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SOME MAXIMUM PRINCIPLES FOR STOCHASTIC EQUATIONS

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In this article we consider diffusion processes $\mathcal{M}$ in a region D which are governed by Ito stochastic equations. The probability that the first exit time of $\mathcal{M}$ from the region D is less than a given number T is important in many problems. Denote by $P(\mathcal{M})$ this probability. We deal with classes of diffusion processes $\mathcal{M}$ for which the drift $a(t, x)$, i.e. the "non-stochastic part" of equation $(1, 1)$ is a given vector function, but the matrix of the local diffusion $B(t, x)$, i.e. the "stochastic part" of $(1, 1)$ can vary in a class. The question arises how the probability $P(\mathcal{M})$ depends on the matrix of the local diffusion $B(t, x)$. A condition will be given which guarantees that the maximum probability $P(\mathcal{M})$ occurs in case of the "greatest" matrix of the local diffusion $B(t, x)$. The exact formulation of this problem is given in Definitions 5 and 6. It is shown that the dependence: "The greater is $B(t, x)$, the greater is $P(\mathcal{M})$" is generally not valid, even for very simple regions. This article is the continuation of [1], [2], but the formulations are different. The greatest difference, however, is that in [1], [2] only the one-dimensional case is considered.

In example 3 the case is considered that D is a circle and $a(t, x) \equiv 0$. The matrix of the local diffusion is found which gives the maximum of $P(\mathcal{M})$ in this case. Nevertheless the construction of the matrix shows that also in this case the strategy of the greatest possible B is not the best one.

Our problem may be reformulated and we could consider it as an optimal control problem [10], but due to the assumptions used there we cannot apply the results of [10] on our problem.

1. Basic definitions and notation. Let R_n denote the n-dimensional Euclidean space with a norm$|\bullet|$. Denote by D a region in R_n and by Q a region in R_{n+1} of the type $Q = (0, T) \times D$ where T is a positive number. Denote by $\bar{D}$ the closure of D and by $\dot{D}$ the boundary of D and by S the set $\langle 0, T\rangle \times \dot{D}$. Let Ω be a space, $\mathcal{F}$ a σ-field of subsets of Ω and P a probability measure defined on $\mathcal{F}$.

Random variables z and random processes $z(t)$ may be considered as $\mathcal{F}$-measurable functions $z(\omega)$, $z(t, \omega)$ on Ω, respectively. We suppose that the structure of Ω, $\mathcal{F}$, P

enables us to express every random variable and every random process as an $\mathcal{F}$-measurable function on Ω.

We shall consider Ito stochastic equation

$$\text{(1,1)} \qquad \mathrm{d}x = a(t, x)\,\mathrm{d}t + B(t, x)\,\mathrm{d}w(t)$$

where $a(t, x)$ is a vector function $a(t, x) \in R_n$, $B(t, x)$ is an $n \times n$ matrix function and $w(t)$ is an n-dimensional Wiener process, i.e. $w(t) = [w_1(t), \ldots, w_n(t)]$ where $w_i(t)$ are independent Wiener processes with $Ew_i(t) = 0$, $Ew_i^2(t) = t$. The letter E denotes the mathematical expectation.

We call a function $f(x)$ a density in D if $f(x)$ is Lebesgue-measurable, nonnegative and $\int_D f(x)\,\mathrm{d}x = 1$. A random variable $x(\omega)$ has a density $f(x)$ if $P\{\omega : x(\omega) \in A\} = \int_A f(x)\,\mathrm{d}x$ for every n-dimensional Borel set A, $A \subset D$. Denote by $x_f(t)\}$ the solution of (1,1) with the initial density $f(x)$, i.e., $x_f(0)$ has density $f(x)$. We assume that (1,1) fulfils some condition guaranteeing that the unique solution exists for every $f(x)$.

Definition 1. Denote by $P(B, a, f, Q)$ the probability that $x_f(t)$ leaves the region D at least once on the interval $\langle 0, T\rangle$, $(Q = (0, T) \times D)$ i.e.

$$\text{(1,2)} \qquad P(B, a, f, Q) = P\{\omega : \exists\{\tau : x_f(\tau, \omega) \notin D,\ \tau \in \langle 0, T\rangle\}\}\,.$$

If the initial value of a solution of (1,1) is nonstochastic $x(0) = x_0 \in D$, then we shall write $\delta(x_0)$ instead of f in (1,2).

Definition 2. A region D fulfils condition (B) if it is bounded and if to every point $\bar{x} \in \dot{D}$ there exists a ball K with the centre at $\bar{x}$ and a system of orthogonal coordinates $y_1, \ldots, y_n$ where y_n has the direction of the outward normal to $\dot{D}$ with respect to D at the point $\bar{x}$ such that the frontier $\dot{D}$ can be expressed in the ball K as a function $y_n = h(y_1, \ldots, y_{n-1})$ for $[y_1, \ldots, y_{n-1}] \in K^+ \subset K^*$ with Hölder continuous second derivatives. The set K^* is defined by $K^* = \{[y_1, \ldots, y_{n-1}] : [y_1, \ldots, y_{n-1}, 0] \in K\}$ and K^+ is an open subset of K^* containing the origin of $y_1 \ldots y_{n-1}$-coordinate system.

We shall pass to the definition of the concept of the solution of (1,1) with an adhesive barrier. Let $a(t, x)$ be a vector function and $B(t, x)$ be an $n \times n$ matrix function which are defined for all $t \geqq 0$, $x \in R_n$ and which are continuous in t and Lipschitz continuous in x. Let D be a region in R_n. By Theorem 4 [5] (similar results are in [9]) there exists a solution $x^*(t, \omega)$ of

$$\text{(1,3)} \qquad x^*(t) = x_0 + \int_0^t a(\tau, x^*(\tau))\,\mathrm{d}\tau + \int_0^t B(\tau, x^*(\tau))\,\mathrm{d}w(\tau)\,,$$

where x_0 is a random variable with the given density $f(x)$ which is independent of $w(t)$. The process $x^*(t, \omega)$ is a solution of (1,3) in the sense that

$$\left|\left|\left| x^*(t) - x_0 - \int_0^t a(\tau, x^*(\tau))\, d\tau - \int_0^t B(\tau, x^*(\tau))\, dw(\tau) \right|\right|\right|_T = 0 .$$

The norm $|||z|||_T$ is defined by $|||z|||_T = \sqrt{E \sup_{\tau \in \langle 0, T \rangle} |z(\tau, \omega)|^2}$.

Definition 3. Let $\tau(\omega)$ be the first exit time of $x^*(t)$ from D. The process $x(t, \omega) = x^*(t, \omega)$ for $0 \leqq t < \tau(\omega)$ is called the part of x^* in D. The process $x_\tau(t, \omega) = x(t, \omega)$ for $t < \tau(\omega)$, $x_\tau(t, \omega) = x^*(\tau(\omega), \omega)$ for $t \geqq \tau(\omega)$ is called the solution of (1,3) with the adhesive barrier $\dot{D}$.

The last statement of 11.14 Chap. 11 [6] yields (similarly as in the conclusion of 11.13 Chap 11) that the processes $x(t, \omega)$ and $x_\tau(t, \omega)$ are independent of the values of $a(t, x)$ and $B(t, x)$ outside of D. We may suppose that $a(t, x)$, $B(t, x)$ are defined in Q only. In this case we extend first the domain of definition of $a(t, x)$, $B(t, x)$ on the whole $\langle 0, T \rangle \times R_n$ and then define $x(t, \omega)$ and $x_\tau(t, \omega)$ as the part of the solution or the solution of (1,3) with the adhesive barrier $\dot{D}$, respectively. Obviously, the probability $P(B, a, f, Q)$ does not depend on the extension of $a(t, x)$, $B(t, x)$ outside of Q.

Definition 4. A vector function $a(t, x)$ and a matrix function $B(t, x)$ defined on $\bar{Q}$ fulfil condition (A) if it holds:

i) $B(t, x)\, B^T(t, x)$ is positive definite in $\bar{Q}$ (B^T is the transpose matrix).

ii) $a(t, x)$, $B(t, x)\, B^T(t, x)$ are Hölder continuous, i.e.

$$|a(t_2, x_2) - a(t_1, x_1)| \leqq M(|x_2 - x_1|^\alpha + |t_2 - t_1|^{\alpha/2})$$

and similarly for BB^T where M is a positive constant and α is a number $0 < \alpha < 1$.

iii) Ito equation (1,1) has the unique solution with the adhesive barrier $\dot{D}$ for every initial density $f(x)$ in D, i.e., there is the unique solution $x_f(t)$ of (1,1) with the adhesive barrier $\dot{D}$ such that $x_f(0)$ has the density $f(x)$.

iv) The parabolic equation

$$(1,4) \qquad \frac{\partial u}{\partial t} = \tfrac{1}{2} \sum_{i,k,j} B_{ik}(T - t, x)\, B_{jk}(T - t, x) \frac{\partial^2 u}{\partial x_i\, \partial x_j} + \sum_i a_i(T - t, x) \frac{\partial u}{\partial x_i}$$

has the unique bounded solution (unique in the class of bounded functions) fulfilling

$$(1,5) \qquad \lim_{t \to 0+} u(t, x) = 0 \quad \text{for} \quad x \in D$$

$$(1,6) \qquad \lim_{x \to \bar{x}} u(t, x) = 1 \quad \text{for} \quad t > 0 , \quad \bar{x} \in \dot{D} .$$

v) The bounded solution $u(t, x)$ fulfils

$$P(B, a, f, Q) = \int_D f(x)\, u(T, x)\, \mathrm{d}x \tag{1,7}$$

for every density $f(x)$ in D and also for $f(x) = \delta(x)$, $x \in D$: $P(B, a, \delta(x), Q) = u(T, x)$.

Item iii) is fulfilled if $a(t, x)$, $B(t, x)$ are Lipschitz continuous in x and continuous in t [9]. Sufficient conditions for iv) and v) are given in Lemmas 4 and 5.

In the following Definitions the problem will be formulated precisely which was only sketched in the introduction.

Definition 5. A matrix function $B(t, x)$ is called maximal with respect to a vector function $a(t, x)$ and to a region Q ($Q = (0, T) \times D$) if

i) $B(t, x)\, B^T(t, x)$ is a diagonal matrix in Q,

ii) $a(t, x)$, $B(t, x)$ fulfil condition (A),

iii) the region D fulfils condition (B),

iv) $P(B, a, f, Q) = \max P(B', a, f, Q)$ for all densities f in D where the maximum is taken over the set of matrix functions $B'(t, x)$ fulfilling conditions i), ii) and $A'_{ii}(t, x) \leqq A_{ii}(t, x)$ where $A'_{ii}(t, x)$, $A_{ii}(t, x)$ are the diagonal elements of the matrix $A'(t, x) = B'(t, x)\, B'^T(t, x)$, $A(t, x) = B(t, x)\, B^T(t, x)$, respectively.

Definition 6. A matrix function $B(t, x)$ is called strongly maximal with respect to a vector function $a(t, x)$ and to a region Q if conditions ii), iii) from Definition 5 are fulfilled and if

iv*) $P(B, a, f, Q) = \max P(B', a, f, Q)$ for all densities f in D where the maximum is taken over the set of matrix functions $B'(t, x)$ fulfilling condition ii) from Definition 5 and $A(t, x) - A'(t, x)$ is a positive semi-definite matrix for every $[t, x] \in Q$ ($A(t, x)$ and $A'(t, x)$ are defined in the same manner as in Definition 5).

If no confusion about $a(t, x)$ and Q may arise, we shall use only the terms "maximal matrix function" or "strongly maximal matrix function", respectively.

In some considerations we shall need more general regions than cylindric regions Q. Let C_0^0, C_T^0 be regions in R_n. Denote $C_0 = \{[0, x] : x \in C_0^0\}$, $C_T = \{[T, x] : x \in C_T^0\}$. We say that a region C in R_{n+1} is regular if the frontier $\dot{C}$ consists of the sets C_0, C_T and of a surface S which is situated in the strip $\langle 0, T \rangle \times R_n$ and has the outside strong sphere property for every $[t, x] \in S$ (for the definition of the outside strong property, see [3]). We can modify the definition of $P(B, a, f, Q)$ and also those of the maximal and strongly maximal matrix function for regular regions (cf. Remark 6).

2. We shall need the following statement about comparison of solutions of parabolic equation.

Lemma 1. *Let C be a regular region in R_{n+1}, let $a_i(t, x)$, $A_{ij}(t, x)$ be defined and Hölder continuous in $\bar{C}$ and let the matrix function $A(t, x)$ be positive definite in $\bar{C}$. Assume that $\varphi_i(t, x)$ are continuous functions defined on $C_0 \cup S$ fulfilling $\varphi_1(t, x) \geqq \geqq \varphi_2(t, x)$. If $v(t, x)$, $u(t, x)$ and the derivatives*

$$\frac{\partial v}{\partial t}, \quad \frac{\partial v}{\partial x_i}, \quad \frac{\partial^2 v}{\partial x_i, \partial x_j}, \quad \frac{\partial u}{\partial t}, \quad \frac{\partial u}{\partial x_i}, \quad \frac{\partial^2 u}{\partial x_i \partial x_j}$$

are continuous in C and

$$\frac{\partial v}{\partial t} \geqq \sum_{i,j} A_{ij}(t, x) \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_i a_i(t, x) \frac{\partial v}{\partial x_i},$$

$v(t, x) = \varphi_1(t, x)$ on $C_0 \cup S$,

$$\frac{\partial u}{\partial t} = \sum_{i,j} A_{ij}(t, x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i a_i(t, x) \frac{\partial u}{\partial x_i},$$

$u(t, x) = \varphi_2(t, x)$ on $C_0 \cup S$, then $v(t, x) \geqq u(t, x)$.

This lemma is an easy consequence of Theorem 16 Chap. II [3].

3. We need some approximation of bounded solutions of (1,4) fulfilling (1,5) and (1,6), by smooth solutions of (1,4).

Lemma 2. *Let conditions (A), (B) be fulfilled. If $u(t, x)$ is a bounded solution of (1,4) fulfilling (1,5) and (1,6), then $u(t, x) = \lim u_m(t, x)$ holds where $u_m(t, x)$ are solutions of (1,4) fulfilling*

$$\lim_{t \to 0+} u_m(t, x) = 0 \quad \text{for} \quad x \in D$$

and $u_m(t, \bar{x}) = 0$ for $0 \leqq t \leqq 2^{-m}$, $u_m(t, \bar{x}) = 2^m(t - 2^{-m})$ for $2^{-m} < t \leqq 2^{-m+1}$, $u_m(t, \bar{x}) = 1$ for $t > 2^{-m+1}$, $\bar{x} \in \dot{D}$. Such solution $u(t, x)$ always exists.

Proof. Denote by $\mathcal{O}(\varrho, \dot{D})$ the ϱ-neighbourhood of $\dot{D}$, i.e. $\mathcal{O}(\varrho, \dot{D}) = \{z : \exists \{x : : |x - z| < \varrho, x \in \dot{D}\}\}$. Let $\varphi_m(x)$ be a sequence of functions such that $\varphi_m(x) = 1$ for $x \in \mathcal{O}(2^{-m}, \dot{D})$, $\varphi_m(x) = 0$ for $x \notin \mathcal{O}(2^{-m+1}, D)$, $\varphi_m(x)$ being defined and continuous in $\bar{D}$, $0 \leqq \varphi_m(x) \leqq 1$ ($\bar{D}$ is the closure of D).

Let $u_{-m}(t, x)$ be the solution of (1,4) fulfilling (1,6) and $u_{-m}(0, x) = \varphi_m(x)$ for $x \in \bar{D}$.

Obviously

$$(3,1) \qquad 0 \leqq \ldots \leqq u_m(t, x) \leqq u_{m+1}(t, x) \leqq \ldots \leqq u_{-m-1}(t, x) \leqq u_{-m}(t, x) \leqq \ldots \leqq 1.$$

Put $u^*(t, x) = \lim_{m\to\infty} u_{-m}(t, x)$, $u^{**}(t, x) = \lim_{m\to\infty} u_m(t, x)$. By (3,1) we have

$$0 \leqq u^{**}(t, x) \leqq u^*(t, x) \leqq 1 . \tag{3,2}$$

Since $0 \leqq \lim_{t\to 0^+} u^{**}(t, x) \leqq \lim_{t\to 0^+} u^*(t, x) \leqq \lim_{t\to 0^+} u_{-m}(t, x) = 0$ for $x \in D$, $x \notin \mathcal{O}(2^{-m+1}, \dot{D})$, the functions u^*, u^{**} fulfil (1,5). Since $1 = \lim_{x\to\bar{x}} u_m(t, x) \leqq \lim_{x\to\bar{x}} u^{**}(t, x) \leqq \lim_{x\to\bar{x}} u^*(t, x) \leqq$ $\leqq 1$ for $t > 2^{-m+1}$, $\bar{x} \in \dot{D}$, the functions u^*, u^{**} fulfil (1,6).

