theory of distributions.

As a principal theorem in this theory it was proved that every single-valued, linear, continuous and time-invariant system T is convolutional, i.e.

$$T[x] = x * f$$

for some distribution f and for all distributions x. On the other hand, in the book by V. DOLEŽAL [4] there are considered operators constructed over the space of distributions which are not time-invariant. In this paper we shall give an analogous theorem about the meaning of convolution for a single-valued, linear and continuous system T which is not time-invariant, i.e.

$$T[x] = \int_{-\infty}^{+\infty} (x, f_a) \, \mathrm{d}a$$

for some set $\{f_a\}$ of distributions depending on a parameter a and for all distributions x.

While a single-valued, linear, continuous and time-invariant system is determined by the response $f = T[\delta]$ to the Dirac's distribution $\delta = \delta_0$, it turns out that a system, which is not time-invariant, is determined by responses $f_a = T[\delta_a]$ to the shifted Dirac's distributions δ_a.

1. INTRODUCTION

The notation and terminology of the book [4] will, with a few minor exceptions, be used throughout. Let $\boldsymbol{K}$ denote the set of all infinitely differentiable real functions

$\varphi(t)$ on the interval $(-\infty, +\infty)$ such that every $\varphi(t)$ vanishes identically outside some finite interval (which in general depends on φ). Further let $\mathbf{D}$ be the corresponding space of all distributions over $\mathbf{K}$ – that is the space of all linear and continuous functionals on $\mathbf{K}$. We shall write $\langle f, \varphi \rangle$ as the value of the functional $f \in \mathbf{D}$ for $\varphi \in \mathbf{K}$.

The linear combination, product of a distribution with an infinitely differentiable function (see p. 40 in [4]), n-th distributional derivative (see p. 45 in [4]) and a shifted distribution (see p. 51 in [4]) will be denoted by $\alpha f + \beta g$, $\alpha(t) f, f^{(n)}$ and $P_b[f]$, respectively. Clearly the set $\mathbf{D}$ is a linear space (see p. 128 in [4]).

Let $f_n \in \mathbf{D}$, $n = 1, 2, \ldots$ be a sequence of distributions. If $f \in \mathbf{D}$ then the symbol $f_n \to f$ means that the sequence converges to the distribution f (see p. 43 in [4]). Let us recall the theorem on completeness of the space $\mathbf{D}$.

Lemma 1.1. *Let $f_n \in \mathbf{D}$, $n = 1, 2, \ldots$ be a sequence of distributions. If $\langle f_n, \varphi \rangle$, $n = 1, 2, \ldots$ is a convergent sequence of real numbers for every $\varphi \in \mathbf{K}$, then there exists (a unique) distribution $f \in \mathbf{D}$ such that $f_n \to f$.*

Proof see p. 457 in [5].

Definition 1.1. Let $n \geqq 0$ be an integer, and let $\mathbf{D}_n$ be the set of all distributions having the following property: if $f \in \mathbf{D}_n$, then there exists a real continuous function $z(t)$ on $(-\infty, +\infty)$ such that $f = z^{(n)}$, i.e.

$$\langle f, \varphi \rangle = (-1)^n \int_{-\infty}^{+\infty} z(t)\, \varphi^{(n)}(t)\, \mathrm{d}t$$

for every $\varphi \in \mathbf{K}$. Let us also remark that in this article $x(t)$ for $x \in \mathbf{D}_0$ always means a *continuous function* on $(-\infty, +\infty)$. Furthermore, let $\mathbf{D}_* = \bigcup_{n=1}^{\infty} \mathbf{D}_n$. Obviously, $\mathbf{D}_n$ $(n = 0, 1, \ldots)$ and $\mathbf{D}_*$ are linear subspaces of $\mathbf{D}$.

Lemma 1.2. *Let $f \in \mathbf{D}$; then there is a sequence $f_n \in \mathbf{D}_*$, $n = 1, 2, \ldots$ such that $f_n \to f$.*

Proof. Let $\alpha_n(t)$, $n = 1, 2, \ldots$ be a sequence of infinitely differentiable functions with properties $\alpha_n(t) = 1$ for $-n \leqq t \leqq n$, $0 < \alpha_n(t) < 1$ for $n < t < n + 1$ and $-n - 1 < t < -n$, $\alpha_n(t) = 0$ for $t \geqq n + 1$ and $t \leqq -n - 1$. The rest of the proof is analogous to that of Lemma 5.4.5 in [4]. We get $\alpha_n f \in \mathbf{D}_*$ and $\alpha_n f \to f$.

If $f, g \in \mathbf{D}$ and b is a real number, then the equality $f = g$ on the interval $(-\infty, b)$ means that $f - g = 0$ on $(-\infty, b)$ (see p. 41 in [4]).

From Lemma 3.1.2 [4] it follows that:

Lemma 1.3. *Let b be a real number, $f \in \mathbf{D}$; then $f = 0$ on the interval $(-\infty, b)$ if and only if $\langle f, \varphi \rangle = 0$ for every $\varphi \in \mathbf{K}$ with $\varphi(t) = 0$ on the interval $(c, +\infty)$, where $c < b$.*

Definition 1.2. Let $\{f_a\}$ be a set of distributions from $\mathbf{D}$ depending on a parameter a, where a is an arbitrary real number. The distribution $g_a \in \mathbf{D}$ will be called the *partial derivative* of f_a with respect to the parameter a, if

$$\langle g_a, \varphi \rangle = \lim_{h \to 0} \left\langle \frac{f_{a+h} - f_a}{h}, \varphi \right\rangle$$

for every $\varphi \in \mathbf{K}$. We shall write $g_a = \partial f_a / \partial a$.