We shall prove that u^*, u^{**} are solutions of (1,4). Choose a point $[t, x]$, $t > 0$, $x \in D$. Let $0 < \bar{t} < t$. Let $G^0(t, x; \tau, \xi)$ be the Green function of (1,4) for Q (Theorem 16 Chap. III [3]) then

$$u_m(t, x) = 1 + \int_D G^0(t, x; \bar{t}, \xi)\,(u_m(\bar{t}, \xi) - 1)\,\mathrm{d}\xi .$$

For $m \to \infty$ we obtain

$$u^{**}(t, x) = 1 + \int_D G^0(t, x; \bar{t}, \xi)\,(u^{**}(t, \xi) - 1)\,\mathrm{d}\xi .$$

The last equality proves that u^{**} and its derivatives

$$\frac{\partial u^{**}}{\partial t}, \quad \frac{\partial u^{**}}{\partial x_i}, \quad \frac{\partial^2 u^{**}}{\partial x_i\,\partial x_j}$$

are continuous in $(\bar{t}, T) \times D$ and that u^{**} satisfies (1,4). The same holds for u^*. According to (3,2) these solutions are bounded and with respect to condition (A) iv) $u^*(t, x) = u^{**}(t, x)$.

4. Now all is prepared to formulate one of the main results.

Theorem 1. *Let a vector function $a(t, x)$, a matrix function $B(t, x)$ and a region D be given such that D fulfils condition* (B), *$a(t, x)$, $B(t, x)$ fulfil condition* (A) *and the matrix $A(t, x) = B(t, x)\,B^T(t, x)$ is diagonal. The matrix function $B(t,x)$ is maximal with respect to the vector function $a(t, x)$ and to the region $Q = (0, T) \times D$ if and only if the bounded solution of*

$$\frac{\partial u}{\partial t} = \tfrac{1}{2} \sum_i A_{ii}(T - t, x)\,\frac{\partial^2 u}{\partial x_i^2} + \sum_i a_i(T - t, x)\,\frac{\partial u}{\partial x_i} \tag{4,1}$$

fulfilling

$$\lim_{t\to 0+} u(t, x) = 0 \quad \textit{for} \quad x \in D , \tag{4,2}$$

$$\lim_{x\to\bar{x}} u(t, x) = 1 \quad \textit{for} \quad t > 0 , \quad \bar{x} \in \dot{D} \tag{4,3}$$

has nonnegative partial derivatives $\partial^2 u/\partial x_i^2 \geqq 0$, $i = 1, \ldots, n$ in Q.

Remark 1. According to condition (A) the bounded solution of (4,1) fulfilling (4,2) and (4,3) exists and is unique.

Proof. With respect to condition (A) (Definition 4 v)) the solution $u(T, x)$ is the probability that the solution of Ito stochastic equation (1,1) with a nonstochastic initial value $x \in D$ crosses the barrier $\dot{D}$ at least once in the interval $\langle 0, T\rangle$. Consider another Ito equation

$$\mathrm{d}x = a(t, x)\,\mathrm{d}t + B'(t, x)\,\mathrm{d}w(t)\,. \tag{4,4}$$

The above mentioned probability is now $v(T, x)$, v being the bounded solution of

$$\frac{\partial v}{\partial t} = \tfrac{1}{2}\sum_i A'_{ii}(T - t, x)\frac{\partial^2 v}{\partial x_i^2} + \sum_i a_i(T - t, x)\frac{\partial v}{\partial x_i} \tag{4,5}$$

fulfilling conditions (4,2), (4,3). We assume still as above that

$$A'(t, x) = B'(t, x)\,B'^T(t, x)$$

is a diagonal matrix. Suppose $\partial^2 u/\partial x_i^2 \geqq 0$; since $A_{ii}(t, x) \geqq A'_{ii}(t, x)$ we obtain

$$\begin{aligned}\frac{\partial u}{\partial t} &= \tfrac{1}{2}\sum_i A_{ii}(T - t, x)\frac{\partial^2 u}{\partial x_i^2} + \sum_i a_i(T - t, x)\frac{\partial u}{\partial x_i} \geqq \\ &\geqq \tfrac{1}{2}\sum_i A'_{ii}(T - t, x)\frac{\partial^2 u}{\partial x_i^2} + \sum_i a_i(T - t, x)\frac{\partial u}{\partial x_i}\,.\end{aligned} \tag{4,6}$$

Lemma 2 implies that the solution $v(t, x)$ is the limit of solutions $v_m(t, x)$ of (4,5) fulfilling $v_m(0, x) = 0$ for $x \in D$, $v_m(t, \bar{x}) = 0$ for $0 \leqq t \leqq 2^{-m}$, $v_m(t, \bar{x}) = 2^m(t - 2^{-m})$ for $2^{-m} < t \leqq 2^{-m+1}$, $v_m(t, \bar{x}) = 1$ for $t > 2^{-m+1}$ $\bar{x} \in \dot{D}$. It means $v_m(2^{-m}, x) = 0$ for $x \in D$. Obviously the solution $u(t, x)$ fulfils $u(2^{-m}, x) \geqq 0$ for $x \in D$, $u(t, \bar{x}) = 1$ for $t \geqq 2^{-m}$, $\bar{x} \in \dot{D}$. Using Lemma 1 in $(2^{-m}, T) \times D$ we get $u(t, x) \geqq v_m(t, x)$. As $v(t, x)$ is the limit of v_m, we have proved $u(t, x) \geqq v(t, x)$. By condition (A) (Definition 4 v)) we have also

$$P(B, a, f, Q) = \int f(x)\,u(T, x)\,\mathrm{d}x \geqq P(B', a, f, Q) = \int f(x)\,v(T, x)\,\mathrm{d}x\,.$$

This prove the first part of the Theorem.

Let the function $u(t, x)$ not fulfil $\partial^2 u/\partial x_i^2 \geqq 0$ in Q, $i = 1, \ldots, n$. Put $K_i = \{[t, x] : \partial^2 u/\partial x_i^2 < 0\}$. The sets K_i are open, $K_i \subset Q$, $\bigcup K_i$ is nonempty. Define $A'(t, x)$ so that the assumptions of the Theorem are fulfilled and that $A'_{ii}(T - t, x) = A_{ii}(T - t, x)$ for $[t, x] \notin K_i$, $0 < A'_{ii}(T - t, x) < A_{ii}(T - t, x)$ in K_i. We compare equations (4,1) and(4,5) once more

$$\frac{\partial u}{\partial t} = \tfrac{1}{2} \sum_i A_{ii}(T - t, x) \frac{\partial^2 u}{\partial x_i^2} + \sum_i a_i(T - t, x) \frac{\partial u}{\partial x_i} \leqq \tag{4,7}$$

$$\leqq \tfrac{1}{2} \sum_i A'_{ii}(T - t, x) \frac{\partial^2 u}{\partial x_i^2} + \sum_i a_i(T - t, x) \frac{\partial u}{\partial x_i}\,.$$

In the same manner as above, we obtain $u(t, x) \leqq v(t, x)$ now. It cannot be $u(t, x) \equiv \equiv v(t, x)$ since $\partial u/\partial t < \partial v/\partial t$ would hold on $\bigcup K_i$.

Let $[t_0, x_0]$ be such point that $u(t_0, x_0) < v(t_0, x_0)$, $0 < t_0 < T$, $x_0 \in D$. Let $y(t, x)$ be the solution of (4,5) fulfilling the initial condition $y(t_0, x) = u(t_0, x)$ for $x \in D$ and $y(t, \bar{x}) = 1$ for $t \geqq t_0$, $\bar{x} \in \dot{D}$. By (4,7) and by Lemma 1 we obtain $u(t, x) \leqq y(t, x)$ for $t \geqq t_0$, $x \in D$. Put $\Delta(t, x) = v(t, x) - y(t, x)$. The function $\Delta(t, x)$ is the solution of (4,5) with $\Delta(t_0, x) \geqq 0$ for $x \in D$, $\Delta(t_0, x_0) > 0$, $\Delta(t, \bar{x}) = 0$ for $t \geqq t_0$, $\bar{x} \in \dot{D}$. Obviously $\Delta(t, x) \geqq 0$ and applying Theorem 5 §2 Chap. II [3] we get $\Delta(t, x) > 0$ for $t > t_0$, $x \in D$, i.e. $u(t, x) < v(t, x)$ for $t > t_0$, $x \in D$. It means that for every density f in D, $P(B, a, f, Q) < P(B', a, f, Q)$ holds.

Remark 2. We have proved some stronger result. If the function $u(t, x)$ does not fulfil $\partial^2 u/\partial x_i^2 \geqq 0$ $i = 1, \ldots, n$ in Q ($u(t, x)$ is determined in Theorem 1) then $B(t, x)$ is not maximal and there exists a matrix function $B'(t, x)$ fulfilling all conditions of Definition 5 and such that $P(B, a, f, Q) < P(B', a, f, Q)$ for all densities $f(x)$ in D.

On the other hand if all assumptions about $u(t, x)$ are fulfilled, then the region D must have the following property: The intersection of D and of a straight line which is parallel to one axis is either empty or an interval.

5. Before formulating the second main result we shall introduce a statement about convex functions. Let $f(x)$ be a function defined in a region D, $D \subset R_n$ such that $f(x)$ has continuous second derivatives. The function $f(x)$ is convex if and only if $\mathrm{d}^2 f/\mathrm{d}l^2(x) \geqq 0$ in D for all vectors $l \neq 0$, where $\mathrm{d}^2 f/\mathrm{d}l^2$ means the second derivative in the direction of the vector l. This condition is equivalent to the following one: The function $f(x)$ is convex if and only if the matrix $\mathrm{d}^2 f/\mathrm{d}x^2$ consisting of the elements $(\partial^2 f/\partial x_i\, \partial x_j)(x)$ is positive semi-definite for all $x \in D$. We shall prove a Lemma which we shall use in the proof of Theorem 2.

Lemma 3. *Let B be a square matrix. The inequality*

$$\sum_{i,j} A_{ij} B_{ij} \geqq 0 \tag{5,1}$$

holds for all symmetric positive semi-definite matrices A if and only if the matrix B is positive semi-definite.

Proof. Let B be a positive semi-definite matrix. We assume that A is a symmetric positive semi-definite matrix. There exist real characteristic values λ_i, $\lambda_i \geqq 0$ and an orthonormal basis consisting of eigenvectors z_i of A. The expression $\sum A_{ij}B_{ij}$ can be written as

$$\sum_{i,j} A_{ij}B_{ij} = \sum_k (Ae_k, Be_k) \tag{5,2}$$

where e_k is the k-th column of the unit matrix J. Since the expression $\sum_k (Ae_k, Be_k)$ is independent of the orthonormal basis we obtain by (5,2) $\sum_{i,j} A_{ij}B_{ij} = \sum_k (Ae_k, Be_k) = = \sum_k (Az_k, Bz_k) = \sum_k \lambda_k(z_k, Bz_k) \geqq 0$, which proves (5,1).

Conversely let (5,1) be fulfilled for all symmetric positive semi-definite matrices A. To prove that B is also a positive semi-definite matrix it is sufficient to choose $A_{ij} = = l_i l_j$ where l_i, $i = 1, \ldots, n$ are arbitrary real numbers. Lemma 5 is proved.

6. Theorem 2. *Let a vector function $a(t, x)$, a matrix function $B(t, x)$ and a region D be given such that D fulfils condition* (B) *and $a(t, x)$, $B(t, x)$ fulfil condition* (A). *The matrix function $B(t, x)$ is strongly maximal with respect to the vector function $a(t, x)$ and to the region $Q = (0, T) \times D$ if and only if the bounded solution of*

$$\frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i,j} A_{ij}(T - t, x) \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \sum_i a_i(T - t, x) \frac{\partial u}{\partial x_i}, \tag{6,1}$$

$$A(t, x) = B(t, x)\, B^T(t, x)$$

fulfilling

$$\lim_{t \to 0+} u(t, x) = 0 \quad \textit{for} \quad x \in D, \tag{6,2}$$

$$\lim_{x \to \bar{x}} u(t, x) = 1 \quad \textit{for} \quad t > 0, \quad \bar{x} \in \dot{D} \tag{6,3}$$

is convex with respect to x in Q.

Remark 3. Since $A(t, x) = B(t, x)\, B^T(t, x)$ the problem (6,1) to (6,3) is the same as (1,4) to (1,6).

Proof. In the same way as in the proof of Theorem 1 to Ito equation (1,1) there corresponds the parabolic equation (6,1) and to Ito equation (4,4) there corresponds now the parabolic equation

$$\frac{\partial v}{\partial t} = \frac{1}{2} \sum_{i,j} A'_{i,j}(T - t, x) \frac{\partial^2 v}{\partial x_i \, \partial x_j} + \sum_i a_i(T - t, x) \frac{\partial v}{\partial x_i}. \tag{6,4}$$

The bounded solutions of (6,1) and (6,4) fulfilling (6,2) and (6,3) have the same meaning as in the proof of Theorem 1. Since $u(t, x)$ is convex, the matrix $d^2u/dx^2(t, x)$ is positive semi-definite. As $A(t, x) - A'(t, x)$ are symmetric positive semi-definite matrices for all t, x, hence according to Lemma 3

$$\sum_{i,j}(A_{ij}(T - t, x) - A'_{ij}(T - t, x)) \frac{\partial^2 u}{\partial x_i \partial x_j}(t, x) \geqq 0$$

so that

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sum_{i,j} A_{ij}(T - t, x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i a_i(T - t, x) \frac{\partial u}{\partial x_i} \geqq \tag{6,5}$$

$$\geqq \frac{1}{2}\sum_{i,j} A'_{ij}(T - t, x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i a_i(T - t, x) \frac{\partial u}{\partial x_i}.$$

In the same manner as in the proof of Theorem 1, we get $u(t, x) \geqq v(t, x)$ and this implies $P(B, a, f, Q) \geqq \sup P(B', a, f, Q)$ which proves that the matrix function is strongly maximal.

Let $u(t, x)$ be not convex in Q. Hence there exists a point $[t, x]$ and a vector l, $l \neq 0$ such that $d^2u/dl^2\ (t, x) < 0$. Put $l_1 = l/|l|$ and let vectors $l_2, \ldots, l_n$ together with l_1 form an orthonormal basis. Let L be the matrix with columns $l_1, \ldots, l_n$. Using the transformation $x = Ly$, $u(t, x) = u^*(t, y)$, we obtain the equation

$$\frac{\partial u^*}{\partial t} = \frac{1}{2}\sum_{i,j} A^*_{ij}(T - t, y) \frac{\partial^2 u^*}{\partial y_i \partial y_j} + \sum_i a^*_i(T - t, y) \frac{\partial u^*}{\partial y_i}. \tag{6,6}$$

The region D is transformed onto a region D^* and conditions (6,2), (6,3) are modified, too. Denote by K the set $K = \{[t, x] : \partial^2 u^*/\partial y_1^2 < 0\}$. We construct the matrix function: $A^{*\prime}_{ij}(T - t, y) = A^*_{ij}(T - t, y)$ for all i, j except $i = j = 1$ and for all $[t, y] \in Q^*$. Moreover, $0 < A^{*\prime}_{11}(T - t, y) < A^*_{11}(T - t, y)$ for $[t, y] \in K$,

$$A^{*\prime}_{11}(T - t, y) = A^*_{11}(T - t, y) \quad \text{outside of } K.$$

Since the matrix A^* is symmetric and positive definite, obviously the matrix $A^{*\prime}$ is symmetric and can be constructed as positive definite (if $A^*_{11} - A^{*\prime}_{11}$ is sufficiently small). Evidently $A^* - A^{*\prime}$ is symmetric and positive semi-definite. From that there follows

$$\frac{\partial u^*}{\partial t} = \frac{1}{2}\sum_{i,j} A^*_{ij}(T - t, y) \frac{\partial^2 u^*}{\partial y_i \partial y_j} + \sum_i a^*_i(T - t, y) \frac{\partial u^*}{\partial y_i} \leqq$$

$$\leqq \frac{1}{2}\sum_{i,j} A^{*\prime}_{ij}(T - t, y) \frac{\partial^2 u^*}{\partial y_i \partial y_j} + \sum_i a^*_i(T - t, y) \frac{\partial u^*}{\partial y_i}$$

and $u^*(t, y) < v^*(t, y)$ as in Theorem 1 which means $u(t, x) < v(t, x)$. The function $v^*(t, y)$ is defined by $v^*(t, y) = v(t, x)$ and it is the solution of

$$\frac{\partial v^*}{\partial t} = \frac{1}{2} \sum_{i,j} A_{ij}^{*\prime}(T - t, y) \frac{\partial^2 v^*}{\partial y_i \, \partial y_j} + \sum_i a_i^*(T - t, y) \frac{\partial v^*}{\partial y_i}.$$

By the inverse transformation L^{-1} the matrix function $A^{*\prime}$ is transformed into $A' = LA^*L^T$, i.e. A' is symmetric and positive definite and $A - A'$ is symmetric and positive semi-definite. There exists a matrix function $B'(t, x)$ such that $A'(t, x) = B'(t, x) \, B'^T(t, x)$. The inequality $u(T, x) < v(T, x)$ implies $P(B, a, f, Q) < P(B', a, f, Q)$ for every density $f(x)$ in D.