Lemma 1.4. *Let $f_a \in \mathbf{D}$ for an arbitrary real number a; then the partial derivative $\partial f_a / \partial a$ exists if and only if the function $\psi(t)$ has a derivative $\psi'(t)$ on the interval $(-\infty, +\infty)$ for every $\varphi \in \mathbf{K}$, where $\psi(a) = \langle f_a, \varphi \rangle$ for every real number a.*

Proof follows from the equation

$$\left\langle \frac{f_{a+h} - f_a}{h}, \varphi \right\rangle = \frac{\psi(a+h) - \psi(a)}{h} \quad (h \neq 0).$$

Note. Clearly

$$\left\langle \frac{\partial f_a}{\partial a}, \varphi \right\rangle = \psi'(a)$$

for every $\varphi \in \mathbf{K}$, where $\psi(a) = \langle f_a, \varphi \rangle$ for every real number a.

Lemma 1.5. *Let $f_a \in \mathbf{D}$ for an arbitrary real number a; then $g_a = \partial f_a / \partial a$ if and only if*

$$(a_n - a)^{-1} (f_{a_n} - f_a) \to g_a$$

for all convergent sequences of real numbers $a_n \to a$ $(a_n \neq a)$, $n = 1, 2, \ldots$

Proof follows immediately from Lemma 1.4.

Definition 1.3. Let $\{f_a\}$ be a set of distributions from $\mathbf{D}$ depending on a parameter a, where a is an arbitrary real number. If $n \geqq 0$ is an integer, then the distribution $\partial^n f_a / \partial a^n \in \mathbf{D}$ will be called the *n-th partial derivative* of f_a with respect to the parameter a, if

$$\frac{\partial^0 f_a}{\partial a^0} = f_a, \quad \frac{\partial^1 f_a}{\partial a^1} = \frac{\partial f_a}{\partial a}, \quad \frac{\partial^n f_a}{\partial a^n} = \frac{\partial}{\partial a} \left(\frac{\partial^{n-1} f_a}{\partial a^{n-1}} \right) \quad (n \geqq 2).$$

Lemma 1.6. *Let $f_a \in \mathbf{D}$ for an arbitrary real number a. If $n \geqq 0$ is an integer, then the n-th partial derivative $\partial^n f_a / \partial a^n$ exists if and only if the function $\psi(t)$ has an n-th derivative $\psi^{(n)}(t)$ on the interval $(-\infty, +\infty)$ for every $\varphi \in \mathbf{K}$, where $\psi(a) = \langle f_a, \varphi \rangle$ for every real number a.*

Proof follows from Lemma 1.4.

Note. Evidently

$$\left\langle \frac{\partial^n f_a}{\partial a^n}, \varphi \right\rangle = \psi^{(n)}(a)$$

for every $\varphi \in \mathbf{K}$, where $\psi(a) = \langle f_a, \varphi \rangle$ for every real number a.

Example. Let δ_a be the Dirac distribution (a being a real number). It is clear that $\psi(a) = \langle \delta_a, \varphi \rangle = \varphi(a)$ for every $\varphi \in \mathbf{K}$ (see p. 40 in [4]). From Lemma 1.6 it follows that

$$\frac{\partial^n \delta_a}{\partial a^n} = (-1)^n \delta_a^{(n)} \tag{1.1}$$

for $n = 0, 1, \ldots$

Definition 1.4. Let a_i, $i = 0, 1, \ldots, m$; ξ_i, $i = 1, 2, \ldots, m$ be real numbers, where $a_{i-1} < a_i$ and $a_{i-1} \leqq \xi_i \leqq a_i$ for $i = 1, 2, \ldots, m$. The set $\mathscr{D} = \{a_0, a_1, \ldots, a_m; \xi_1, \xi_2, \ldots, \xi_m\}$ will be called a *division* (partial and with the points ξ_i) of the interval $(-\infty, +\infty)$. Let us put

$$\mu(\mathscr{D}) = \max_{i=1,2,\ldots,m} (a_i - a_{i-1}), \quad l(\mathscr{D}) = a_0 \quad \text{and} \quad r(\mathscr{D}) = a_m .$$

Let $\{f_a\}$ be a set of distributions from $\mathbf{D}$ depending on a parameter a, where a is an arbitrary real number. The distribution

$$s(\mathscr{D}) = \sum_{i=1}^{m} (a_i - a_{i-1}) f_{\xi_i}$$

will be called the *integral sum* of $\{f_a\}$ with respect to the division $\mathscr{D}$.

The sequence $\mathscr{D}_n$, $n = 1, 2, \ldots$ of divisions of the interval $(-\infty, +\infty)$ will be called the *zero sequence*, if $\mu(\mathscr{D}_n) \to 0$, $l(\mathscr{D}_n) \to -\infty$ and $r(\mathscr{D}_n) \to +\infty$. Let $\{f_a\}$ be a set of distributions from $\mathbf{D}$ depending on a parameter a, where a is an arbitrary real number. The distribution $g \in \mathbf{D}$ will be called the *integral* of $\{f_a\}$, if $s(\mathscr{D}_n) \to g$ for an arbitrary zero sequence of divisions of the interval $(-\infty, +\infty)$. We shall write $g = \int_{-\infty}^{+\infty} f_a \, da$.

Lemma 1.7. *Let $f_a \in \mathbf{D}$ for an arbitrary real number a. If ψ is a continuous function on the interval $(-\infty, +\infty)$ and ψ vanishes identically outside some finite interval for every $\varphi \in \mathbf{K}$, where $\psi(a) = \langle f_a, \varphi \rangle$, then there exists the integral $\int_{-\infty}^{+\infty} f_a \, da$.*

Proof. Let $\mathscr{D}_n$, $n = 1, 2, \ldots$ be an arbitrary zero sequence of divisions of $(-\infty, +\infty)$. If we put $g_n = s(\mathscr{D}_n) = \sum_{i=1}^{m} (a_i - a_{i-1}) f_{\xi_i}$, then we get

$$\langle g_n, \varphi \rangle = \sum_{i=1}^{m} \psi(\xi_i)(a_i - a_{i-1}) \tag{1.2}$$

for $\varphi \in \mathbf{K}$. By the assumption there exist real numbers c, d such that $\psi(t) = 0$ on $(-\infty, c) \cup (d, +\infty)$. Thus there exists a positive integer n_0 such that $\mathrm{l}(\mathscr{D}_n) \leqq c$ and $\mathrm{r}(\mathscr{D}_n) \geqq d$ for every positive integer $n > n_0$. From (1.2) it follows that

$$\langle g_n, \varphi \rangle \to \int_c^d \psi(a)\, \mathrm{d}a = \int_{-\infty}^{+\infty} \psi(a)\, \mathrm{d}a\,.$$

According to Lemma 1.1 there exists a distribution $g \in \mathbf{D}$ such that $g_n \to g$. Hence $\langle g, \varphi \rangle = \int_{-\infty}^{+\infty} \psi(a)\, \mathrm{d}a$ for every $\varphi \in \mathbf{K}$. Consequently, the distribution g does not depend on the choice of the zero sequence $\mathscr{D}_n$, $n = 1, 2, \ldots$ of divisions. Therefore $g = \int_{-\infty}^{+\infty} f_a\, \mathrm{d}a$.