Remark 4. As in the case of Theorem 1 we have proved a stronger result. If the function $u(t, x)$ of Theorem 2 is not convex in Q with respect to x then there exists a matrix function $B'(t, x)$ fulfilling all condition of Definition 6 and such that $P(B, a, f, Q) < P(B', a, f, Q)$ for all densities $f(x)$ in D.

If the function $u(t, x)$ determined in Theorem 2 is convex in Q then the region D is convex.

7. We shall introduce some explicit conditions under which $a(t, x)$, $B(t, x)$ fulfil condition (A). (Definition 4). We have already mentioned that item iii) is fulfilled if $a(t, x)$, $B(t, x)$ are continuous in all variables and Lipschitz continuous in x. In this paragraph we shall deal with item iv) of Definition 4, i.e. we shall introduce conditions for the unicity of bounded solutions of parabolic equations with noncontinuous boundary values. (Item v) will be investigated in the next paragraph.) The existence of bounded solution of (1,4) fulfilling (1,5) and (1,6) was proved in the proof of Lemma 2. Actually, we proved that the function $u^*(t, x)$ is solution of (1,4) under conditions i), ii) of Definition 4. These conditions i), ii) are included in assumptions of Lemma 5 and Lemma 6 so that the existence of such solutions is guaranteed.

Let N be a subset of S. The set N is almost everywhere in S if to every $\bar{x} \in \dot{D}$ the set $\{[t, y_1, \ldots, y_{n-1}] : [t, y_1, \ldots, y_{n-1}, h(y_1, \ldots, y_{n-1})] \in N\}$ is almost everywhere in $\langle 0, T \rangle \times K^+$ where K^+ is some open subset of K^* containing $\bar{x}$ and where $y_1, \ldots, y_n$ are local coordinates at $\bar{x}$ which together with h and K^* fulfil all conditions of Definition 2.

Lemma 4. *Let $a(t, x)$, $B(t, x)$ be defined on $\bar{Q} = \langle 0, T \rangle \times \bar{D}$ where D fulfils condition* (B). *Let the matrix $A(t, x) = B(t, x) \, B^T(t, x)$ be positive definite matrix for every $[t, x] \in \bar{Q}$ and let $\partial A_{ij}/\partial x_j$, $\partial^2 A_{ij}/\partial x_i \, \partial x_j$, $\partial a_i/\partial x_i$ exist and*

$$|A_{ij}(t_2, x_2) - A_{ij}(t_1, x_1)| \leqq M(|x_2 - x_1|^\alpha + |t_2 - t_1|^{\alpha/2}),$$

$$\left| \frac{\partial A_{ij}}{\partial x_j}(t_2, x_2) - \frac{\partial A_{ij}}{\partial x_j}(t_1, x_1) \right| \leqq M(|x_2 - x_1|^\alpha + |t_2 - t_1|^{\alpha/2}),$$

$$\left|\frac{\partial^2 A_{ij}}{\partial x_i\,\partial x_j}(t_2, x_2) - \frac{\partial^2 A_{ij}}{\partial x_i\,\partial x_j}(t_1, x_1)\right| \leqq M(|x_2 - x_1|^\alpha + |t_2 - t_1|^{\alpha/2}),$$

$$|a_i(t_2, x_2) - a_i(t_1, x_1)| \leqq M(|x_2 - x_1|^\alpha + |t_2 - t_1|^{\alpha/2}),$$

$$\left|\frac{\partial a_i}{\partial x_i}(t_2, x_2) - \frac{\partial a_i}{\partial x_i}(t_1, x_1)\right| \leqq M(|x_2 - x_1|^\alpha + |t_2 - t_1|^{\alpha/2})$$

for $[t_s, x_s] \in \bar{Q}$, $s = 1, 2$ *for all indices* i, j *where* α *is a positive number* $\alpha < 1$ *and* M *is a positive number. Let* $u_k(t, x)$, $k = 1, 2$ *be given functions on* Q *such that their derivatives*

$$\frac{\partial u_k}{\partial t}(t, x), \quad \frac{\partial u_k}{\partial x_i}(t, x), \quad \frac{\partial^2 u_k}{\partial x_i\,\partial x_j}(t, x)$$

are continuous on Q *and let a function* $\varphi(t, x)$ *be given on* $\{[0, x] : x \in D\} \cup S$ $(S = \langle 0, T\rangle \times \dot{D})$.

If $u_k(t, x)$ *are bounded on* Q *and fulfil*

$$\frac{\partial u}{\partial t} = \sum_{i,j} A_{ij}(t, x)\frac{\partial^2 u}{\partial x_i\,\partial x_j} + \sum_i a_i(t, x)\frac{\partial u}{\partial x_i} \tag{7,1}$$

$$\lim_{t\to 0^+} u_k(t, x) = \varphi(0, x) \quad \textit{for almost all} \quad x \in D \tag{7,2}$$

$$\lim_{x\to \bar{x}} u_k(t, x) = \varphi(t, \bar{x}) \quad \textit{almost everywhere on } S \tag{7,3}$$

then $u_1(t, x) = u_2(t, x)$.

The proof of Lemma 4 is in Appendix.

8. Lemma 5. *Let all assumptions of Lemma 4 be fulfilled and let* $a(t, x)$, $B(t, x)$ *be continuous in all variables and Lipschitz continuous in* x. *If* $f(x)$ *is a given density in* D *then*

$$P(B, a, f, Q) = \int_D f(x)\,u(T, x)\,\mathrm{d}x \tag{8,2}$$

where $u(t, x)$ *is the bounded solution of*

$$\frac{\partial u}{\partial t} = \tfrac{1}{2}\sum_{i,j,k} B_{ik}(T - t, x)\,B_{jk}(T - t, x)\frac{\partial^2 u}{\partial x_i\,\partial x_j} + \sum_i a_i(T - t, x)\frac{\partial u}{\partial x_i} \tag{8,3}$$

fulfilling

$$\lim_{t\to 0^+} u(t, x) = 0 \quad \textit{for} \quad x \in D, \tag{8,4}$$

$$\lim_{x\to \bar{x}} u(t, x) = 1 \quad \textit{for} \quad t > 0, \quad \bar{x} \in \dot{D}. \tag{8,5}$$

This statement is valid also for $f(x) = \delta(x) : P(B, y, \delta(x), Q) = u(T, x)$.

Remark 5. The unicity of the bounded solution u is guaranteed by Lemma 4. The proof of Lemma 5 is in Appendix.

9. Remark 6. Let a region C be regular (see definition in 1). In paragraph 1 there was suggested the concept of the strongly maximal matrix function $B(t, x)$ with respect to a vector function $a(t, x)$ and to the region C. The question arises if Theorem 2 can be adapted to this case. Assume that there exists a linear regular transformation $y = \psi(t)x + \varphi(t)$ which maps C onto a cylindric region $Q = (0, T) \times D$ such that $\mathrm{d}\psi(t)/\mathrm{d}t$, $\mathrm{d}\varphi(t)/\mathrm{d}t$ are Hölder continuous. Equation (1,1) is transformed into

$$(9,1)\quad \mathrm{d}y = \left[\frac{\mathrm{d}\psi(t)}{\mathrm{d}t}\,\psi^{-1}(t)\,(y - \varphi(t)) + \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} + \psi(t)\,a(t, \psi^{-1}(t)\,(y - \varphi(t)))\right]\mathrm{d}t + $$
$$+ \psi(t)\,B(t, \psi^{-1}(t)\,(y - \varphi(t)))\,\mathrm{d}w(t)\,.$$

If the coefficients in (9,1) fulfil the assumptions of Lemma 5 and the region D fulfils (B) then we can apply Lemma 5. We obtain that $P(B, y, \delta(x_0), C) = u(T, \psi(0, x_0))$ where $\delta(x_0)$ is the Dirac function concentrated into the point x_0 and $u(t, y)$ is the bounded solution of the parabolic differential equation corresponding to (9,1) according to Theorem 2 and fulfilling the initial and boundary conditions given in Theorem 2. The coefficients at $(\partial^2 u/\partial y_i\, \partial y_j)(t, y)$ in this parabolic equation are the elements of the matrix $\psi(T - t)\, B(T - t, \psi^{-1}(T - t)\,(y - \varphi(T - t)))\, B^T(T - t, \psi^{-1}(T - t)\,(y - \varphi(T - t)))\, \psi^T(T - t)$. If A is a symmetric matrix then $\psi BB^T\psi^T - \psi A\psi^T$ is positive definite if and only if the matrix $BB^T - A$ is positive definite. Hence Theorem 2 can be applied in the following manner. If the bounded solution $u(t, y)$ of the parabolic equation corresponding to (9,1) according to Theorem 2 (or Lemma 5) fulfilling (6,2), (6,3) is convex as the function of y then the matrix function $B(t, x)$ is strongly maximal with respect to the vector function $a(t, x)$ and to the region C. The convexity is a necessary condition again. The matrix $B(t, x)$ is strongly maximal with respect to a and C if $P(B, a, f, C) \geqq P(B', a, f, C)$ for every density f in C_0 and every matrix function $B'(t, x)$ fulfilling all assumptions formulated above and if $BB^T - B'B'^T$ is positive semi-definite.

$$P(B, a, f, C) = P\{\omega : \exists\, \{\tau : [\tau, x_f(\tau, \omega)] \notin C,\ \tau \in \langle 0, T\rangle\}\}\,.$$

10. Remark 7. Let Ito stochastic equation (1,1) be given and let a, B, D fulfil the conditions of Theorem 2. Let $\tilde{\sigma}$ be the set of the nonstochastic solutions of

$$(10,1)\qquad \dot{x} = a(t, x)$$

for which there exists τ, $0 < \tau \leqq T$ such that $x(0) \in D$, $x(\tau) \in \dot{D}$. Let σ be a set of points $[t, x]$, $0 \leqq t \leqq \tau$ lying in Q and simultaneously on some solution of $\tilde{\sigma}$. Denote by $u(t, x)$ the bounded solutions of (6,1) fulfilling (6,2), (6,3). If the interior σ^0 of σ is nonempty then $u(t, x)$ cannot be convex on σ with respect to x. We have

$$P(B, a, f, Q) \to 1 \quad \text{for} \quad B \to 0 \tag{10,2}$$

if the initial density $f(x)$ is concetrated on the interior of the set $\{[t, x] : t = 0\} \cap \sigma$. In this case we can the l.u.b. of $P(B, a, f, Q)$ only approximate, choosing the elements of $B(t, x)$ sufficiently small on σ^0.

Proof. If $B \to 0$ then the solution $x_B(t)$ of (1,1) converges to the solution $x_0(t)$ of (10,1) (which has the same intial values as $x_B(t)$) in the sense $|||x_B(t) - x_0(t)|||_T \to 0$. Since the solution of (10,1) passing through a point $[t, x] \in \sigma$ certainly leaves D on $\langle 0, T \rangle$ we have (10,2). If $u(t, x)$ were convex on σ then by Theorem 2 the corresponding l.u.b. of $P(B, a, f, Q)$ would occur for given B. By the maximum principle there is $u(t, x) < 1$ for $t > 0$, $x \in D$ and consequently $P(B, a, f, Q) < 1$. This proves the Remark.

11. Theorems 1 and 2 can be applied in the one-dimensional case $n = 1$ and they are identical then. But in the one-dimensional case explicit conditions are presented which cannot be generalized directly to $n > 1$.

Theorem 3. *Let a region Q be defined by $Q = (0, T) \times (x_1, x_2)$ where x_1, x_2 are numbers $x_1 < x_2$. Let a function $B(t, x)$ be defined on $\bar{Q}$ so that $B(t, x)$ is continuous in t, Lipschitz continuous in x, $B(t, x) \neq 0$ on $\bar{Q}$, $B^2(t, x)$ and $\partial^2 B^2(t, x)/\partial x^2$ are Hölder continuous in t, x (as in Lemma 4 with $A(t, x) = B^2(t, x)$). Let $\alpha(t)$, $\beta(t)$ be Hölder continuous functions on $\langle 0, T \rangle$. Put $a(t, x) = \alpha(t) + \beta(t)\, x$. If $a(t, x_2) \leqq 0$, $a(t, x_1) \geqq 0$ then the function $B(t, x)$ is (strongly) maximal with respect to the function $a(t, x)$ and to the region Q.*

Proof. By Lemma 4 and 5 condition (A) is fulfilled ($n = 1$). According to Theorem 2 it is sufficient to prove that the bounded solution of

$$\frac{\partial u}{\partial t} = \tfrac{1}{2} B^2(T - t, x) \frac{\partial^2 u}{\partial x^2} + a(T - t, x) \frac{\partial u}{\partial x} \tag{11,1}$$

fulfilling

$$\lim_{t \to 0^+} u(t, x) = 0 \quad \text{for} \quad x \in (x_1, x_2) \tag{11,2}$$

$$\lim_{x \to x_i} u(t, x) = 1 \quad \text{for} \quad t > 0\,, \quad i = 1, 2 \tag{11,3}$$

is convex in x. Obviously the following Lemma implies this statement.

Lemma 6. *Let a function $b(t, x)$ be defined on $\bar{Q} = \langle 0, T\rangle \times \langle x_1, x_2\rangle$, let $b(t, x) > 0$ on $\bar{Q}$ and*

$$|b(t_2, x_2) - b(t_1, x_1)| \leqq M(|x_2 - x_1|^\alpha + |t_2 - t_1|^{\alpha/2}),$$

$$\left|\frac{\partial^2 b}{\partial x^2}(t_2, x_2) - \frac{\partial^2 b}{\partial x^2}(t_1, x_1)\right| \leqq M(|x_2 - x_1|^\alpha + |t_2 - t_1|^{\alpha/2}), \quad [t_s, x_s] \in \bar{Q},$$

$$s = 1, 2, \quad M > 0, \quad 0 < \alpha < 1.$$

Let $a(t, x)$ be defined as in Theorem 3 *and let $a(t, x)$ fulfil the conditions of Theorem* 3. *The bounded solution of*

$$\text{(11,4)} \qquad \frac{\partial u}{\partial t} = \tfrac{1}{2}\, b(t, x) \frac{\partial^2 u}{\partial x^2} + a(t, x) \frac{\partial u}{\partial x}$$

fulfilling (11,2), (11,3) *is convex in x.*

Proof of Lemma 6. Let $\varphi(x)$ be a convex function on $\langle x_1, x_2\rangle$, with Hölder continuous third derivative and

$$\varphi(x_i) = 1, \quad \tfrac{1}{2}\, b(0, x_i) \frac{d^2\varphi}{dx^2}(x_i) + a(0, x_i) \frac{d\varphi}{dx}(x_i) = 0 \quad \text{for} \quad i = 1, 2.$$

We shall prove that a solution $v(t, x)$ of (11,4) fulfilling

$$\text{(11,5)} \qquad v(0, x) = \varphi(x) \quad \text{for} \quad x \in \langle x_1, x_2\rangle, \quad v(t, x_i) = 1 \quad \text{for} \quad t \geqq 0, \quad i = 1, 2$$

is a convex function in x.

We shall prove this using an approximation method well-known in the numerical mathematics. We divide the interval $\langle x_1, x_2\rangle$ by equidistant points $x_1 + kh$, $h = (x_2 - x_1)/n$ where n is an arbitrary integer. Put

$$\text{(11,6)} \qquad 0 < \tau < \min\left[\frac{h}{3 \max |\alpha(t) + \beta(t)\, x_i|}, \frac{h^2}{3 \max b(t, x)}, \frac{1}{3 \max |\beta(t)|}\right]$$

where $t \in \langle 0, T\rangle$, $x \in \langle x_1, x_2\rangle$ $i = 1, 2$.

Let $v^*(t, x)$ be defined at points $x_1 + kh$, $l\tau$, k, l being integers and let $v^*(t, x)$ be the solution of linear algebraic equations

$$\text{(11,7)} \qquad v^*(t + \tau, x) = v^*(t, x) - v(t, x) \frac{\tau}{h} (\alpha(t) + \beta(t)\, x) [v^*(t, x) -$$

$$- v^*(t, x + v(t, x)\, h] + \frac{\tau}{2h^2}\, b(t, x) [v^*(t, x + h) - 2v^*(t, x) + v^*(t, x - h)]$$

fulfilling

$$(11,8) \qquad v^*(0, x) = \varphi(x) \quad \text{for} \quad x = x_1 + kh$$

$$(11,9) \qquad v^*(t, x_i) = 1 \quad \text{for} \quad \text{for} \quad t = l\tau\,, \quad i = 1, 2\,.$$

The function $v(t, x)$ is defined by $v(t, x) = \operatorname{sgn}(\alpha(t) + \beta(t)\,x)$. Since this method is only slightly different from the well known method used in numerical mathematics (e.g. [8]) the proof that $v^*(t, x)$ converges to $v(t, x)$ for $n \to \infty$ will be sketched only.