Note. Clearly

$$\left\langle \int_{-\infty}^{+\infty} f_a\, \mathrm{d}a, \varphi \right\rangle = \int_{-\infty}^{+\infty} \langle f_a, \varphi \rangle\, \mathrm{d}a$$

for every $\varphi \in \mathbf{K}$.

Example. We have

$$\int_{-\infty}^{+\infty} \delta_a\, \mathrm{d}a = 1\,,$$

where $\langle 1, \varphi \rangle = \int_{-\infty}^{+\infty} \varphi(t)\, \mathrm{d}t$ for every $\varphi \in \mathbf{K}$.

Let $\mathscr{F}$ be the set of all distributions $\{f_a\}$ from $\mathbf{D}$ depending on a parameter a (where a is an arbitrary real number) such that $\psi \in \mathbf{K}$ for every $\varphi \in \mathbf{K}$, where $\psi(a) = \langle f_a, \varphi \rangle$.

Theorem 1.1. *Let $f_a \in \mathbf{D}$ for an arbitrary real number a. The necessary and sufficient condition that $\{f_a\} \in \mathscr{F}$ is the fulfillment of conditions:*

1. *The partial derivative $\partial^n f_a / \partial a^n$ exists for every positive integer n.*
2. *$\alpha_n f_{a_n} \to 0$ for every two sequences of real numbers $\alpha_n, a_n, |a_n| \to +\infty$, $n = 1, 2, \ldots$*

Proof. By Lemma 1.6 condition 1 is satisfied if and only if the function ψ is infinitely differentiable on the interval $(-\infty, +\infty)$ for every $\varphi \in \mathbf{K}$, where $\psi(a) = \langle f_a, \varphi \rangle$. We shall prove that condition 2 is satisfied if and only if the function ψ vanishes identically outside some finite interval for every $\varphi \in \mathbf{K}$.

Let ψ vanish identically outside some finite interval for every $\varphi \in \mathbf{K}$. Let $\alpha_n, a_n, n = 1, 2, \ldots$ be two sequences of real numbers and let $|a_n| \to +\infty$. Then $\langle \alpha_n f_{a_n}, \varphi \rangle = \alpha_n \langle f_{a_n}, \varphi \rangle = \alpha_n\, \psi(a_n) \to 0$ for every $\varphi \in \mathbf{K}$. Hence $\alpha_n f_{a_n} \to 0$.

Let now $\alpha_n f_{a_n} \to 0$ for every two sequences of real numbers $\alpha_n, a_n, |a_n| \to +\infty$, $n = 1, 2, \ldots$ If there exists $\varphi \in \mathbf{K}$ such that the corresponding function ψ does not vanish identically outside some finite interval, then there exists a sequence of real numbers a_n, $|a_n| \to +\infty$, $n = 1, 2, \ldots$ such that $\psi(a_n) \neq 0$. Put $\alpha_n^{-1} = \psi(a_n)$. Hence $\langle \alpha_n f_{a_n}, \varphi \rangle = \alpha_n \langle f_{a_n}, \varphi \rangle = \alpha_n \psi(a_n) = 1 \nrightarrow 0$, which is a contradiction. The Theorem is thus proved.

Theorem 1.2. *Let* $\{f_a\}, \{g_a\} \in \mathscr{F}$.

1. *If* α, β *are real numbers, then* $\{\alpha f_a, + \beta g_a\} \in \mathscr{F}$.
2. *If* n *is a positive integer, then* $\{\partial^n f_a / \partial a^n\} \in \mathscr{F}$.
3. *If* b *is a real number, then* $\{h_a\} \in \mathscr{F}$, *where* $h_a = P_b[f_a]$.
4. *If* b *is a real number, then* $\{h_a\} \in \mathscr{F}$, *where* $h_a = f_{a-b}$.

Proof. Put $\psi(a) = \langle f_a, \varphi \rangle$ and $\chi(a) = \langle g_a, \varphi \rangle$ for every $\varphi \in \mathbf{K}$. Evidently $\psi, \chi \in \mathbf{K}$.

1. If α, β are real numbers, then $\omega(a) = \langle \alpha f_a + \beta g_a, \varphi \rangle = \alpha \, \psi(a) + \beta \, \chi(a)$. Thus $\omega = \alpha \psi + \beta \chi \in \mathbf{K}$. Hence $\{\alpha f_a + \beta g_a\} \in \mathscr{F}$.

2. If n is a positive integer, then from Lemma 1.6 it follows that $\omega(a) = \langle \partial^n f_a / \partial a^n, \varphi \rangle = \psi^{(n)}(a)$. Thus $\omega = \psi^{(n)} \in \mathbf{K}$. Hence $\{\partial^n f_a / \partial a^n\} \in \mathscr{F}$.

3. If b is a real number, then $\omega(a) = \langle h_a, \varphi \rangle = \langle P_b[f_a], \varphi \rangle = \langle f_a, \varphi(t + b) \rangle$. Clearly $\varphi(t + b) \in \mathbf{K}$. Thus, we have $\omega \in \mathbf{K}$. Hence $\{h_a\} \in \mathscr{F}$.