As usually, we see that $v(t, x)$ fulfils

$$\Big|v(t + \tau, x) - v(t, x) + v(t, x)\frac{\tau}{h}(\alpha(t) + \beta(t)\,x)\,[v(t, x) - v(t, x + v(t, x)\,h] -$$

$$- \frac{\tau}{2h^2}\,b(t, x)\,[v(t, x + h) - 2v(t, x) + v(t, x - h)]\Big| \leqq M\tau(h + \sqrt{\tau})$$

where M is a constant. We use the fact that $\partial^3 v/\partial x^3$ is bounded and $|(\partial v/\partial t)(t, x) - (\partial v/\partial t)(t', x)| \leqq M|t - t'|^{1/2}$ (cf. Theorem 5.2 §5 Chap. IV [4] where due to the assumptions on $\varphi(x)$, $l > 1$ can be chosen). If we denote $\Delta(t, x) = v^*(t, x) - v(t, x)$ then $\Delta(t, x)$ is a solution fulfilling $\Delta(0, x) = 0$, $\Delta(t, x_i) = 0$ $i = 1, 2$ of the linear algebraic system

$$(11,10) \quad \Delta(t + \tau, x) = \left[\frac{\tau}{2h^2}\,b(t, x) + v^+(t, x)\,\frac{\tau}{h}\,(\alpha(t) + \beta(t)\,x)\right]\Delta(t, x + h) +$$

$$+ \left[1 - v(t, x)\frac{\tau}{h}(\alpha(t) + \beta(t)\,x) - \frac{\tau}{h^2}\,b(t, x)\right]\Delta(t, x) +$$

$$+ \left[\frac{\tau}{2h^2}\,b(t, x) + v^-(t, x)\,\frac{\tau}{h}\,(\alpha(t) + \beta(t)\,x)\right]\Delta(t, x - h) + \psi(t, x)$$

where

$$v^+(t, x) = \max(0, v(t, x))\,, \quad v^-(t, x) = \min(0, v(t, x)) \quad \text{and} \quad \psi(t, x)$$

is a function fulfilling $|\psi(t, x)| \leqq M\tau(h + \sqrt{\tau})$. With respect to (11,6) we obtain

$$\max\{|\Delta(t + \tau, x); x = x_1 + kh\} \leqq \max\{|\Delta(t, x)|; x = x_1 + kh\} +$$
$$+ \max\{|\psi(t, x)|; x = x_1 + kh\}\,.$$

Thus $|\Delta(t, x)| \leqq M(h + \sqrt{\tau})$ and the convergence $v^*(t, x) \to v(t, x)$ for $n \to \infty$ is proved.

We shall prove that the functions $v^*(t, x)$ are convex in x. Put $\delta(t, x) = \frac{1}{2}[\operatorname{sgn}(\alpha(t) + \beta(t)\,x + \beta(t)\,h) + \operatorname{sgn}(\alpha(t) + \beta(t)\,x - \beta(t)\,h)]$. The function $\delta(t, x)$ can assume values -1, $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1. The values $-\frac{1}{2}$, $\frac{1}{2}$ will not be further considered since these cases can be reduced to $\delta = 0$ by means of arbitrary small changes of $\alpha(t)$, $\beta(t)$.

With respect to (11,7), we obtain

$$(11,11) \qquad v^*(t + \tau, x + h) - 2v^*(t + \tau, x) + v^*(t + \tau, x - h) =$$

$$= \left[1 - \nu \frac{\tau}{h} (\alpha(t) + \beta(t) x) - \frac{\tau}{h^2} b(t, x) + \tau\beta(t)\right] [v^*(t, x + h) -$$

$$- 2v^*(t, x) + v^*(t, x - h)] + \delta \frac{\tau}{h} [\alpha(t) + \beta(t) x + \delta\beta(t) h] \times$$

$$\times [v^*(t, x + \delta h + h) - 2v^*(t, x + \delta h) + v^*(t, x + \delta h - h)] +$$

$$+ \frac{\tau}{2h^2} b(t, x + h) [v^*(t, x + 2h) - 2v^*(t, x + h) + v^*(t, x)] +$$

$$+ \frac{\tau}{2h^2} b(t, x - h) [v^*(t, x) - 2v^*(t, x - h) + v^*(t, x - 2h)]$$

for $|\delta| \neq \frac{1}{2}$ and for $x = x_1 + kh$ such that $x_1 \leqq x \leqq x_2$. For the sake of simplicity denote $\nu = \nu(t, x)$, $\delta = \delta(t, x)$ and put $v^*(t, x_i + (-1)^i h) = 2 - v^*(t, x_i - (-1)^i h) + 2h [|a(t, x_i)|/b(t, x_i)] (1 - v^*(t, x_i - (-1)^i h))$ for $i = 1, 2$ i.e. (11,7) is fulfilled for $x = x_i$, $i = 1, 2$.

For $t = 0$, $v^*(0, x)$ is convex since $v^*(0, x) = \varphi(x)$ and $\varphi(x)$ is assumed convex. Let $v^*(t, x)$ be convex in x then we shall prove that $v^*(t + \tau, x)$ is convex in x. Since $v^*(t, x)$ is assumed convex we obtain by (11,11)

$$(11,12) \qquad v^*(t + \tau, x + h) - 2v^*(t + \tau, x) + v^*(t + \tau, x - h) \geqq$$

$$\geqq \left[1 - \nu \frac{\tau}{h} (\alpha(t) + \beta(t) x) - \frac{\tau}{h^2} b(t, x) + \tau\beta(t)\right] \times$$

$$\times [v^*(t, x + h) - 2v^*(t, x) + v^*(t, x - h)] + \delta \frac{\tau}{h} [\alpha(t) + \beta(t) x + \delta\beta(t) h] \times$$

$$\times [v^*(t, x + \delta h + h) - 2v^*(t, x + \delta h) + v^*(t, x + \delta h - h)]$$

for x fulfilling $x \pm h \neq x_i$, $i = 1, 2$, $|\delta| \neq \frac{1}{2}$. Since the convexity of $v^*(t, x)$ implies $v^*(t, x_2 - h) \leqq v^*(t, x_2) = 1$, $v^*(t, x_1 + h) \leqq v^*(t, x_1) = 1$ the inequality (11,12) holds also for the points $x = x_1 + h$ and $x = x_2 - h$. By means of (11,6) we obtain

$$1 - \nu \frac{\tau}{h} (\alpha + \beta x) - \frac{\tau}{h^2} b + \tau\beta > 0$$

and using the definition of δ we get

$$\delta \frac{\tau}{h} (\alpha + \beta x + \delta\beta h) \geqq 0 .$$

From (11,12) then follows that $v^*(t + \tau, x)$ is convex in x. Since $v(t, x)$ is the limit of $v^*(t, x)$ for $n \to \infty$, the function $v(t, x)$ is convex in x, too.

We choose a sequence of functions $\varphi_m(x)$ fulfilling conditions formulated above (i.e. $\varphi_m(x)$ are convex, they have Hölder continuous third derivative and

$$\varphi_m(x_i) = 1\,, \quad \tfrac{1}{2}b(0, x_i)\,\frac{d^2\varphi_m}{dx^2}(x_i) + a(0, x_i)\,\frac{d\varphi_m}{dx}(x_i) = 0\,,$$

$i = 1, 2$) and, moreover fulfilling the condition: functions $\varphi_m(x)$ converge uniformly to zero on every compact subinterval of (x_1, x_2). As in the proof of Lemma 2 (and using Lemma 4) we see that the corresponding solutions $v_m(t, x)$ converge to $u(t, x)$, $u(t, x)$ being a convex function in x. Lemma 6 and Theorem 3 are proved.

12. Similarly as in Remark 6 we can extend Theorem 3 on a more general class of regions.

Remark 8. Let $\alpha(t)$, $\beta(t)$, $h_i(t)$, $(dh_i/dt)(t)$, $i = 1, 2$ be defined and Hölder continuous on $\langle 0, T\rangle$, $h_1(t) < h_2(t)$ for $t \in \langle 0, T\rangle$. Denote by C the region of points $[t, x] : h_1(t) < x < h_2(t)$, $0 < t < T$. Denote $a(t, x) = \alpha(t) + \beta(t)\,x$ in C and assume that $B(t, x)$ is defined on $\bar{C}$ and fulfils there the conditions of Theorem 3 (where Q is replaced by C). If $a(t, h_2(t)) \leqq h_2'(t)$, $a(t, h_1(t)) \geqq h_1'(t)$ then the function $B(t, x)$ is (strongly) maximal with respect to the function $a(t, x)$ and to the region C.

The proof follows from Remark 6 and from the fact that the required transformation on a region $(0, T) \times (0, 1)$ has the form $y = \psi(t, x) = (x - h_1)/(h_2 - h_1)$. Hence the coefficients at dt in (9,1) are linear in y again and we easily establish that the conditions of Theorem 3 are valid.

13. We introduce now three examples. Examples 1, 2 show that Theorem 3 cannot be directly generalized onto a multidimensional case. In Example 3 an important case when D is a circle is investigated. In Examples 1, 2 regions D do not fulfil condition (B). But in these cases we can calculate Green functions explicitly and it is possible to show that in these cases the preceding results can be applied as if condition (B) were satisfied.

Example 1. Let $n = 2$, the region D be a square $(0, l) \times (0, l)$, $l > 0$, $a(t, x, y) \equiv 0$, and $B(t, x, y)$ be a unit matrix for all t, x, y. We shall show that B is maximal with respect to $a \equiv 0$ and to the square D. To be able to use Theorem 1, we must prove that the bounded solution of

$$\frac{\partial u}{\partial t} = \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{13,1}$$

fulfilling

(13,2) $$\lim_{t\to 0^+} u(t, x, y) = 0 \quad \text{for} \quad [x, y] \in D\,, \qquad \lim_{[x,y]\to[\bar{x},\bar{y}]} u(t, x, y) = 1 \quad \text{for} \quad t > 0\,,\quad [\bar{x}, \bar{y}] \in \dot{D}$$

has nonnegative second derivatives $\partial u/\partial x^2 \geqq 0$, $\partial^2 u/\partial y^2 \geqq 0$ in Q. Let $v(t, x)$ be the bounded solution of

(13,3) $$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2}$$

fulfilling

(13,4) $$\lim_{t\to 0^+} v(t, x) = 0 \quad \text{for} \quad x \in (0, l)\,, \quad v(t, 0) = v(t, l) = 1 \quad \text{for} \quad t > 0\,.$$

The solution $u(t, x, y)$ can be written as

(13,5) $$u(t, x, y) = v(t, x) + v(t, y) - v(t, x)\, v(t, y)\,.$$

According to Theorem 3 (Lemma 6) the function $v(t, x)$ is convex in x and since $0 \leqq v(t, x) \leqq 1$ we obtain actually $\partial^2 u/\partial x^2 \geqq 0$, $\partial^2 u/\partial y^2 \geqq 0$ in Q.

On the other hand we shall prove that the unit matrix B is not strongly maximal with respect to $a(t, x, y) = 0$ and with respect to the square D. According to Theorem 2 it suffices to prove that the solution $u(t, x, y)$ is not convex with respect to the spatial variables x, y. The bounded solution of (13,3) fulfilling (13,4) is given by the well known formula

$$v(t, x) = 1 + \sum_k (-1)^{k+1} \frac{1}{\sqrt{(2\pi t)}} \int_0^l \exp\left\{- \frac{(\tilde{x} - x + kl)^2}{2t}\right\} d\tilde{x}$$

and by (13,5)

(13,6) $$u(t, x, y) = 1 + \sum_{k,m}(-1)^{k+m+1} \frac{1}{2\pi t} \int_0^l\int_0^l \exp\left\{- \frac{(\tilde{x} - x + kl)^2}{2t} - \frac{(\tilde{y} - y + ml)^2}{2t}\right\} d\tilde{x}\, d\tilde{y}\,.$$

Provided that $u(t, x, y)$ is convex for all $l > 0$, $T > 0$, $[t, x, y] \in (0, T) \times (0, l) \times \times (0, l)$ we get a contradiction. The function $u(t, x, y)$ must then have a bounded limit $\bar{u}(t, x, y)$ for $l \to \infty$. The function $\bar{u}(t, x, y)$ must be convex for all $x > 0$, $y > 0$. (13,6) implies

$$\bar{u}(t, x, y) = 1 - \frac{1}{2\pi t} \int_0^\infty\int_0^\infty \exp\left\{- \frac{(\tilde{x} - x)^2}{2t} - \frac{(\tilde{y} - y)^2}{2t}\right\} d\tilde{x}\, d\tilde{y} +$$

$$+\frac{1}{2\pi t}\int_0^\infty\int_0^\infty \exp\left\{-\frac{(\tilde{x}+x)^2}{2t}-\frac{(\tilde{y}-y)^2}{2t}\right\}d\tilde{x}\,d\tilde{y}+$$

$$+\frac{1}{2\pi t}\int_0^\infty\int_0^\infty \exp\left\{-\frac{(\tilde{x}-x)^2}{2t}-\frac{(\tilde{y}+y)^2}{2t}\right\}d\tilde{x}\,d\tilde{y}-$$

$$-\frac{1}{2\pi t}\int_0^\infty\int_0^\infty \exp\left\{-\frac{(\tilde{x}+x)^2}{2t}-\frac{(\tilde{y}+y)^2}{2t}\right\}d\tilde{x}\,d\tilde{y}\,.$$

By means of this formula we deduce

$$\frac{\partial^2\bar{u}}{\partial x^2}=\frac{x}{\pi t^2}\exp\left\{-\frac{x^2}{2t}\right\}\int_{-y}^{y}\exp\left\{-\frac{\xi^2}{2t}\right\}d\xi\,,$$

$$\frac{\partial^2\bar{u}}{\partial y^2}=\frac{y}{\pi t^2}\exp\left\{-\frac{y^2}{2t}\right\}\int_{-x}^{x}\exp\left\{-\frac{\xi^2}{2t}\right\}d\xi\,,\quad \frac{\partial^2\bar{u}}{\partial x\,\partial y}=-\frac{2}{\pi t}\exp\left\{-\frac{x^2+y^2}{2t}\right\}.$$

The function $\bar{u}$ is convex if

$$\left|\frac{\partial^2\bar{u}}{\partial x\,\partial y}\right|^2\leqq\frac{\partial^2\bar{u}}{\partial x^2}\cdot\frac{\partial^2\bar{u}}{\partial y^2}$$

which means

$$4\leqq\frac{xy}{t}\int_{-x/\sqrt{t}}^{x/\sqrt{t}}\exp\left\{-\frac{\xi^2}{2}\right\}d\xi\int_{-y/\sqrt{t}}^{+y/\sqrt{t}}\exp\left\{-\frac{\xi^2}{2}\right\}d\xi\,\exp\left\{\frac{x^2+y^2}{2t}\right\}.$$

For every t we can find positive x, y, such small that the last inequality is not valid, i.e. the function $\bar{u}(t,x,y)$ is not convex in x, y. This example shows that it is not possible to generalize Theorem 3 to the multidimensional case in the sense of Theorem 2. However, it could still be a possibility to generalize Theorem 3 in the sense of Theorem 1 where only $\partial^2u/\partial x_i^2\geqq 0$ is demanded. To show that there is not such a possibility we introduce another example.

14. Example 2. Let the region D be a triangle: $D=\{[x,y]:0<y<\sqrt{3}\min(x,1-x)\}$, let $a(t,x,y)\equiv 0$ and let $B(t,x,y)$ be the unit matrix on D. We shall prove that the unit matrix B is not maximal with respect to $a(t,x,y)\equiv 0$ and to the triangle D.

With regard to Theorem 1 we need to prove that the bounded solution of (13,1) fulfilling (13,2) does not fulfil $\partial^2u/\partial x^2\geqq 0$, $\partial^2u/\partial y^2\geqq 0$ on Q. Denote

$$D_{kl}=\left\{[x,y]:\frac{l-k-1}{2}\sqrt{3}<y<\sqrt{3}\min(x-k,l-x)\right\}$$

$$D^*_{k,l}=\left\{[x,y]:\frac{l-k+1}{2}\sqrt{3}>y>\sqrt{3}\max(x-k,l-x)\right\}.$$

For bounded solutions of (13,1) fulfilling (13,2), we have

$$u(t, x, y) = 1 - \frac{1}{2\pi t}\left[\sum_{k,l}\iint_{D_{kl}} \exp\left\{-\frac{(\tilde{x} - x)^2}{2t} - \frac{(\tilde{y} - y)^2}{2t}\right\} d\tilde{x}\, d\tilde{y} - \right.$$

$$\left. - \sum_{k,l}\iint_{D^*_{kl}} \exp\left\{-\frac{(\tilde{x} - x)^2}{2t} - \frac{(\tilde{y} - y)^2}{2t}\right\} d\tilde{x}\, d\tilde{y}\right].$$

Denote

$$G = G(t, \tilde{x}, \tilde{y}, x, y) = \exp\left\{-\frac{(\tilde{x} - x)^2}{2t} - \frac{(\tilde{y} - y)^2}{2t}\right\}$$

and

$$u_{kl}(t, x, y) = \iint_{D_{kl}} G\, d\tilde{x}\, d\tilde{y}\,, \quad u^*_{kl}(t, x, y) = \iint_{D^*_{kl}} G\, d\tilde{x}\, d\tilde{y}$$

Evidently

$$\frac{\partial^2 u_{kl}}{\partial x^2}(t, x, y) = -\int\left[\frac{\partial G}{\partial x}\right]_d^h d\tilde{y}$$

where h, d are the y-coordinates of the intersections of the straight line $\tilde{x} = x$ with $\dot{D}_{kl}$, $d < h$ and

$$\left[\frac{\partial G}{\partial x}\right]_d^h = \left.\frac{\partial G}{\partial x}\right|_{\tilde{x}=h} - \left.\frac{\partial G}{\partial x}\right|_{\tilde{x}=d}.$$

The domain of integration is the interval $\langle\frac{1}{2}(l - k - 1)\sqrt{3}, \frac{1}{2}(l - k)\sqrt{3}\rangle$.

Similarly

$$\frac{\partial^2 u^*_{kl}}{\partial x^2}(t, x, y) = -\int\left[\frac{\partial G}{\partial x}\right]_d^h d\tilde{y}$$

where d, h are the y-coordinates of the intersections of the straight line $\tilde{x} = x$ with $\dot{D}^*_{kl}$ and the domain of integration is $\langle\frac{1}{2}(l - k)\sqrt{3}, \frac{1}{2}(l - k + 1)\sqrt{3}\rangle$. Using these relations, we obtain

$$\frac{\partial^2 u}{\partial x^2}(t, x, y) = -\frac{1}{\pi t}\left(\sum_k \int_{-\infty}^{\infty} \left.\frac{\partial G}{\partial x}\right|_{\tilde{x}=k+\tilde{y}/\sqrt{3}} d\tilde{y} - \sum_l \int_{-\infty}^{\infty} \left.\frac{\partial G}{\partial x}\right|_{\tilde{x}=l-\tilde{y}/\sqrt{3}} d\tilde{y}\right).$$

Since

$$\int_{-\infty}^{\infty} \left.\frac{\partial G}{\partial x}\right|_{\tilde{x}=k+\tilde{y}/\sqrt{3}} d\tilde{y} = \tfrac{3}{4}\sqrt{\pi}\,\frac{y - x\sqrt{3} + k\sqrt{3}}{\sqrt{(2t)}} \exp\left\{-\frac{(y - x\sqrt{3} + k\sqrt{3})^2}{8t}\right\}$$

and

$$\int_{-\infty}^{\infty} \left.\frac{\partial G}{\partial x}\right|_{\tilde{x}=l-\tilde{y}/\sqrt{3}} d\tilde{y} = -\tfrac{3}{4}\sqrt{\pi}\,\frac{y + x\sqrt{3} - l\sqrt{3}}{\sqrt{(2t)}} \exp\left\{-\frac{(y + x\sqrt{3} - l\sqrt{3})^2}{8t}\right\}$$

we have

(14,1)

$$\frac{\partial^2 u}{\partial x^2}(t, x, y) = -\frac{3}{2t\sqrt{\pi}}\left(\sum_l \frac{y + x\sqrt{3} - l\sqrt{3}}{\sqrt{(8t)}} \exp\left\{-\frac{(y + x\sqrt{3} - l\sqrt{3})^2}{8t}\right\} + \right.$$

$$\left. + \sum_k \frac{y - x\sqrt{3} + k\sqrt{3}}{\sqrt{(8t)}} \exp\left\{-\frac{(y - x\sqrt{3} + k\sqrt{3})^2}{8\,t}\right\}\right).$$

The right hand side of (14,1) cannot be positive for all t, x, y. If we take t, x, y positive but sufficiently small then the sign of $\partial^2 u/\partial x^2$ depends most essentially on two terms: $l = 0$, $k = 0$. Hence $\partial^2 u/\partial x^2 > 0$ in every neighbourhood of $[t, 0, 0]$, $t > 0$ if and only if $\exp(xy\sqrt{3}/(2t)) > (x\sqrt{3} + y)/(x\sqrt{3} - y)$. The last inequality is not valid for $0 < x < \sqrt{(2t/3)}$, $(e - 1)\,x\sqrt{3}/(e + 1) < y < x\sqrt{3}$, i.e. in every neighbourhood of $[t, 0, 0]$, $t > 0$, t sufficientlly small, there exist points $[t, x, y]$, $(\partial^2 u/\partial x^2)(t, x, y) < 0$. On the other hand, we obtain $\partial^2 u/\partial x^2 > 0$ in a sufficiently small neighbourhood of $[0, \frac{1}{2}, \frac{3}{2}]$. To prove this, we perform the transformation $x = \frac{1}{2} + \xi$, $y = \sqrt{3}/2 - \eta$. For sufficiently small t, ξ, η the most important terms are $l = 1$, $k = 0$. In this case $\partial^2 u/\partial x^2 > 0$ if and only if $\exp(\xi\eta\sqrt{3}/(2t)) > (\xi\sqrt{3} + \eta)\,.\,(\xi\sqrt{3} - \eta)^{-1}$. Since $\xi\sqrt{3} - \eta < 0 < \xi\sqrt{3} + \eta$ we have $\partial^2 u/\partial x^2 > 0$ there.

15. The last example deals with the case when D is a circle $D = \{[x, y] : x^2 + y^2 < R^2\}$, $R > 0$ and $a(t, x, y) \equiv 0$. We shall show that the unit matrix is not strongly maximal with respect to a and Q. However, the prescription will be given how to choose a generalized strongly maximal matrix function B^* with respect to a and Q so that $J - B^*(t, x, y)\, B^{*T}(t, x, y)$ is positive semi-definite. The notion of the generalized strongly maximal matrix function is introduced in the following way: A matrix function $B^*(t, x, y)$ is called generalized strongly maximal with respect to a and Q if it fulfils all conditions of Definition 6 except ii) (i.e. B^*B^{*T} need not be smooth and positive definite).

If we require condition (A) to be fulfilled then the generalized strongly maximal matrix can be approximated only. This result can be immediately modified to the multidimensional case. The next Lemma is a generalization of the well-known theorem about the removable singularity of the harmonic function. It is interesting that such result may be proved using Lemma 6 (Theorem 3).

Lemma 7. *Let $w_i(t)$ $i = 1, 2$ be independent Wiener processes fulfilling $Ew_i(t) = 0$, $Ew_i^2(t) = t$, $i = 1, 2$. We have*

(15,1) $$P\{\omega : \exists\, \{t \in \langle 0, T\rangle,\ w_1(t, \omega) = 1,\ w_2(t, \omega) = 0\}\} = 0\,.$$

Let $u(t, x, y)$, $u_{\varepsilon,R}(t, x, y)$ be the bounded solution of (13,1) *fulfilling*

$$(15,2) \qquad \lim_{t\to 0+} u(t, x, y) = 0 \quad \text{for} \quad 0 \leqq x^2 + y^2 < R^2\,, \quad u(t, \bar{x}, \bar{y}) = 1 \quad \text{for} \quad t > 0\,, \quad \bar{x}^2 + \bar{y}^2 = R^2$$

and

$$(15,3) \quad \lim_{t\to 0+} u_{\varepsilon,R}(t, x, y) = 0 \quad \text{for} \quad \varepsilon^2 < x^2 + y^2 < R^2\,, \quad u_{\varepsilon,R}(t, \bar{x}, \bar{y}) = 1 \quad \text{for} \quad \bar{x}^2 + \bar{y}^2 = \varepsilon^2 \quad \text{or} \quad \bar{x}^2 + \bar{y}^2 = R^2\,, \quad t > 0\,.$$

Then $u_{\varepsilon,R}(t, x, y) \to u(t, x, y)$ for $\varepsilon \to 0$ and for fixed $t, x, y, t \geqq 0, 0 < x^2 + y^2 \leqq R^2$.

Proof. Lemma 5 implies

$$u(t, x, y) \quad = P\{\omega : \exists\, \{\tau : \tau \in \langle 0, t\rangle, \xi^2(\tau) + \eta^2(\tau) = R^2\}\}\,,$$
$$u_{\varepsilon,R}(t, x, y) = P\{\omega : \exists\, \{\tau : \tau \in \langle 0, t\rangle, \xi^2(\tau) + \eta^2(\tau) = R^2 \quad \text{or} \quad \xi^2(\tau) + \eta^2(\tau) = \varepsilon^2\}\}$$

where $\xi(t) = x + w_1(t)$, $\eta(t) = y + w_2(t)$, $\varepsilon^2 < x^2 + y^2 < R^2$ (i.e. ξ, η is the solution of $d\xi = dw_1$, $d\eta = dw_2$ respectively and (13,1) corresponds to this Ito equation). Obviously

$$u(t, x, y) \leqq u_{\varepsilon,R}(t, x, y) \leqq u(t, x, y) + P\{\omega : \exists\, \{\tau : \tau \in \langle 0, t\rangle, \xi^2(\tau) + \eta^2(\tau) = \varepsilon^2\}\,.$$

Thus the second part of Lemma 7 is evidently a consequence of the first part of Lemma 7.

We pass to the proof of the first part of the Lemma. Put $R = \varepsilon^{-1}$. There is

$$(15,4) \qquad \begin{aligned} &P\{\omega : \exists\, \{\tau : \tau \in \langle 0, t\rangle, \xi^2(\tau) + \eta^2(\tau) = \varepsilon^2\} \leqq \\ &\leqq P\{\omega : \exists\, \{\tau : \tau \in \langle 0, t\rangle, \xi^2(\tau) + \eta^2(\tau) = \varepsilon^2 \\ &\quad \text{or} \quad \xi^2(\tau) + \eta^2(\tau) = \varepsilon^{-2}\}\} = u_{\varepsilon,\varepsilon^{-1}}(t, x, y)\,. \end{aligned}$$

It is sufficient to prove $u_{\varepsilon,\varepsilon^{-1}}(t, x, y) \to 0$ for $\varepsilon \to 0$ and $x^2 + y^2 > 0$. According to Lemma 5, the function $u_{\varepsilon,\varepsilon^{-1}}(t, x, y)$ is again the bounded solution of (13,1) fulfilling $u_{\varepsilon,\varepsilon^{-1}}(t, \bar{x}, \bar{y}) = 1$ for $t > 0$, $\bar{x}^2 + \bar{y}^2 = \varepsilon^2$ or $\bar{x}^2 + \bar{y}^2 = \varepsilon^{-2}$. The function $u_{\varepsilon,\varepsilon^{-1}}(t, x, y)$ depends only on t and $r = \sqrt{(x^2 + y^2)}$ since (13,1) as well as the initial and boundary values are independent of the rotation of plane. Put $\theta_\varepsilon(t, z) = u_{\varepsilon,\varepsilon^{-1}}(t, x, y)$, $z = \lg \sqrt{(x^2 + y^2)}$ for $\varepsilon > 0$. The function $\theta_\varepsilon(t, z)$ is the bounded solution of

$$(15,5) \qquad \frac{\partial \theta}{\partial t} = \tfrac{1}{2}\, e^{-2z} \frac{\partial^2 \theta}{\partial z^2}$$

fulfilling $\theta_\varepsilon(0, z) = 0$ for $\lg \varepsilon < z < -\lg \varepsilon$, $\theta_\varepsilon(t, \pm\lg_\varepsilon) = 1$ for $t > 0$. Using Lemma 6 we see that $\theta_\varepsilon(t, z)$ is a convex function in z. We shall prove $\lim_{\varepsilon\to 0} \theta_\varepsilon(t, z) = 0$. Since

$\theta_\varepsilon(t, z)$ is convex in z and $0 \leqq \theta_\varepsilon(t, z) \leqq 1$ on $\lg \varepsilon \leqq z \leqq -\lg \varepsilon$ the derivatives $(\partial\theta_\varepsilon/\partial z)(t, \pm 1) \to 0$ for $\varepsilon \to 0$ uniformly in t. It implies $\int_{-1}^{1} (\partial^2\theta_\varepsilon/\partial z^2)(t, z)\, dz \to 0$ for $\varepsilon \to 0$ uniformly in t and $\int_0^t\int_{-1}^{1} (\partial^2\theta_\varepsilon^{-1}/\partial z^2)(\tau, z)\, d\tau\, dz \to 0$ for $\varepsilon \to 0$. Since $\theta_\varepsilon(t, z)$ is convex in z nad (15,5) holds, we conclude $\int_0^t\int_{-1}^{1} (\partial\theta_\varepsilon/\partial t)(\tau, z)\, d\tau\, dz = \int_{-1}^{1} \theta_\varepsilon(t, z)\, dz \to 0$ for $\varepsilon \to 0$. If we consider the facts that $\theta_\varepsilon(t, z)$ is convex in z and bounded on $\langle \lg \varepsilon, -\lg \varepsilon \rangle$ again, we obtain $\theta_\varepsilon(t, z) \to 0$ for $\varepsilon \to 0$. Together with (15,4) and $u_{\varepsilon,\varepsilon^{-1}}(t, x, y) = \theta_\varepsilon(t, z)$, the equality $\lim_{\varepsilon\to 0} \theta_\varepsilon(t, z) = 0$ proves the first part of Lemma 7.

Example 3. Let D be a cricle $D = \{[x, y] : x^2 + y^2 < 1\}$, $T < 1$, $a(t, x, y) \equiv 0$ and let $B(t, x, y)$ be the unit matrix for all t, x, y. First of all we shall prove that the unit matrix is not strongly maximal. Then we shall find a generalized strongly maximal matrix function fulfilling: $J - B^*(t, x, y)\, B^{*T}(t, x, y)$ is positive semi-definite. To describe such generalized strongly maximal matrix function we introduce a regular region C defined by: $C = \{[t, x, y] : x^2 + y^2 < 1 - T + t,\ 0 < t < T\}$. The region C is a subregion of $Q = (0, T) \times D$. The generalized strongly maximal matrix function $B^*(t, x, y)$ is defined by $B^*_{11}(t, x, y) = B^*_{21}(t, x, y) = 0$, $B^*_{12}(t, x, y) = -\sin\varphi$, $B^*_{22}(t, x, y) = \cos\varphi$ where $\varphi = \operatorname{arctg}(y/x)$ in $Q - C$ and $B^*(t, x, y)$ is the unit matrix in C. In $Q - C$ the process is governed by Ito equation $dx = -\sin\varphi\, dw_2$, $dy = \cos\varphi\, dw_2$ where $w_2(t)$ is an one-dimensional Wiener process. The solutions of this Ito equation have the following properties: The local diffusion in the direction of the radius vector is zero (in $Q - C$) and the local diffusion in the perpendicular direction is constant (in entire Q). If the initial values of a solution of Ito equation $dx = -\sin\varphi\, dw_2$, $dy = \cos\varphi\, dw_2$ are concentrated on a circle $x^2 + y^2 = r^2$, $r^2 > 1 - T$, i.e. $P\{\omega : x(0, \omega)^2 + y(0, \omega)^2 = r^2\} = 1$ then the solution $x(t), y(t)$ is concentrated on the circle $x^2 + y^2 = r^2 + t$ for $0 \leqq t \leqq 1 - r^2$. This implies that the process governed by our Ito equation in $Q - C$ moves away from the origin in the deterministic manner and it reaches the boundary $\dot{D}$ at $t = 1 - r^2 < T$. Nevertheless, it is a random process since the distributions of the solution on the circle $x^2 + y^2 = r^2 + t$ may change with t, if the intial distribution was not uniform on the circle $x^2 + y^2 = r^2$. Thus we can consider the surface $S : x^2 + y^2 = 1 - T + t$ for $0 \leqq t \leqq T$, which is a part of the frontier $\dot{C}$, as an adhesive barrier. (The conditional probability that the solution will reach $\dot{D}$ under the condition that at some moment t_0 this solution assumes values from $Q - C$ only is one.) This enables us to reformulate our problem. We take the unit matrix only in the region C and consider the surface S as an adhesive barrier. Actually we shall prove that the unit matrix is strongly maximal with respect to C. This part of the proof is the most complicated one and will be performed in several steps. We shall not deal with the existence of a process governed by Ito equation where the matrix of local diffusion is a generalized strongly maximal matrix function, since we want to use only the matrix functions fulfilling condition (A). The generalized strongly maximal matrix function may be approximated by matrix functions fulfilling condition (A). Let us consider Ito equation $dx = \gamma(t, r) \cos \varphi \, dw_1 - \sin \varphi \, dw_2$, $dy = \gamma(t, r) \sin \varphi \, dw_1 + \cos \varphi \, dw_2$ where $\varphi = \operatorname{arctg}(y/x)$, $r = \sqrt{(x^2 + y^2)}$ and $\gamma(t, r)$ is a given function. If $\gamma(t, r)$ is sufficiently smooth and positive then the corresponding (by Theorem 2) parabolic system is (15,13). Let $\gamma^*(t, r)$ be defined by: $\gamma^*(t, r) = 1$ for $r < \sqrt{(1 - T + t)}$ and $\gamma^*(t, r) = 0$ for $\sqrt{(1 - T + t)} \leqq r \leqq 1$ (i.e., it equals to one on C and equals to zero on $Q - C$ if this function is expressed in the variables x, y). For $\gamma = \gamma^*$ we obtain a generalized strongly maximal matrix function (the systems $dx = \cos \varphi \, dw_1 - \sin \varphi \, dw_2$, $dy = \sin \varphi \, dw_1 + \cos \varphi \, dw_2$ and $dx = dw_1$, $dy = dw_2$ are equivalent – cf. Proposition in Appendix.) If $\gamma_m \to \gamma^*$ then $P(B_m, 0, f, Q)$ converges to the least upper bound of $P(B', 0, f, Q)$ where $J - B'(t, x, y) B'^T(t, x, y)$ is positive semi-definite and B' fulfils condition (A).