4. If b is a real number, then $\omega(a) = \langle h_a, \varphi \rangle = \langle f_{a-b}, \varphi \rangle = \psi(a - b)$. Thus $\omega \in \mathbf{K}$. Hence $\{h_a\} \in \mathscr{F}$.

Theorem 1.3. Let $\{f_a\} \in \mathscr{F}$ and $\alpha \in \mathbf{D}_0$; then there exists the integral

$$g = \int_{-\infty}^{+\infty} \alpha(a) f_a \, \mathrm{d}a$$

and for every $\varphi \in \mathbf{K}$ *we have*

$$\langle g, \varphi \rangle = \int_{-\infty}^{+\infty} \alpha(a) \, \psi(a) \, \mathrm{d}a \,,$$

where $\psi(a) = \langle f_a, \varphi \rangle$.

Proof follows from Lemma 1.7 because $\langle \alpha(a) f_a, \varphi \rangle = \alpha(a) \, \psi(a)$ for every $\varphi \in \mathbf{K}$.

Example. Evidently $\{\delta_a\} \in \mathscr{F}$ and for every $\alpha \in \mathbf{D}_0$ we have

$$\int_{-\infty}^{+\infty} \alpha(a) \, \delta_a \, \mathrm{d}a = \alpha \,.$$

Theorem 2.1. *Let $\{f_a\} \in \mathscr{F}$ and $x \in \mathbf{D}$; then there is a unique distribution $y \in \mathbf{D}$ such that*

$$\langle y, \varphi \rangle = \langle x, \psi \rangle \tag{2.1}$$

for every $\varphi \in \mathbf{K}$, where $\psi(a) = \langle f_a, \varphi \rangle$ for every real number a.

Proof. 1. Let $x \in \mathbf{D}_n$ $(n \geqq 0)$; then there exists a distribution $z \in \mathbf{D}_0$ such that $z^{(n)} = x$. By Theorem 1.2 it follows that $\{\partial^n f_a/\partial a^n\} \in \mathscr{F}$. According to Theorem 1.3 there exists a distribution $y = (-1)^n \int_{-\infty}^{+\infty} z(a) (\partial^n f_a/\partial a^n) \, da$. We have therefore $\langle y, \varphi \rangle = \int_{-\infty}^{+\infty} \chi(a) \, da$ for every $\varphi \in \mathbf{K}$, where $\chi(a) = \langle (-1)^n z(a) (\partial^n f_a/\partial a^n), \varphi \rangle = (-1)^n z(a) \langle (\partial^n f_a/\partial a^n), \varphi \rangle = (-1)^n z(a) \psi^{(n)}(a)$. Hence $\langle y, \varphi \rangle = (-1)^n \int_{-\infty}^{+\infty} z(a) \, . \, \psi^{(n)}(a) \, da = \langle z, (-1)^n \psi^{(n)} \rangle = \langle z^{(n)}, \psi \rangle = \langle x, \psi \rangle$ for every $\varphi \in \mathbf{K}$.

2. Let $x \in \mathbf{D}$; then according to Lemma 1.2 there is a sequence $x_n \in \mathbf{D}_*$, $n = 1, 2, \ldots$ such that $x_n \to x$. By the first part of the proof there exist $y_n \in \mathbf{D}$ such that $\langle y_n, \varphi \rangle = \langle x_n, \psi \rangle$ for every $\varphi \in \mathbf{K}$. However, since $\psi \in \mathbf{K}$, then $\langle y_n, \varphi \rangle = \langle x_n, \psi \rangle \to \langle x, \psi \rangle$. By Lemma 1.1 there exists a distribution $y \in \mathbf{D}$ such that $\langle y_n, \varphi \rangle \to \langle y, \varphi \rangle$. Hence $\langle y, \varphi \rangle = \langle x, \psi \rangle$ for every $\varphi \in \mathbf{K}$.

3. The uniqueness of distribution y follows from (2.1)

Definition 2.1. Let $\{f_a\} \in \mathscr{F}$ and $x \in \mathbf{D}$; then the distribution y (see Theorem 2.1) will be denoted by $\int_{-\infty}^{+\infty} (x, f_a) \, da$.

Note. Clearly

$$\left\langle \int_{-\infty}^{+\infty} (x, f_a) \, da, \varphi \right\rangle = \langle x, \psi \rangle \tag{2.2}$$

for every $\varphi \in \mathbf{K}$, where $\psi(a) = \langle f_a, \varphi \rangle$ for every real number a.

Examples. 1. Evidently

$$x = \int_{-\infty}^{+\infty} (x, \delta_a) \, da \tag{2.3}$$

for every $x \in \mathbf{D}$. The proof follows from $\varphi(a) = \langle \delta_a, \varphi \rangle$ for every $\varphi \in \mathbf{K}$ and for every real number a.

2. Let $\{f_a\} \in \mathscr{F}$. Using (2.2) we have $\langle \int_{-\infty}^{+\infty} (\delta_b, f_a) \, da, \varphi \rangle = \langle \delta_b, \psi \rangle = \psi(b) = \langle f_b, \varphi \rangle$ for every $\varphi \in \mathbf{K}$ and for every real number b. Thus

$$f_b = \int_{-\infty}^{+\infty} (\delta_b, f_a) \, da \, . \tag{2.4}$$

Theorem 2.2. *If α, β are real numbers, $\{f_a\} \in \mathcal{F}$ and $x, y \in \mathbf{D}$, then*

$$\int_{-\infty}^{+\infty} (\alpha x + \beta y, f_a)\,\mathrm{d}a = \alpha \int_{-\infty}^{+\infty} (x, f_a)\,\mathrm{d}a + \beta \int_{-\infty}^{+\infty} (y, f_a)\,\mathrm{d}a\,.$$

Proof. Put $w = \int_{-\infty}^{+\infty} (\alpha x + \beta y, f_a)\,\mathrm{d}a$, $u = \int_{-\infty}^{+\infty} (x, f_a)\,\mathrm{d}a$ and $v = \int_{-\infty}^{+\infty} (y, f_a)\,\mathrm{d}a$. Then $\langle w, \varphi\rangle = \langle \alpha x + \beta y, \psi\rangle = \alpha\langle x, \psi\rangle + \beta\langle y, \psi\rangle = \alpha\langle u, \varphi\rangle + \beta\langle v, \varphi\rangle$ for every $\varphi \in \mathbf{K}$, where $\psi(a) = \langle f_a, \varphi\rangle$ for every real number a. Hence $w = \alpha u + \beta v$ which completes the proof.