Now we shall begin with the first part of the proof and show that the unit matrix is not strongly maximal with respect to Q.

Theorem 2 shows that we shall deal with bounded solution of

$$\frac{\partial u}{\partial t} = \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{15,6}$$

fulfilling

$$u(0, x, y) = 0 \quad \text{for} \quad x^2 + y^2 < 1\,, \quad u(t, \bar{x}, \bar{y}) = 1 \quad \text{for} \quad t > 0\,, \quad \bar{x}^2 + \bar{y}^2 = 1\,. \tag{15,7}$$

With respect to Lemma 7 we can approximate $u(t, x, y)$ by $u_\varepsilon(t, x, y) = u_{\varepsilon,1}(t, x, y)$. For $v_\varepsilon(t, r) = u_\varepsilon(t, x, y)$, $r = \sqrt{(x^2 + y^2)}$, $\varepsilon > 0$ we get that $v_\varepsilon(t, r)$ is the bounded solution of

$$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial r^2} + \frac{1}{2r}\frac{\partial v}{\partial r} \tag{15,8}$$

fulfilling

$$v_\varepsilon(0, r) = 0 \quad \text{for} \quad \varepsilon < r < 1\,, \quad v_\varepsilon(t, \varepsilon) = v_\varepsilon(t, 1) = 1 \quad \text{for} \quad t > 0\,. \tag{15,9}$$

By Lemma 5 $v_\varepsilon(t, r) = P\{\omega : \exists \{\tau : \tau \in \langle 0, t\rangle, \zeta(\tau) = \varepsilon \text{ or } \zeta(\tau) = 1\}$ where $\zeta(t)$ is the solution of Ito equation

$$\zeta(t) = r + \int_0^t \frac{d\tau}{2\zeta(\tau)} + w(t)\,,$$

$w(t)$ is some Wiener process, $Ew(t) = 0$, $Ew^2(t) = t$. With respect to Remark 7, we consider the equation $\dot{\zeta} = 1/(2\zeta)$ which has solutions $\zeta(t) = \sqrt{(r^2 + t)}$. The interior of the set σ (Remark 7) is a nonempty set: $\sigma^0 = \{[t, r] : 1 > r > \sqrt{(1 - T + t)}, 0 < t < T\}$. Remark 7 implies that $v_\varepsilon(t, r)$ cannot be convex in r. If we consider a class of Ito stochastic equations

$$d\zeta = \frac{dt}{2\zeta} + \gamma(t, \zeta(t)) \, dw(t) \tag{15,10}$$

then, by Remark 7, the extremal case occurs for $\gamma = 0$ on σ^0. We shall show the meaning of this assertion in the wording of the matrices B. By means of condition (A) or Lemma 5 to (15,10) corresponds the parabolic equation

$$\frac{\partial v}{\partial t} = \tfrac{1}{2}\gamma^2(T - t, r)\frac{\partial^2 v}{\partial r^2} + \frac{1}{2r}\frac{\partial v}{\partial r}. \tag{15,11}$$

Let $v_\varepsilon(t, r; \gamma)$ be the bounded solution of (15,11) fulfilling (15,9), then $v_\varepsilon(t, r; \gamma) = P\{\omega : \exists\, \{\tau \in \langle T - t, T\rangle, \tilde{\zeta}(\tau) = \varepsilon \text{ or } \tilde{\zeta}(\tau) = 1\}\}$ where $\tilde{\zeta}(\tau)$ is the solution of (15,10) fulfilling $\tilde{\zeta}(T - t) = r$. By Remark 7 this probability converges to one if $\gamma \to 0$ and if $[t, r] \in \sigma^0$ i.e.

$$v_\varepsilon(t, r; \gamma) \to 1 \quad \text{for} \quad \gamma \to 0\,, \quad r > \sqrt{(1 - T + t)}\,, \quad 0 \leqq t \leqq T\,. \tag{15,12}$$

Put $u_\varepsilon(t, x, y; \gamma) = v_\varepsilon(t, r; \gamma)$, $r = \sqrt{(x^2 + y^2)}$. The function $u_\varepsilon(t, x, y; \gamma)$ is the bounded solution of

$$\frac{\partial u}{\partial t} = \tfrac{1}{4}[\gamma^2 + 1 + (\gamma^2 - 1)\cos 2\varphi]\frac{\partial^2 u}{\partial x^2} + \tfrac{1}{2}(\gamma^2 - 1)\sin 2\varphi\,\frac{\partial^2 u}{\partial x\,\partial y} + \tag{15,13}$$

$$+ \tfrac{1}{4}[\gamma^2 + 1 - (\gamma^2 - 1)\cos 2\varphi]\frac{\partial^2 u}{\partial y^2}$$

fulfilling (15,3) (where $R = 1$), $\varphi = \operatorname{arctg}(y/x)$.

Let x_0, y_0 be a given point $\varepsilon^2 < x_0^2 + y_0^2 < 1$. Using a new coordinate system r, k in a neighbourhood of $[x_0, y_0]$ where r has the direction of the radius vector of $[x_0, y_0]$ and k is the straight line perpendicular to r and passing through x_0, y_0, (15,13) will take the form

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial k^2} + \frac{\gamma^2}{2}\frac{\partial^2 u}{\partial r^2}.$$

The relation $\gamma \to 0$ means that the local diffusion in the direction of r converges to zero and the local diffusion in the direction of k is constant. We cannot directly put $\gamma = 0$ (on $x^2 + y^2 > 1 - T + t$, $0 \leqq t \leqq T$) since we should obtain irregular parabolic equation (condition (A) is not fulfilled). But from (15,12) we obtain $u_\varepsilon(t, x, y; \gamma) \to 1$ if $\gamma \to 0$ on $x^2 + y^2 > 1 - T + t$, $0 \leqq t \leqq T$ and similarly as in Lemma 7 also $u(t, x, y; \gamma) \to 1$ if $\gamma \to 0$ for the same t, x, y. The function $u(t, x, y; \gamma)$ is the solution of (15,13) fulfilling (15,7) (we can assume that γ is a constant for this moment). Hence the surface $x^2 + y^2 = 1 - T + t$ can be actually considered as an adhesive barrier and our problem can be reformulated. Ito equations corresponding to (15,13) are for example

$$\mathrm{d}x = \gamma(t, r)\cos\varphi\,\mathrm{d}w_1 - \sin\varphi\,\mathrm{d}w_2$$
$$\mathrm{d}y = \gamma(t, r)\sin\varphi\,\mathrm{d}w_1 + \cos\varphi\,\mathrm{d}w_2$$

The first two steps are finished i.e. we proved that the unit matrix is not strongly maximal and we suggested a way of constructing B^* in $Q - C$.

Remind that $0 < T < 1$, $C = \{[t, x, y] : x^2 + y^2 < 1 - T + t, \; 0 < t < T\}$, $S = \{[t, x, y] : x^2 + y^2 = 1 - T + t, 0 \leqq t \leqq T\}$, $C_0^0 = \{[x, y] : x^2 + y^2 < 1 - T\}$, $C_T^0 = \{[x, y] : x^2 + y^2 < 1\}$. The third part of the problem will be solved in accordance with Remark 6. First we must transform the region C onto the region $(0, T) \times D$. This transformation has the form $\xi^* = (1 - T + t)^{-1/2} \xi$, $\eta^* = (1 - T + t)^{-1/2} \eta$. Ito stochastic equation $\mathrm{d}\xi = \mathrm{d}w_1$, $\mathrm{d}\eta = \mathrm{d}w_2$ (which corresponds to (15,6) will be transformed (by Ito formula or by (9,1)) onto

$$\text{(15,14)} \qquad \begin{aligned} \mathrm{d}\xi^* &= -\tfrac{1}{2}(1 - T + t)^{-1} \xi^* \, \mathrm{d}t + (1 - T + t)^{-1/2} \, \mathrm{d}w_1 \\ \mathrm{d}\eta^* &= -\tfrac{1}{2}(1 - T + t)^{-1} \eta^* \, \mathrm{d}t + (1 - T + t)^{-1/2} \, \mathrm{d}w_2 \,. \end{aligned}$$

We can apply Theorem 2 to (15,14) in $Q = (0, T) \times D$. With respect to this Theorem and (15,14) we obtain the parabolic equation

$$\text{(15,15)} \quad \frac{\partial u}{\partial t} = -\tfrac{1}{2}(1 - t)^{-1} \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) + \tfrac{1}{2}(1 - t)^{-1} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

We want to show that the bounded solution of (15,15) fulfilling (15,2) ($R = 1$) is convex in x, y for $0 < t < T < 1$. Then there follows by Theorem 2 and Remark 6 that the unit matrix B is strongly maximal with respect to $a \equiv 0$ and to C. The transformation $t = 1 - \mathrm{e}^{-\tau}$ transforms (15,15) to

$$\text{(15,16)} \qquad \frac{\partial u}{\partial t} = -\frac{1}{2} \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) + \frac{1}{2} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

The interval $\langle 0, T \rangle$ is transformed to $\langle 0, -\lg(1 - T) \rangle$. Since the property of u being convex in x, y does not depend on this transformation we shall use the same notation u for the bounded solution of (15,16) fulfilling (15,2) ($R = 1$) in the sequel. First we shall prove that $u(t, x, y)$ is convex in a neighbourhood of the cylinder $t \geqq 0$, $x^2 + y^2 = 1$. This will be done by means of functions $u_\varepsilon(t, x, y)$ and $v_\varepsilon(t, r)$. By Lemma 7, we can approximate $u(t, x, y)$ by $u_\varepsilon(t, x, y)$. The function $u_\varepsilon(t, x, y)$ is the bounded solution of (15,16) fulfilling (15,3) ($R = 1$). We can put $v_\varepsilon(t, r) = u_\varepsilon(t, x, y)$, $r = \sqrt{(x^2 + y^2)}$, $\varepsilon > 0$ again. The function $v_\varepsilon(t, r)$ is the bounded solution of

$$\text{(15,17)} \qquad \frac{\partial v}{\partial t} = \frac{1 - r^2}{2r} \frac{\partial v}{\partial r} + \frac{1}{2} \frac{\partial^2 v}{\partial r^2}$$

fulfilling (15,9).

Let $\psi(z)$ be the solution of $\mathrm{d}\psi/\mathrm{d}z = \psi \exp \tfrac{1}{2}(1 - \psi^2)$, $\psi(0) = 1$. There exists a number $\delta(\varepsilon) < 0$ for every $0 < \varepsilon < 1$ such that $\psi(\delta(\varepsilon)) = \varepsilon$, $\psi'(z) > 0$, $\psi''(z) > 0$ on $\langle \delta(\varepsilon), 0)$, $\psi'(0) = 1$.

By the transformation $v_\varepsilon(t, r) = \theta_\varepsilon(t, z)$, $r = \psi(z)$, equation (15,17) is transformed into

$$\frac{\partial\theta}{\partial t} = \frac{1}{2\psi'^2(z)} \frac{\partial^2\theta}{\partial z^2} \tag{15,18}$$

and $\theta_\varepsilon(t, z)$ is obviously the bounded solution of (15,18) fulfilling $\theta_\varepsilon(0, z) = 0$ for $\delta(\varepsilon) < z < 0$, $\theta_\varepsilon(t, \delta(\varepsilon)) = \theta_\varepsilon(t, 0) = 1$ for $t > 0$. According to Lemma 6 the function $\theta_\varepsilon(t, z)$ is convex in z. Since $\lim_{\varepsilon\to 0} \delta(\varepsilon) = -\infty$ there exists $\theta(t, z) = \lim_{\varepsilon\to 0} \theta_\varepsilon(t, z)$ for every $z \leqq 0$. The function $\theta(t, z)$ is convex in z and bounded. This implies that $\theta(t, z)$ is nondecreasing. Since $v_\varepsilon(t, r) = \theta_\varepsilon(t, z)$ there exists $v(t, r) = \lim_{\varepsilon\to 0} v_\varepsilon(t, r)$ for $0 < r \leqq 1$. Since $(\partial\theta/\partial z)(t, z) = (\partial v/\partial r)(t, r)\,\psi'(z)$, $r = \psi(z)$, $\psi'(z) > 0$, the function $v(t, r)$ is also nondecreasing in r. Since $(\partial^2\theta_\varepsilon/\partial z^2)(t, z) = (\partial^2 v_\varepsilon/\partial r^2)(t, r)\,\psi'^2(z) + (\partial v_\varepsilon/\partial r)(t, r)\,\psi''(z)$, $r = \psi(z)$, $\psi''(z) > 0$ and θ_ε is convex in z the function $v_\varepsilon(t, r)$ is convex in x at all points $[t, r]$ for which $(\partial v_\varepsilon/\partial r)(t, r) \leqq 0$ i.e.

$$\frac{\partial v}{\partial r}(t, r) \geqq 0 \quad \text{for} \quad 0 \leqq t \leqq T, \quad 0 < r \leqq 1 \tag{15,19}$$

$$\frac{\partial^2 v_\varepsilon}{\partial r^2}(t, r) \geqq 0 \quad \text{for} \quad [t, r] \quad \text{for which} \quad \frac{\partial v_\varepsilon}{\partial r}(t, r) \leqq 0\,.$$

To solve the problem whether v_ε is convex also for $\partial v_\varepsilon/\partial r > 0$ we need the following

Proposition. *Let $\psi(z)$ be the solution of*

$$\frac{d^2\psi}{dz^2} = \frac{1 - \psi^2}{\psi}\left(\frac{d\psi}{dz}\right)^2 + a_\varepsilon z \left(\frac{d\psi}{dz}\right)^3, \tag{15,20}$$

$a_\varepsilon = \pi^2(2 - \varepsilon)^{-2}$, $\varepsilon > 0$, *fulfilling the initial conditions* $\psi(0) = 1$, $\psi'(0) = -1$. *Then $\psi(z)$ is defined on an interval* $\langle 0, z_0\rangle$, $z_0 > 0$, $\psi(z_0) = \varepsilon$ *and* $\psi'(z) < 0$ *for* $z \in \langle 0, z_0\rangle$, $\psi''(z) < 0$ *for those* $z > 0$ *for which* $\psi(z) > (a_\varepsilon - 1)^{-1/2}$.