Theorem 2.3. *If α, β are real numbers, $\{f_a\}, \{g_a\} \in \mathcal{F}$ and $x \in \mathbf{D}$, then*

$$\int_{-\infty}^{+\infty} (x, \alpha f_a + \beta g_a)\,\mathrm{d}a = \alpha \int_{-\infty}^{+\infty} (x, f_a)\,\mathrm{d}a + \beta \int_{-\infty}^{+\infty} (x, g_a)\,\mathrm{d}a\,.$$

Proof. According to Theorem 1.2, $\{\alpha f_a + \beta g_a\} \in \mathcal{F}$. Denote $w = \int_{-\infty}^{+\infty} (x, \alpha f_a + \beta g_a)\,\mathrm{d}a$, $u = \int_{-\infty}^{+\infty} (x, f_a)\,\mathrm{d}a$ and $v = \int_{-\infty}^{+\infty} (x, g_a)\,\mathrm{d}a$. Then $\langle w, \varphi\rangle = \langle x, \alpha\psi + \beta\chi\rangle = \alpha\langle x, \psi\rangle + \beta\langle x, \chi\rangle = \alpha\langle u, \varphi\rangle + \beta\langle v, \varphi\rangle$ for every $\varphi \in \mathbf{K}$, where $\psi(a) = \langle f_a, \varphi\rangle$ and $\chi(a) = \langle g_a, \varphi\rangle$ for every real number a. Hence $w = \alpha u + \beta v$ which completes the proof.

Theorem 2.4. *If $x, x_n \in \mathbf{D}$, $n = 1, 2, \ldots$, $x_n \to x$ and $\{f_a\} \in \mathcal{F}$, then*

$$\int_{-\infty}^{+\infty} (x_n, f_a)\,\mathrm{d}a \to \int_{-\infty}^{+\infty} (x, f_a)\,\mathrm{d}a\,.$$

Proof. Denote $y = \int_{-\infty}^{+\infty} (x, f_a)\,\mathrm{d}a$ and $y_n = \int_{-\infty}^{+\infty} (x_n, f_a)\,\mathrm{d}a$ for $n = 1, 2, \ldots$ Then $\langle y_n, \varphi\rangle = \langle x_n, \psi\rangle \to \langle x, \psi\rangle = \langle y, \varphi\rangle$ for every $\varphi \in \mathbf{K}$, where $\psi(a) = \langle f_a, \varphi\rangle$ for every real number a. We have $y_n \to y$, q.e.d.

Theorem 2.5. *If $x \in \mathbf{D}$ and $\{f_a\} \in \mathcal{F}$, then*

$$\int_{-\infty}^{+\infty} (x^{(n)}, f_a)\,\mathrm{d}a = (-1)^n \int_{-\infty}^{+\infty} \left(x, \frac{\partial^n f_a}{\partial a^n}\right)\mathrm{d}a\,.$$

Proof. By Theorem 1.2 we have $\{\partial^n f_a/\partial a^n\} \in \mathcal{F}$. Put $u = \int_{-\infty}^{+\infty} (x^{(n)}, f_a)\,\mathrm{d}a$ and $v = \int_{-\infty}^{+\infty} (x, \partial^n f_a/\partial a^n)\,\mathrm{d}a$. Then $\langle u, \varphi\rangle = \langle x^{(n)}, \psi\rangle = \langle x, (-1)^n \psi^{(n)}\rangle = (-1)^n \cdot \langle x, \psi^{(n)}\rangle = (-1)^n \langle v, \varphi\rangle$ for every $\varphi \in \mathbf{K}$, where $\psi(a) = \langle f_a, \varphi\rangle$ and $\psi^{(n)}(a) = \langle \partial^n f_a/\partial a^n, \varphi\rangle$ for every real number a. Therefore $u = (-1)^n v$, q.e.d.

Example. We have

$$x^{(n)} = \int_{-\infty}^{+\infty} (x, \delta_a^{(n)})\,\mathrm{d}a$$

for every $x \in \mathbf{D}$. By (2.3), Theorem 2.5, Theorem 2.3 and (1.1),

$$x^{(n)} = \int_{-\infty}^{+\infty} (x^{(n)}, \delta_a)\, \mathrm{d}a = (-1)^n \int_{-\infty}^{+\infty} \left(x, \frac{\partial^n \delta_a}{\partial a^n}\right) \mathrm{d}a = \int_{-\infty}^{+\infty} (x, \delta_a^{(n)})\, \mathrm{d}a\,.$$

Theorem 2.6. *If b is a real number, $\{f_a\} \in \mathscr{F}$ and $x \in \mathbf{D}$, then*

$$P_b[y] = \int_{-\infty}^{+\infty} (x, g_a)\, \mathrm{d}a\,,$$

where $y = \int_{-\infty}^{+\infty} (x, f_a)\, \mathrm{d}a$ and $g_a = P_b[f_a]$.

Proof. According to Theorem 1.2, $\{g_a\} \in \mathscr{F}$. Then $\langle P_b[y], \varphi\rangle = \langle y, \varphi(t + b)\rangle = \langle x, \chi\rangle$ for every $\varphi \in \mathbf{K}$, where $\chi(a) = \langle f_a, \varphi(t + b)\rangle = \langle P_b[f_a], \varphi\rangle = \langle g_a, \varphi\rangle$. Hence it follows that $P_b[y] = \int_{-\infty}^{+\infty} (x, g_a)\, \mathrm{d}a$. The theorem is proved.

Example. From Theorem 2.6 and (2.3) it follows that

$$P_b[x] = \int_{-\infty}^{+\infty} (x, \delta_{a+b})\, \mathrm{d}a$$

for every $x \in \mathbf{D}$ and for an arbitrary real number b.

Theorem 2.7. *If b is a real number, $\{f_a\} \in \mathscr{F}$ and $x \in \mathbf{D}$, then*

$$\int_{-\infty}^{+\infty} (x, f_a)\, \mathrm{d}a = \int_{-\infty}^{+\infty} (P_b[x], f_{a-b})\, \mathrm{d}a\,.$$

Proof. By Theorem 1.2, $\{g_a\} \in \mathscr{F}$, where $g_a = f_{a-b}$. Denote $y = \int_{-\infty}^{+\infty} (x, f_a)\, \mathrm{d}a$ and $z = \int_{-\infty}^{+\infty} (P_b[x], f_{a-b})\, \mathrm{d}a$. Then $\langle y, \varphi\rangle = \langle x, \psi\rangle = \langle P_b[x], \psi(t - b)\rangle = \langle P_b[x], \chi\rangle = \langle z, \varphi\rangle$ for every $\varphi \in \mathbf{K}$, where $\psi(a) = \langle f_a, \varphi\rangle$ and $\chi(a) = \psi(a - b) = \langle f_{a-b}, \varphi\rangle$ for every real number a. Thus $y = z$, q.e.d.

Theorem 2.8. *Let $\{f_a\} \in \mathscr{F}$ and let f_a vanish on $(-\infty, a)$ for every real number a. If b is a real number, $x \in \mathbf{D}$ and x vanishes on $(-\infty, b)$, then $\int_{-\infty}^{+\infty} (x, f_a)\, \mathrm{d}a$ vanishes on $(-\infty, b)$.*

Proof. Let $\varphi \in \mathbf{K}$. If $\varphi(t) = 0$ on $(c, +\infty)$, where $c < b$, then by Lemma 1.3 it follows that $\psi(a) = \langle f_a, \varphi\rangle = 0$ for $a > c$. Since $\psi(t) = 0$ on $(c, +\infty)$, we have $\langle x, \psi\rangle = 0$. If we put $y = \int_{-\infty}^{+\infty} (x, f_a)\, \mathrm{d}a$, then, by (2.1), $\langle y, \varphi\rangle = 0$. From Lemma 1.3 it follows that y vanishes on $(-\infty, b)$.

Let $\mathbf{P}$ be a non-empty subset of the set $\mathbf{D}$ of all distributions. A mapping T of $\mathbf{P}$ into $\mathbf{D}$ (i.e. a rule whereby to each $x \in \mathbf{P}$ a unique distribution $T[x] \in \mathbf{D}$ is assigned) will be called the *operator* in $\mathbf{P}$. The set $\mathbf{P}$ will be termed the *domain* of the operator T.

Definition 3.1. Let $\mathbf{P}$ be a non-empty subset of $\mathbf{D}$. An operator T on $\mathbf{P}$ will be called *continuous* if the following implication holds:

(3.1) *If* $x, x_n \in \mathbf{P}$, $n = 1, 2, \ldots, x_n \to x$, *then* $T[x_n] \to T[x]$.

Definition 3.2. Let $\mathbf{P}$ be a linear subspace of $\mathbf{D}$. An operator T on $\mathbf{P}$ will be called *linear* if the following condition holds:

(3.2) *If* α, β *are real numbers and* $x, y \in \mathbf{P}$, *then* $T[\alpha x + \beta y] = \alpha\, T[x] + \beta\, T[y]\cdot$

Note. From Definition 3.2 it follows that $O \in \mathbf{P}$ and $T[O] = O$.

Theorem 3.1. *Let T be a linear and continuous operator on a linear subspace $\mathbf{P} \subset \mathbf{D}$. Let $\{f_a\} \in \mathscr{F}$ and $\partial^n f_a/\partial a^n \in \mathbf{P}$ for every $n = 0, 1, 2, \ldots$ and for every real number a. Then $\{g_a\} \in \mathscr{F}$ and $\partial^n g_a/\partial a^n = T[\partial^n f_a/\partial a^n]$, where $g_a = T[f_a]$.*

Proof. Let $a_n \to a$ $(a_n \neq a)$, $n = 1, 2, \ldots$ be a convergent sequence of real numbers. By Lemma 1.5 we have

$$(a_n - a)^{-1} (f_{a_n} - f_a) \to \frac{\partial f_a}{\partial a}\,.$$

From this and (3.1), (3.2) it follows that $(a_n - a)^{-1} (g_{a_n} - g_a) = (a_n - a)^{-1}\,. \,.\,(T[f_{a_n}] - T[f_a]) = T[(a_n - a)^{-1} (f_{a_n} - f_a)] \to T[\partial f_a/\partial a]$, where $g_a = T[f_a]$. According to Lemma 1.5 there exists $\partial g_a/\partial a$ and $\partial g_a/\partial a = T[\partial f_a/\partial a]$. Similarly we obtain that there exists $\partial^n g_a/\partial a^n$ and $\partial^n g_a/\partial a^n = T[\partial^n f_a/\partial a^n]$ for every $n = 2, 3, \ldots$

Let a_n, α_n, $n = 1, 2, \ldots$ be two sequences of real numbers and let $|a_n| \to +\infty$. By Theorem 1.1 we have $\alpha_n f_{a_n} \to 0$. Then it follows from (3.1) and (3.2) that $\alpha_n g_n = \alpha_n\, T[f_{a_n}] = T[\alpha_n f_{a_n}] \to 0$. Finally from Theorem 1.1 it follows that $\{g_a\} \in \mathscr{F}$.