Proof of Proposition. Let G be the region of poins $[\psi, \psi'] : \varepsilon < \psi < 1$, $\psi' < 0$ in R_2. The frontier $\dot{G}$ consists of $\dot{G}_1 = \{[1, \psi'] : \psi' < 0\}$, $\dot{G}_2 = \{[\psi, 0] : \varepsilon \leqq \psi \leqq 1\}$, $\dot{G}_3 = \{[\varepsilon, \psi'] : \psi' < 0\}$. We shall say that a solution $\psi(z)$ is in G for some z or intersects $\dot{G}_i$ if $[\psi(z), \psi'(z)]$ belongs to G for this z or $[\psi(z), \psi'(z)] \in \dot{G}_i$, respectively. The solution $\psi(z)$ enters G for $z > 0$ since $\psi'' < 0$ for $z > 0$, $\psi \geqq 1$, $\psi' < 0$ by (15,20). We shall prove that $\psi(z)$ cannot leave G but by intersecting $\dot{G}_3$ and, simultaneously, it cannot stay in G for all z. Obviously $\psi(z)$ is defined on some maximal interval $\langle 0, z_1)$, $z_1 > 0$ ($z_1 = \infty$ is possible). We assume that $\psi(z)$ does not intersect $\dot{G}_3$ on $\langle 0, z_1)$. Since $\psi(z)$ cannot intersect neither $\dot{G}_1$ for $z > 0$ (there is $\psi' < 0$) nor $\dot{G}_2$ ($\dot{G}_2$ consists of solutions $\psi(z) = \text{const}$ and the unicity conditions are fulfilled) and since we assume that $\psi(z)$ does not intersect $\dot{G}_3$, the solution $\psi(z)$ must stay in G for $z \in \langle 0, z_1)$.

First we eliminate the case $z_1 = \infty$. If we multiply (15,20) by $(\mathrm{d}\psi/\mathrm{d}z)^{-1}$ and integrate, we obtain

$$(15,21) \quad \psi'(z) = -\psi(z) \exp\left\{\frac{1-\psi^2(z)}{2} + a_\varepsilon \int_0^z \eta\, \psi'^2(\eta)\, \mathrm{d}\eta\right\} \quad \text{for} \quad 0 \leqq z < z_1 .$$

As $\varepsilon < \psi(z) < 1$ for $z \geqq 0$, we obtain $\psi'(z) \leqq -\psi(z)$ and finally $\psi(z) \leqq e^{-z}$ for $z \in \langle 0, z_1)$. Certainly it cannot be $z_1 = \infty$.

Secondly we shall prove that $\psi(z)$ must intersect $\dot{G}_3$. Since $[\psi(z), \psi'(z)] \in G$ for $z \in \langle 0, z_1)$, we have $\psi'(z) < 0$ and there exists $\lim_{z \to z_1 -} \psi(z) \geqq \varepsilon$. As $z_1 < \infty$, $[\psi(z), \psi'(z)]$ cannot stay in any bounded set and this implies $\liminf_{z \to z_1 -} \psi'(z) = -\infty$. According to (15,21) this means $\lim_{z \to z_1 -} \psi'(z) = -\infty$. As $\lim_{z \to z_1} \psi(z) \geqq \varepsilon$ the integral $\int_0^{z_1} (1 - \psi^2(\eta)) (\psi(\eta))^{-1} (-\psi'(\eta))^{-1} \mathrm{d}\eta$ converges and $(1 - \psi^2(z)) (\psi(z))^{-1} (-\psi'(z))^{-1}$ is a continuous function on $\langle 0, z_1 \rangle$ if its value for $z = z_1$ is zero. We shall prove

$$(15,22) \qquad 1 - a_\varepsilon z_1^2 + 2 \int_0^{z_1} \frac{1 - \psi^2(\eta)}{\psi(\eta)\,(-\psi'(\eta))}\, \mathrm{d}\eta = 0 .$$

Equation (15,20) can be rewritten in the form

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}z}\right)^{-3} \frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} = a_\varepsilon z + \frac{1 - \psi^2}{\psi} \left(\frac{\mathrm{d}\psi}{\mathrm{d}z}\right)^{-1} .$$

If the last term is considered as a known function, we obtain

$$(15,23) \quad \psi'(z) = -\left(1 - a_\varepsilon z^2 + 2 \int_0^z \frac{1 - \psi^2}{\psi(-\psi')}\, \mathrm{d}\eta\right)^{-1/2} \quad \text{for} \quad 0 \leqq z < z_1 ,$$

$$(15,24) \quad \psi(z) = 1 - \int_0^z \left(1 - a_\varepsilon \xi^2 + 2 \int_0^\xi \frac{1 - \psi^2}{\psi(-\psi')}\, \mathrm{d}\eta\right)^{-1/2} \mathrm{d}\xi \quad \text{for} \quad 0 \leqq z < z_1 .$$

Equation (15,23) implies $1 - a_\varepsilon z^2 + 2 \int_0^z (1 - \psi^2)\, \psi^{-1}(-\psi')^{-1}\, \mathrm{d}\eta > 0$ for $z < z_1$. The left hand side of (15,22) must be nonnegative. In fact, supposing the left hand side of (15,22) is positive then there exists $z_2 < z_1$ such that $1 - a_\varepsilon z^2 + 2 \int_0^z (1 - \psi^2)\, \psi^{-1}(-\psi')^{-1}\, \mathrm{d}\eta \geqq \delta > 0$ for $z \in (z_2, z_1)$ where $\delta > 0$ is some number. With respect to (15,23) it implies $\psi'(z) \geqq -\delta^{-1/2}$ for $z \in (z_2, z_1)$ but this is a contradiction with $\lim_{z \to z_1} \psi'(z) = -\infty$. (15,22) is proved. Using $\varepsilon < \psi(z) \leqq 1$, $\psi'(z) < 0$ for $z \in \langle 0, z_1)$ and (15,24), we obtain

$$\psi(z) \leqq 1 - \int_0^z \left(1 - a_\varepsilon \xi^2 + 2 \int_0^{z_1} \frac{1 - \psi^2}{\psi(-\psi')}\, \mathrm{d}\eta\right)^{-1/2} \mathrm{d}\xi =$$

$$= 1 - \frac{1}{\sqrt{a_\varepsilon}} \arcsin \left[z \sqrt{a_\varepsilon} \left(1 + 2 \int_0^{z_1} \frac{1 - \psi^2}{\psi(-\psi')}\, \mathrm{d}\eta\right)^{-1/2}\right] = 1 - \frac{1}{\sqrt{a_\varepsilon}} \arcsin \frac{z}{z_1}$$

for $0 \leqq z < z_1$. The last equality holds with regard to (15,22). Hence

$$\lim_{z\to z_1-} \psi(z) \leqq 1 - \frac{\pi}{2\sqrt{a_\varepsilon}} = \frac{\varepsilon}{2}.$$

This contradiction proves that $[\psi(z), \psi'(z)]$ must intersect $\dot{G}_3$ on $(0, z_1)$, i.e. there exists $z_0 > 0$ such that $\psi(z_0) = \varepsilon$ and $[\psi(z), \psi'(z)] \in G$ for $z \in (0, z_0)$. Simultaneously we have proved $\psi'(z) < 0$ for $z \in \langle 0, z_0 \rangle$.

Further, we shall deal with $\psi''(z)$. (15,20) can be rewritten by $\psi'' = a_\varepsilon z \psi'^3 [1 + (1 - \psi^2) a_\varepsilon^{-1} \psi^{-1} \psi'^{-1} z^{-1}]$ and since $\lim_{z\to 0^+} (1 - \psi^2)/(a_\varepsilon \psi \psi' z) = -2/a_\varepsilon > -1$ the second derivative $\psi''(z)$ is negative for sufficiently small positive z. Let z_3 be the first number greater than 0 for which $\psi''(z) = 0$. Assume $\psi(z_3) > (a_\varepsilon - 1)^{-1/2}$ then the inequality $\psi'' < \psi'^2 \psi (1 - \psi^2)^{-1} (a_\varepsilon - 1 - \psi^{-2})$ holds on $(0, z_3)$ (its right hand side being positive). This inequality gives $-a_\varepsilon + 1 + \psi^{-2} + (1 - \psi^2) \psi^{-1} \psi'^{-2} \psi'' < 0$ on $(0, z_3)$ and by integration $-a_\varepsilon z_3 + (1 - \psi^2(z_3))/(\psi(z_3)(-\psi'(z_3))) < 0$. This inequality is equivalent to $\psi''(z_3) < 0$ according to (15,20). This contradiction implies $\psi(z_3) \leqq (a_\varepsilon - 1)^{-1/2}$ and Proposition is proved.

We return to equation (15,17) and perform a transformation $v_\varepsilon(t, r) = \theta_\varepsilon(t, z)$, $r = \psi(z)$. The function $\theta_\varepsilon(t, z)$ is the bounded solution of

$$\frac{\partial \theta}{\partial t} = \frac{1}{2\psi'^2(z)} \frac{\partial^2 \theta}{\partial z^2} - \frac{a_\varepsilon}{2} z \frac{\partial \theta}{\partial z}$$

fulfilling $\theta_\varepsilon(0, z) = 0$ for $0 < z < z_0$, $\theta_\varepsilon(t, 0) = \theta(t, z_0) = 1$ for $t > 0$. Applying Lemma 6 we conclude that $\theta_\varepsilon(t, z)$ is a convex function in z. Since $(\partial^2 \theta_\varepsilon / \partial z^2)(t, z) = (\partial^2 v_\varepsilon / \partial r^2)(t, r) \psi'^2(z) + (\partial v_\varepsilon / \partial r)(t, r) \psi''(z)$, $r = \psi(z)$ and $\psi''(z) < 0$ for $\psi(z) > (a_\varepsilon - 1)^{-1/2}$ the function $v_\varepsilon(t, r)$ is convex as the function of r at all points t, r for which $(\partial v_\varepsilon / \partial r)(t, r) \geqq 0$, $r \geqq (a_\varepsilon - 1)^{-1/2}$. (15,19) implies that $v_\varepsilon(t, r)$ is convex as the function of r at all points $[t, r]$, $r \geqq (a_\varepsilon - 1)^{-1/2}$. Hence also the limit $v(t, r)$ is convex as the function of r at all points $[t, r]$, $r \geqq 2(\pi^2 - 4)^{-1/2}$ $(a \to \pi^2/4$ for $\varepsilon \to 0)$. Since $u(t, x, y) = v(t, r)$, $r = \sqrt{(x^2 + y^2)}$ and considering the first part of (15,19) we conclude that $u(t, x, y)$ is convex as the function of x, y for $[t, x, y] : x^2 + y^2 \geqq 4(\pi^2 - 4)^{-1}$.

To prove that $u(t, x, y)$ is convex at all points of $(0, -\lg(1 - T)) \times D$ we put $\lambda(t, x, y) = (\partial^2 u / \partial x^2)(t, x, y)$. By (15,16) the function $\lambda(t, x, y)$ is the solution of

$$\frac{\partial \lambda}{\partial t} = -\frac{1}{2}\left(2\lambda + x \frac{\partial \lambda}{\partial x} + y \frac{\partial \lambda}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial^2 \lambda}{\partial x^2} + \frac{\partial^2 \lambda}{\partial y^2}\right)$$

fulfilling $\lambda(0, x, y) = 0$ for $x^2 + y^2 \leqq \varrho^2$, $\lambda(t, \bar{x}, \bar{y}) = \partial^2 u / \partial x^2$ for $t \geqq 0$, $\bar{x}^2 + \bar{y}^2 = \varrho^2$ where $\varrho = \frac{1}{2} + (\pi^2 - 4)^{-1/2}$. Since we proved that $(\partial^2 u / \partial x^2)(t, \bar{x}, \bar{y}) \geqq 0$ for $\bar{x}^2 + \bar{y}^2 = \varrho^2$ we obtain by the maximum principle $\lambda(t, x, y) = (\partial^2 u / \partial x^2)(t, x, y) \geqq$

$\geqq 0$. Since (15,16) is independent of the rotation of the plane we have proved simultaneously $(\partial^2 u/\partial l^2)(t, x, y) \geqq 0$ for every vector $l \neq 0$, i.e. the function $u(t, x, y)$ is convex as the function of x, y.

APPENDIX

This part is devided into two paragraphs which are devoted to the proofs of Lemma 4 and Lemma 5.

16. The proof of Lemma 4.

Let $u_k(t, y)$, $k = 1, 2$ be the bounded solution of

$$(16,1) \qquad \frac{\partial u}{\partial t} = \sum_{i,j} A_{ij}(t, x) \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \sum_i a_i(t, x) \frac{\partial u}{\partial x_i}$$

fulfilling

$$(16,2) \qquad \lim_{t \to 0^+} u_k(t, x) = \varphi(0, x) \quad \text{for almost all} \quad x \in D\,,$$

$$(16,3) \qquad \lim_{x \to \bar{x}} u_k(t, x) = \varphi(t, \bar{x}) \quad \text{almost everywhere in } S\,.$$

We put $u(t, x) = u_2(t, x) - u_1(t, x)$. The function $u(t, x)$ is the bounded solution of (16,1) fulfilling

$$(16,4) \qquad \lim_{t \to 0^+} u(t, x) = 0 \quad \text{for almost all} \quad x \in D\,,$$

$$(16,5) \qquad \lim_{x \to \bar{x}} u(t, x) = 0 \quad \text{almost everywhere in} \quad S\,.$$

We shall prove $u(t, x) \equiv 0$. On the contrary, let a point $[\tilde{t}, \tilde{x}]$ exist such that $u(\tilde{t}, \tilde{x}) > 0$ (if $u(\tilde{t}, \tilde{x}) < 0$ we consider $-u(t, x)$). Certainly $0 < \tilde{t} \leqq T$. Without loss of generality we can suppose $T = \tilde{t}$. First, we construct a sequence of regions D_m such that $\tilde{x} \in D_0 \subset \bar{D}_0 \subset \ldots \subset D_m \subset \bar{D}_m \subset \ldots$, $\bigcup_m D_m = D$. To every point $\bar{x} \in \dot{D}$ there exists a ball K and local coordinates $y_1, \ldots, y_n$ such that besides assumptions of Definition 2 the following conditions are fulfilled: $\dot{D}_m \cap K$ can be expressed by means of functions $h_m(y_1, \ldots, y_{n-1})$. The domains of definition of h_m contain a neighbourhood K^+ of the origin of y-coordinate system which is part of K^* (K^* and K^+ are independent of m), $\lim_{m \to \infty} h_m = h$, $\lim_{m \to \infty} \partial^2 h_m/\partial y_i \, \partial y_j = \partial^2 h/\partial y_i \, \partial y_j$ uniformly onK^+, $\sup_m |\dot{D}_m|^{(2+\alpha)} < \infty$ where $|\dot{D}_m|^{(2+\alpha)} = \sup_{K^+} |h_m|_{K^+}^{(2+\alpha)}$ and the norm $|h_m|_{K^+}^{(2+\alpha)}$ is defined in [4] Chap. I §1 sect. 2 (1,9). Let ε_m be a monotonically decreasing sequence of positive numbers $\varepsilon_m \to 0$ for $m \to \infty$, $2\varepsilon_0 < T$. The adjoint equation to (16,1) is

$$(16,6) \qquad \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} (A_{ij}(t, x) v) - \sum_i \frac{\partial}{\partial x_i} (a_i(t, x) v) + \frac{\partial v}{\partial t} = 0 .$$

Let G_m be the Green function of (16,6) for the region $(0, T) \times D_m$. Denote $U_m = \{[\varepsilon_m, x] : x \in D_m\} \cup \langle \varepsilon_m, T \rangle \times \dot{D}_m$. If we express G_m as $G_m(t, x) = p(T, \tilde{x}; t, x) - z_m(t, x)$ where $p(\tau, \xi; t, x)$ is the fundamental solution of (16,6) and $z_m(t, x)$ is some solution of (16,6) ($z_m(T, x) = 0$, $z_m(t, x) = -p(T, \tilde{x}; t, x)$ on $\langle 0, T \rangle \times \dot{D}_m$) we obtain by means of (13,1) to (13,3) and by Theorem 5,4 Chap. IV [4] that G_m and $\partial G_m / \partial x_i$ are bounded on U_m independently of m (cf. Remark 9).

For $u(t, x)$ and $G_m(t, x)$ the Green formula

$$(16,7) \; \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[\sum_{j=1}^{n} \left(G_m A_{ij} \frac{\partial u}{\partial x_j} - u A_{ij} \frac{\partial G_m}{\partial x_j} - u G_m \frac{\partial A_{ij}}{\partial x_j} \right) + a_i u G_m \right] = \frac{\partial}{\partial t} (u G_m)$$

holds. Integrating (16,7) over $(\varepsilon_m, T - \varepsilon_m) \times D_m$ we obtain

$$(16,8) \qquad \int_{D_m} u(T - \varepsilon_m, x) \, G_m(T - \varepsilon_m, x) \, dx = \int_{D_m} u(\varepsilon_m, x) \, G_m(\varepsilon_m, x) \, dx - $$
$$- \sum_{i,j=1}^{n} \cos(\nu, x_i) \int_{\varepsilon_m}^{T - \varepsilon_m} \int_{\dot{D}_m} u A_{ij} \frac{\partial G_m}{\partial x_j} \, d\tau \, ds$$

where ν is the outwardly directed normal to $\dot{D}_m$ and ds is the surface element on $\dot{D}_m (S_m = (\varepsilon_m, T - \varepsilon_m) \times \dot{D}_m$, $G_m = 0$ on S_m). Since G_m, $\partial G_m / \partial x_j$, u are bounded and (16,4), (16,5) are fulfilled we get from Lebesgue Theorem that the right hand side of (16,8) converges to zero for $m \to \infty$. As G_m are Green functions ($\lim_{m \to \infty} z_m(T - \varepsilon_m, x) = 0$) the left hand side of (16,8) converges to $u(T, x)$ for $m \to \infty$. This contradiction ($u(T, \tilde{x}) > 0$) implies $u(t, x) \equiv 0$.