Theorem 3.2. *Let T be a linear and continuous operator on a linear subspace $\mathbf{P} \subset \mathbf{D}$. Let $\{f_a\} \in \mathscr{F}$ and $\partial^n f_a/\partial a^n \in \mathbf{P}$ for every $n = 0, 1, 2, \ldots$ and for every real number a. If $x \in \mathbf{D}_0$ and $y \in \mathbf{P}$, where $y = \int_{-\infty}^{+\infty} x(a)\, f_a\, \mathrm{d}a$, then $T[y] = \int_{-\infty}^{+\infty} x(a)\, g_a\, \mathrm{d}a$, where $g_a = T[f_a]$.*

Proof. According to Theorem 3.1, $\{g_a\} \in \mathscr{F}$. By Theorem 1.3 there exists $u = \int_{-\infty}^{+\infty} x(a)\, g_a\, \mathrm{d}a$. Let $\mathscr{D}$ be an arbitrary division of the interval $(-\infty, +\infty)$. Using the notation of Definition 1.4 we have for the integral sums $s_1(\mathscr{D}) = \sum_{i=1}^{m} (a_i - a_{i-1})\,.$

$\cdot\, x(\xi_i) f_{\xi_i}$, $s_2(\mathscr{D}) = \sum_{i=1}^{m} (a_i - a_{i-1})\, x(\xi_i)\, g_{\xi_i}$. From (3.2) it follows that $T[s_1(\mathscr{D})] = T[\sum_{i=1}^{m} (a_i - a_{i-1})\, x(\xi_i) f_{\xi_i}] = \sum_{i=1}^{m} (a_i - a_{i-1})\, x(\xi_i)\, T[f_{\xi_i}] = s_2(\mathscr{D})$. If $\{\mathscr{D}_n\}$ is an arbitrary zero sequence of divisions of the interval $(-\infty, +\infty)$, then $s_1(\mathscr{D}_n) \to y$ and $s_2(\mathscr{D}_n) \to u$. By (3.1) we have $s_2(\mathscr{D}_n) = T[s_1(\mathscr{D}_n)] \to T[y]$. Hence $u = T[y]$, q.e.d.

Theorem 3.3. *Let T be a linear and continuous operator on a linear subspace $\mathbf{P} \subset \mathbf{D}$. Let $\{f_a\} \in \mathscr{F}$ and $\partial^n f_a/\partial a^n \in \mathbf{P}$ for every $n = 0, 1, 2, \ldots$ and for every real number a. If $x \in \mathbf{D}_*$ and $y \in \mathbf{P}$, where $y = \int_{-\infty}^{+\infty} (x, f_a)\, da$, then $T[y] = \int_{-\infty}^{+\infty} (x, g_a)\, da$, where $g_a = T[f_a]$.*

Proof. By Theorem 3.1 we have $\{g_a\} \in \mathscr{F}$. Thus, $\int_{-\infty}^{+\infty} (x, g_a)\, da$ exists. From the proof of Theorem 2.1 it follows that there exists $z \in \mathbf{D}_0$ $(z^{(n)} = x)$ such that $y = (-1)^n \int_{-\infty}^{+\infty} z(a)\, (\partial^n f_a/\partial a^n)\, da$. Using Theorem 1.2, Theorem 3.1, Theorem 3.2, (3.2) and Theorem 2.1 we get $T[y] = (-1)^n \int_{-\infty}^{+\infty} z(a)\, (\partial^n g_a/\partial a^n)\, da = \int_{-\infty}^{+\infty} (x, g_a)\, da$.

Note. If the operator T on a non-empty subset $\mathbf{P} \subset \mathbf{D}$ has the form

$$T[x] = \int_{-\infty}^{+\infty} (x, f_a)\, da \quad (x \in \mathbf{P})$$

and if $\delta_a \in \mathbf{P}$ for every real number a, then from (2.4) it follows that $f_a = T[\delta_a]$.

Theorem 3.4. *Let $\mathbf{P} \subset \mathbf{D}_*$ be a linear subspace and let $\delta_a^{(n)} \in \mathbf{P}$ for every $n = 0, 1, 2, \ldots$ and for every real number a. An operator T on $\mathbf{P}$ is linear and continuous if and only if it has the form*

$$T[x] = \int_{-\infty}^{+\infty} (x, f_a)\, da \quad (x \in \mathbf{P}) \tag{3.3}$$

and $f_a = T[\delta_a]$.

Proof follows from Theorem 2.2, Theorem 2.4, (2.3), Theorem 3.3 and (2.4).

Theorem 3.5. *Let $\mathbf{P}$ $(\mathbf{D}_* \subset \mathbf{P} \subset \mathbf{D})$ be a linear space. An operator T on $\mathbf{P}$ is linear and continuous if and only if it has the form*

$$T[x] = \int_{-\infty}^{+\infty} (x, f_a)\, da \quad (x \in \mathbf{P}) \tag{3.3}$$

and $f_a = T[\delta_a]$.

Proof. If the operator T has the form (3.3), then by Theorem 2.2 and Theorem 2.4 it is linear and continuous. Conversely, let T be a linear and continuous operator on $\mathbf{P}$. Since $\delta_a^{(n)} \in \mathbf{D}_*$ ($\mathbf{D}_*$ is a linear space) for every $n = 0, 1, 2, \ldots$ and for every real number a, then by Theorem 3.4 the operator T has the form (3.3) for every $x \in \mathbf{D}_*$. If $x \in \mathbf{P}$ then according to Lemma 1.2 there is a sequence $x_n \in \mathbf{D}_*$, $n = 1, 2, \ldots$ such that $x_n \to x$. From (3.1) we have $\int_{-\infty}^{+\infty} (x_n, f_a)\, da = T[x_n] \to T[x]$. Finally from Theorem 2.4 it follows that the formula (3.3) holds for every $x \in \mathbf{P}$. This completes the proof.

Corollary. *If T_1, T_2 are two linear and continuous operators on $\mathbf{D}$ and $T_1[x] = T_2[x]$ for every $x \in \mathbf{D}_*$, then $T_1[x] = T_2[x]$ for every $x \in \mathbf{D}$.*

Example. Let $\alpha(t)$ be a real function having all derivatives in $(-\infty, +\infty)$; then

$$\alpha x = \int_{-\infty}^{+\infty} (x, \alpha(a)\, \delta_a)\, da$$

for every $x \in \mathbf{D}$, because $\alpha(t)\, \delta_a = \alpha(a)\, \delta_a$.