17. Now we shall prove Lemma 5.

The following proposition will be needed.

Proposition. *Let $a(x)$, $B^{(i)}(x)$, $i = 1, 2$ be defined, Lipschitz continuous and bounded in the whole R_n and*

$$B^{(1)}(x) \, B^{(1)T}(x) = B^{(2)}(x) \, B^{(2)T}(x) .$$

Denote $Z(x) = B^{(2)}(x) \, B^{(2)T}(x)$. We assume that $Z(x)$ is either positive definite or it *is the zero matrix and that* $a(x) = 0$ *at all points at which* $Z(x)$ *is the zero matrix. If* $x^{(i)}(t)$ *are solutions of*

$$\text{(17,1)} \qquad dx = a(x)\,dt + B^{(i)}(x)\,dw(t)$$

with the same initial value then $x^{(1)}(t), x^{(2)}(t)$ *are equivalent.*

Proof. Let $\hat{D}$ be the component of the open set where $Z(x)$ is positive definite. Let from the beginning $x^{(i)}(0)$ have values from $\hat{D}$ only. We can easily prove that $x^{(i)}(t)$ are $\hat{C}$-processes in $\hat{D}$ (see [6]). It can be done by means of the semi-group of operators associated to $x^{(i)}(t)$ $(T_t^{(i)}(g(x_0)) = E(g(\tilde{x}^{(i)}(t)))$ where $\tilde{x}^{(i)}(t)$ are solutions of (17,1) with a nonstochastic initial value x_0) and by means of the fact that $Z(x) = 0$ and $a(x) = 0$ on the frontier of $\hat{D}$. Since $x^{(i)}(t)$ have the same differential operator, Conclusion 5,24 §6 Chap. 5 [6] implies that $x^{(i)}(t)$ are equivalent in this case. Since for the values x_0 for which $Z(x_0) = 0$ the solution of (17,1) is $x(t) = x_0$ the Proposition is true in the general case, too.

Proof of Lemma 5. Let $u(t, x)$ be the bounded solution of

$$\text{(17,3)} \qquad \frac{\partial u}{\partial t} = \tfrac{1}{2} \sum_{i,j,k} B_{ik}(T - t, x)\, B_{jk}(T - t, x) \frac{\partial^2 u}{\partial x_i\, \partial x_j} + \\ + \sum_i a_i(T - t, x) \frac{\partial u}{\partial x_i}$$

fulfilling

$$\text{(17,4)} \qquad \lim_{t \to 0^+} u(t, x) = 0 \quad \text{for} \quad x \in D\,,$$

$$\text{(17,5)} \qquad \lim_{x \to \bar{x}} u(t, x) = 1 \quad \text{for} \quad t > 0\,, \quad \bar{x} \in \dot{D}\,.$$

Let $G(t, x; \tau, y)$ be the Green function of (17,3) for the region Q. Let $\varphi(x)$ be defined in D, $\varphi(x) = 0$ in a neighbourhood of $\dot{D}$ and let $\varphi(x)$ have Hölder continuous second derivatives. Put $\bar{v}(t, x) = \int G(t, x; 0, y)\, \varphi(y)\, dy$ and $v(t_0, x_0) = E(\varphi(x_\tau(T)))$ where x_τ is the solution of (1,1) with the adhesive barrier $\dot{D}$ (Definition 3) and with the initial condition $x_\tau(T - t_0) = x_0$. We shall prove $v(t, x) = \bar{v}(t, x)$.

α) First, we consider the case that $a(x)$, $B(x)$ do not depend on t. Let $a(x)$, $B^{(1)}(x)$ be an extension of $a(x)$, $B(x)$ onto R_n such that the assumptions of Proposition are fulfilled. Since $Z(x) = B^{(1)}(x)\, B^{(1)T}(x)$ is positive definite or the zero matrix a symmetric positive definite matrix function $B^{(2)}(x)$ exists such that $B^{(2)}(x)\, B^{(2)T}(x) = Z(x)$ and all other assumptions of Proposition are fulfilled. Finally, we find a matrix function $B^{(3)}(x)$ which is uniformly positive definite and uniformly Hölder continuous in the whole R_n and $B^{(3)}(x) = B^{(2)}(x)$ for $x \in D$. Denote by $x^{(i)}(t)$ the solution of (17,1) for $i = 1, 2, 3$ with the initial value $x^{(i)}(T - t_0) = x_0$. By Proposition and by Conclusion 11.13 Chap. 11 [6] the solutions $x^{(i)}(t)$ are equivalent. If $x(t)$ is the part of $x^{(1)}(t)$ in D then $x(t)$ may be considered as the part of $x^{(3)}(t)$ in D. In the case $i = 3$ we can apply Theorems 5.11 and 13.18 from [6]. *These Theorems imply that there exists a transition density of $x(t)$ and this transition density is the Green function of*

$$\frac{\partial u}{\partial t} = \tfrac{1}{2} \sum_{i,j,k} B_{ik}(x)\, B_{jk}(x) \frac{\partial^2 u}{\partial x_i\, \partial x_j} + \sum_i a_i(x) \frac{\partial u}{\partial x_i} \tag{17,6}$$

for the region Q. (The notation in [6] is different, e.g. $p(t, x, y)$ is the Green function. The relation between this and our notation is $p(t, x, y) = G(T - t, x; T, y)$.)

β) Now we pass to the general case when a, B may depend on t. In this case we construct the extension $B^{(3)}(t, x)$ such that

$$\left|B^{(3)}_{ij}(t_1, x) - B^{(3)}_{ij}(t_2, x)\right| \leqq M \max_{x \in \bar{D}, i, j} \left|B_{ij}(t_1, x) - B_{ij}(t_2, x)\right| \tag{17,7}$$

where $M > 0$. We divide the interval $\langle 0, T\rangle$ into m subintervals $\langle kT/m, (k + 1)\, T/m\rangle$ $k = 0, \ldots, m - 1$ for every m. Put $\tilde{a}^{(m)}(t, x) = a(kT/m, x)$, $\tilde{B}^{(m)}(t, x) = B^{(3)}(kT/m, x)$ for $kT/m \leqq t < (k + 1)\, T/m$. Let $\tilde{x}^{(m)}(t)$ be the solution of

$$\mathrm{d}x = \tilde{a}^{(m)}(t, x)\, \mathrm{d}t + \tilde{B}^{(m)}(t, x)\, \mathrm{d}w(t) \tag{17,8}$$

fulfilling the nonstochastic initial value x_0, $x_0 \in D$ $(x^{(m)}(T - t_0) = x_0)$. Let $\mathring{x}^{(m)}(t)$ be the part of $\tilde{x}^{(m)}(t)$ in D and $x^{(m)}_\tau$ the solution of (17,8) with the same initial value and with the adhesive barrier $\dot{D}$. Put $v_m(t_0, x_0) = E\varphi(x^{(m)}_\tau(T))$ $(x^{(m)}_\tau(T - t_0) = x_0)$. Let $\tilde{x}(t)$ be the solution of (1,1) with $B(t, x) = B^{(3)}(t, x)$ fulfilling $\tilde{x}(T - t_0) = x_0$. Since $\tilde{x}(t) - \tilde{x}^{(m)}(t) = \int^t_{T-t_0} (\tilde{a}^{(m)}(\tau, \tilde{x}(\tau)) - \tilde{a}^{(m)}(\tau, \tilde{x}^{(m)}(\tau)))\, \mathrm{d}\tau + \int^t_{T-t_0} (\tilde{B}^{(m)}(\tau, \tilde{x}(\tau)) - \tilde{B}^{(m)}(\tau, \tilde{x}^{(m)}(\tau)))\, \mathrm{d}w(\tau) + \int^t_{T-t_0} (a(\tau, \tilde{x}(\tau)) - \tilde{a}^{(m)}(\tau, \tilde{x}(\tau)))\, \mathrm{d}\tau + \int^t_{T-t_0} B^{(3)}(\tau, \tilde{x}(\tau)) - \tilde{B}^{(m)}(\tau, \tilde{x}(\tau)))\, \mathrm{d}w(\tau)$ we obtain from (6,2) and (6,1) [5] (for $q = 1$, $F(t) = t$) $|||\tilde{x} - \tilde{x}^{(m)}|||_t \leqq M \int^t_{T-t_0} |||\tilde{x} - \tilde{x}^{(m)}|||_\tau\, \mathrm{d}\tau + M \sqrt{\int^t_{T-t_0} |||\tilde{x} - \tilde{x}^{(m)}|||^2_\tau\, \mathrm{d}\tau} + M \max \left|a(t, x) - \tilde{a}^{(m)}(t, x)\right| + M \max \left|B^{(3)}_{ij}(t, x) - \tilde{B}^{(m)}_{ij}(t, x)\right|$. This inequality yields $|||\tilde{x} - \tilde{x}^{(m)}|||_t = \sqrt{E \sup_{\langle T - t_0, t\rangle} \left|\tilde{x}(\tau) - \tilde{x}^{(m)}(\tau)\right|^2} \to 0$ for $m \to \infty$ as $\max_{t,x} \left|a(t, x) - \tilde{a}^{(m)}(t, x)\right| \to 0$, $\max_{t,x} \left|B^{(3)}_{ij}(t, x) - \tilde{B}^{(m)}_{i}(t, x)\right| \to 0$ for $m \to \infty$ (cf. (17,7)).

Denote by $\tau_m(\omega)$ the first exist time of $\tilde{x}^{(m)}$ from D. Since (cf. Remark 9)

$$P\{\omega : \exists\, \{\xi : \tilde{x}(\xi) \notin \bar{D},\ \xi \in \langle \tau(\omega), \tau(\omega) + h\rangle\},\ \tau(\omega) < T\} = P\{\omega : \tau(\omega) < T\} \tag{17,9}$$

for every $h > 0$ where $\tau(\omega)$ is the first exit time of $\tilde{x}$ from D, we obtain $\tau_m(\omega) \to \tau(\omega)$ in probability. Hence $|||x_\tau - x^{(m)}_\tau|||_t \to 0$ for $m \to \infty$. (Processes $\tilde{x}_\tau$ and x_τ are equivalent.) We have proved $v_m(t,x) \to v(t,x)$ for $m \to \infty$. On the other hand, by the definition of v_m and φ we have

$$(17,10) \qquad v_m(t,x) = \int G_m(T-t, x; T, y)\, \varphi(y)\, dy =$$

$$= \int G_m(T-t, x; T-kT/m, y) \int G_m(T-kT/m, y; T, z)\, \varphi(z)\, dz\, dy =$$

$$= \int G_m(T-t, x; T-kT/m, y)\, v_m(kT/m, y)\, dy \quad \text{for} \quad kT/m \leqq t\,,$$

where $G_m(t, x; \tau, y)$ is the transition density of $\mathring{x}^{(m)}(x_\tau^{(m)}$ is the solution of (17,8) with the adhesive barrier $\dot{D}$, $\mathring{x}^{(m)}$ is the part of $\tilde{x}^{(m)}$ in D). Recalling the result of section α) we find that $v_m(t,x)$ is the solution of

$$(17,11) \qquad \frac{\partial v}{\partial t} = \tfrac{1}{2} \sum_{i,j,k} \tilde{B}_{ik}^{(m)}(T-t,x)\, \tilde{B}_{jk}^{(m)}(T-t,x) \frac{\partial^2 v}{\partial x_i\, \partial x_j} +$$

$$+ \sum_i a_i^{(m)}(T-t,x) \frac{\partial v}{\partial x_i}$$

for $kT/m < t \leqq (k+1)\,T/m$ fulfilling $v_m(t,\bar{x}) = 0$ for $\bar{x} \in \dot{D}$ the initial values $v_m(Tk/m, x)$ being given by the solution of (17,11) which was determined in the preceding interval $\langle (k-1)\,T/m, kT/m \rangle$. Finally $v_m(0,x) = \varphi(x)$. The function $\Delta_m(t,x) = \bar{v}(t,x) - v_m(t,x)$ is a continuous function which is on every $(kT/m, (k+1)\,T/m)$ the solution of

$$\frac{\partial \Delta}{\partial t} = \tfrac{1}{2} \sum_{i,j,k} \tilde{B}_{ik}^{(m)}(T-t,x)\, \tilde{B}_{jk}^{(m)}(T-t,x) \frac{\partial^2 \Delta}{\partial x_i\, \partial x_j} + \sum_i \tilde{a}_i^{(m)}(T-t,x) \frac{\partial \Delta}{\partial x_i} +$$

$$+ \tfrac{1}{2} \sum_{i,j,k} \left(B_{ik}^{(3)}(T-t,x)\, B_{jk}^{(3)}(T-t,x) - \tilde{B}_{ik}^{(m)}(T-t,x)\, \tilde{B}_{jk}^{(m)}(T-t,x)\right) \frac{\partial^2 \bar{v}}{\partial x_i\, \partial x_j} +$$

$$+ \sum_i \left(a_i(T-t,x) - \tilde{a}_i^{(m)}(T-t,x)\right) \frac{\partial \bar{v}}{\partial x_i}$$

with $\Delta_m(t,x) = 0$ for $\bar{x} \in \dot{D}$ and $\Delta_m(0,x) = 0$ for $x \in D$. Recalling the assumption about $\varphi(x)$, we find (Theorem 5.2 §5 Chap. IV [4]) that $\partial^2 \bar{v}/\partial x_i\, \partial x_j$ and $\partial \bar{v}/\partial x_i$ are bounded. It implies $\Delta_m(t,x) \to 0$ for $m \to \infty$. We have proved $v(t,x) = \bar{v}(t,x)$. We deduce by the maximum principle that the same is valid if $\varphi(x)$ is continuous, $\varphi(\bar{x}) = 0$ for $\bar{x} \in \dot{D}$ only. Put $\bar{u}(t,x) = 1 - \bar{v}(t,x)$, $u(t,x) = 1 - v(t,x)$. The function $\bar{u}(t,x)$ is again the solution of (17,3) fulfilling (17,5) and the initial condition $u(0,x) = \psi(x)$ where $\psi(x)$ is continuous and $\psi(\bar{x}) = 1$ for $\bar{x} \in \dot{D}$. Analogously $u(t_0, x_0) = E(\psi(\tilde{x}_\tau(T)))$ (where $\tilde{x}(t)$ is the solution of (1,1) with $\tilde{x}(T-t_0) = x_0$). Again $u(t,x) = \bar{u}(t,x)$. Denote by $\hat{\psi}(x)$ the function: $\hat{\psi}(x) = 0$ for $x \in D$ and $\hat{\psi}(\bar{x}) = 1$ for $\bar{x} \in \dot{D}$.

The function $\hat{\psi}(x)$ can be approximated by functions $\psi_m(x)$, $\psi_m(x)$ continuous, $0 \leqq \leqq \psi_m(x) \leqq 1$, $\psi_m(x) = 0$ outside of $\mathcal{O}(2^{-m+1}, \dot{D})$ (2^{-m+1} – neighbourhood of $\dot{D}$) and $\psi_m(x) = 1$ in $\mathcal{O}(2^{-m}, \dot{D})$. The corresponding solutions $\bar{u}_m(t, x)$ converge as in the proof of Lemma 2 to the solution $\bar{u}(t, x)$ given by (17,4) and (17,5). The values $u_m(T, x)$ converge to the probability $P(B, a, \delta(x), Q)$. From this immediately follows that Lemma 5 is true for every density $f(x)$ in D.

Remark 9. Throughout the proof of Lemma 4 we needed to choose the number c in (5,13) [4] Chap. IV independently of m. From the proof of Theorem 5,4 Chap. IV there follows that it is possible. We divide the regions D_m on $\Omega^{(k)m}$, $\omega^{(k)m}$ as in [4]. For $k \in \mathfrak{M}$ these regions do not change and for $k \in \mathfrak{N}$ the functions h is replaced by h_m only. Since the functions $\zeta^{(k)}(x)$ may be constructed so that they are independent of m the parameter m occurs in the proof of (5,13) and (5,14) only in $Z_k^{(m)}$. However, the frontiers $\dot{D}_m$ were chosen just in the manner that all necessary estimates of $Z_k^{(m)}$ are independent of m.

The successive construction of $B^{(i)}(x)$ in section α) of the proof of Lemma 5 is used since it would be impossible to construct directly $B^{(3)}(x)$ which is uniformly positive definite, Hölder continuous and $B^{(3)}(x) = B(x)$ on D.

The proof of (17,9) is a modification of the proof of Lemma 1 §6 Chap. VIII [7].

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