Note. Let $f \in \mathbf{D}$. If f vanishes outside some finite interval, then the operator

$$T[x] = x * f \quad (x \in \mathbf{D})$$

is linear and continuous (see p. 137 in [5]). From Theorem 3.5 it follows that

$$T[x] = \int_{-\infty}^{+\infty} (x, f_a)\, da\,, \quad (x \in \mathbf{D})\,,$$

where $\{f_a\} \in \mathscr{F}$ and $f_a = T[\delta_a] = \delta_a * f = P_a[f]$ for every real number a.

Definition 3.3. A non-empty subset $\mathbf{P} \subset \mathbf{D}$ will be called *time-invariant*, if the following implication holds for every real number b:

$$\textit{If } x \in \mathbf{P}\,, \textit{ then } P_b[x] \in \mathbf{P}\,.$$

Let $\mathbf{P}$ be a time-invariant subset of $\mathbf{D}$. An operator T on $\mathbf{P}$ will be called *time-invariant* if the condition

$$T[P_b[x]] = P_b[T[x]] \tag{3.4}$$

holds for every real number b and for every $x \in \mathbf{P}$.

Example. If we put

$$T_1[x] = x' + x = \int_{-\infty}^{+\infty} (x, \delta'_a + \delta_a)\, da\,, \quad (x \in \mathbf{D})$$

then clearly the operator T_1 is time-invariant. On the other hand, it is obvious that the operator T_2 given by the formula

$$T_2[x] = tx = \int_{-\infty}^{+\infty} (x, a\delta_a)\, da \quad (x \in \mathbf{D})$$

is not time-invariant.

Theorem 3.6. *Let* $\mathbf{P} \subset \mathbf{D}$ *be a time-invariant linear subspace and let* $\{f_a\} \in \mathcal{F}$.

1. *If* $f_a = P_a[f]$ *for some* $f \in \mathbf{D}$, *then the operator* T *on* $\mathbf{P}$ *given by* (3.3) *is time-invariant.*

2. *If the operator* T *on* $\mathbf{P}$ *given by* (3.3) *is time-invariant and* $\delta \in \mathbf{P}$, *then* $f_a = P_a[f]$, *where* $f = T[\delta]$.

Proof. 1. If $f_a = P_a[f]$ for some $f \in \mathbf{D}$, then $f_{a+b} = P_b[f_a]$ for every real number b. From Theorem 2.6, Theorem 2.7 and (3.3) it follows that $T[P_b[x]] = \int_{-\infty}^{+\infty} (P_b[x], f_a) \,.\, da = \int_{-\infty}^{+\infty} (x, f_{a+b})\, da = \int_{-\infty}^{+\infty} (x, P_b[f_a])\, da = P_b[T[x]]$ for every $x \in \mathbf{P}$. Thus, by Definition 3.3, the operator T is time-invariant.

2. If the operator T on $\mathbf{P}$ given by (3.3) is time-invariant and $\delta \in \mathbf{P}$, then from (3.4) it follows that $T[\delta_a] = T[P_a[\delta]] = P_a[T[\delta]] = P_a[f]$, where $f = T[\delta]$. Hence, Theorem 3.6 is proved.

Definition 3.4. Let $\mathbf{P}$ be a non-empty subset of $\mathbf{D}$. An operator T on $\mathbf{P}$ will be called *causal*, if for every pair of distributions $x, y \in \mathbf{P}$, the following implication holds:

(3.5) *If* $x = y$ *on the interval* $(-\infty, b)$, *then* $T[x] = T[y]$ *on the same interval.*

Example. If we put

$$T_1[x] = x'' = \int_{-\infty}^{+\infty} (x, \delta''_a)\, da \quad (x \in \mathbf{D}),$$

then the operator T_1 is causal. On the other hand, the operator T_2, given by

$$T_2[x] = x + P_{-1}[x] = \int_{-\infty}^{+\infty} (x, \delta_a + \delta_{a-1})\, da \quad (x \in \mathbf{D}),$$

is evidently not causal.

Lemma 3.1. *Let* $\mathbf{P} \subset \mathbf{D}$ *be a linear subspace. A linear operator* T *on* $\mathbf{P}$ *is causal if and only if the following implication holds*:

If $x \in \mathbf{P}$ *vanishes on the interval* $(-\infty, b)$, *then* $T[x]$ *vanishes on the same interval.*

Proof. The necessity is obvious. In order to prove the sufficiency, observe that $x = y$ on the interval $(-\infty, b)$ with $x, y \in \mathbf{P}$ implies that $x - y$ vanishes on the interval $(-\infty, b)$. By supposition $T[x - y]$ vanishes on the interval $(-\infty, b)$. From (3.2) it follows that $T[x] = T[y]$ on the interval $(-\infty, b)$. Hence according to (3.5) the operator T is causal.

Theorem 3.7. *Let* $\mathbf{P} \subset \mathbf{D}$ *be a linear subspace and let* $\{f_a\} \in \mathscr{F}$.

1. *If* f_a *vanishes on* $(-\infty, a)$ *for every real number* a, *then the operator* T *on* $\mathbf{P}$ *given by* (3.3) *is causal.*

2. *If the operator* T *on* $\mathbf{P}$ *given by* (3.3) *is causal and* $\delta_a \in \mathbf{P}$ *for every real number* a, *then* f_a *vanishes on* $(-\infty, a)$.

Proof follows from Theorem 2.8 and Lemma 3.1.

Theorem 3.8. *Let* $\mathbf{P} \subset \mathbf{D}$ *be a time-invariant linear subspace and let* $\{f_a\} \in \mathscr{F}$.

1. *If* $f_a = P_a[f]$ *for some* $f \in \mathbf{D}$, *where* f *vanishes on* $(-\infty, 0)$, *then the operator* T *on* $\mathbf{P}$ *given by* (3.3) *is time-invariant and causal.*

2. *If the operator* T *on* $\mathbf{P}$ *given by* (3.3) *is time-invariant and causal, and if* $\delta \in \mathbf{P}$, *then* $f_a = P_a[f]$, *where* f *vanishes on* $(-\infty, 0)$ *and* $f = T[\delta]$.

Proof follows from Theorem 3.7 and Theorem 3.6.

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