Ladislav Bican

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SOME PROPERTIES OF COMPLETELY DECOMPOSABLE TORSION FREE ABELIAN GROUPS

LADISLAV BICAN, Praha

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In this paper there are proved some theorems giving the solution of two following problems:

1. To find the most general sufficient conditions for the type set of a given completely decomposable torsion free abelian group H such that $G \simeq H$ whenever $H \subseteq G$ or $G \subseteq H$ and the embedding of H in G or G in H, respectively, has a given property.

2. For a completely decomposable torsion free abelian group H with the type set of given properties to find the necessary and sufficient conditions for the embedding of H in G or G in H such that $G \simeq H$, whenever H or G is embedded in G or H, respectively, under these conditions.

For this reason, three notions generalizing the boundedness of the factor-group G/H are introduced. Some theorems from [6] and [8] are simple consequences of the theorems solving the problems mentioned above. Further consequences are, for example, some new criteria for the complete decomposability of a torsion free abelian group.

By the word "group" we shall always mean an additively written abelian group. If M is a set of elements of a torsion free group G then $\{M\}_*^G$ denotes the pure closure of M in G. If τ is a height, then $\hat{\tau}$ will be the type to which the height τ belongs. $\hat{R}$ denotes the greatest element of the lattice of all types. If G is a torsion free group, then $\mathfrak{T}(G)$ denotes the set of types of all elements from G. By $T(G)$ we shall denote the set of the types of all direct summands J_α of the completely decomposable group $G = \sum_{d,\, \alpha \in A} J_\alpha$. The p-height of an element g of the group G is denoted by $h_p^G(g)$, and the type of the element g in the group G is denoted by $\hat{\tau}^G(g)$. A group G is called p-divisible if the equation $p^k x = g$ is solvable in G for all $g \in G$ and all positive integers k, p-reduced if it contains no p-divisible subgroup.

In general, we shall adopt the notation used in [1].

Definition 1. Let H be a subgroup of a torsion free group G. We say that H is *regular in* G if the factor-group $S/S \cap H$ is finite for every rank one subgroup S pure in G.

Definition 2. Let H be a subgroup of a torsion free group G. We say that H is *strongly regular in* G if the factor-group $S/S \cap H$ is finite for every rank finite subgroup S pure in G.

Definition 3. Let H be a subgroup of a torsion free group G. We say that H is *fully regular in* G if the factor-group $S/\{S \cap H, T\}$ is finite for every two subgroups $T \subseteq S$ pure in G such that S/T is of a finite rank.

Remark. It is easy to see that if H is fully regular in G or strongly regular in G, then H is strongly regular or regular in G, respectively. Wang in his paper [13] has introduced the notion of the regularity of a subgroup H in a group G. This notion together with the condition that G/H is torsion is equivalent to the notion introduced by Definition 1.

Lemma 1. *Let G be a torsion free group containing a subgroup $H = H_1 \dotplus H_2$ where H_2 is divisible. If G/H is a torsion group then*

$$G = \{H_1\}_*^G \dotplus H_2 \,. \tag{1}$$

Proof. It is easy to see that $\{H_1\}_*^G \cap H_2 = 0$. On the other hand, to an arbitrary $g \in G$ there exists a non-zero integer m such that $mg \in H$, i.e. $mg = h_1 + h_2$, $h_i \in H_i$, $i = 1, 2$. The divisibility of H_2 implies the existence of an element $h_2' \in H_2$, $mh_2' = h_2$. Then $g - h_2' \in \{H_1\}_*^G$ because $m(g - h_2') = h_1$. Now $g = (g - h_2') + h_2'$ and the proof is finished.

Theorem 1. *Let G be a torsion free group containing a completely decomposable subgroup H such that*

(α) *$T(H)$ satisfies the maximum condition,*

(β) *for any two incomparable types $\hat{\tau}_1, \hat{\tau}_2$ from $T(H)$ there is $\hat{\tau}_1 \vee \hat{\tau}_2 = \hat{R}$.*
If H is fully regular in G then $G \simeq H$.

Proof. We shall prove this theorem in several steps. Due to Lemma 1 we may restrict ourselves to a reduced group H.

a) Let $H = \sum_{d,\, \alpha \in A} J_\alpha$. Let us decompose the set A into disjoint classes $A(\hat{\tau})$ in the following way:

$$\alpha \in A(\hat{\tau}) \Leftrightarrow \hat{\tau}(J_\alpha) = \hat{\tau} \,. \tag{2}$$

We define an arbitrary well-ordering $\prec_{\hat{\tau}}$ on any set $A(\hat{\tau})$ and then partial ordering $\prec$

on the set A as follows:

$$\text{(3)} \qquad \alpha \prec \beta \Leftrightarrow \text{either } \alpha \in A(\hat{\tau}), \quad \beta \in A(\hat{\sigma}) \quad \text{and} \quad \hat{\tau} > \hat{\sigma}, \qquad \text{or} \quad \alpha, \beta \in A(\hat{\tau}) \quad \text{and} \quad \alpha \prec_{\hat{\tau}} \beta .$$

It is easy to see that the set A with the relation $\prec$ satisfies the minimum condition and that there holds

$$\text{(4)} \qquad \alpha \prec \beta \Rightarrow \hat{\tau}(J_\alpha) \geqq \hat{\tau}(J_\beta) .$$

For an arbitrary $\alpha \in A$ we shall introduce the following notation:

$$H_\alpha = \sum_{\beta \prec \alpha} {}_d J_\beta , \quad \tilde{H}_\alpha = H_\alpha \dotplus J_\alpha , \quad G_\alpha = \{H_\alpha\}_*^G , \quad \tilde{G}_\alpha = \{\tilde{H}_\alpha\}_*^G .$$

Evidently

$$\text{(5)} \qquad \alpha \prec \beta \Leftrightarrow \tilde{G}_\alpha \subseteq G_\beta .$$

b) First we shall prove that G_α is a direct summand of $\tilde{G}_\alpha$ for every $\alpha \in A$. There is $H_\alpha \cap J_\alpha = 0$ and hence $G_\alpha \cap J_\alpha = 0$, too, according to the torsion free character of G. Thus we can put $K = G_\alpha \dotplus J_\alpha$. If $g_1, g_2 \in \tilde{G}_\alpha \dot{-} G_\alpha$ are arbitrary elements, then there exists a non-zero integer m such that $mg_i \in \tilde{H}_\alpha \dot{-} H_\alpha$, $i = 1, 2$. In view of the fact that $\tilde{H}_\alpha / H_\alpha$ is of rank one, there exist non-zero integers λ_1, λ_2 such that $\lambda_1 m g_1 + \lambda_2 m g_2 \in H_\alpha \subseteq G_\alpha$. From this there follows immediately that the factor-group $\tilde{G}_\alpha / G_\alpha$ is of rank one. By Definition 3 the factor-group

$$\text{(6)} \qquad \tilde{G}_\alpha / \{\tilde{G}_\alpha \cap H, G_\alpha\} = \tilde{G}_\alpha / \{\tilde{H}_\alpha, G_\alpha\} = \tilde{G}_\alpha / K$$

is finite, so that $m . (\tilde{G}_\alpha / G_\alpha) \subseteq K / G_\alpha$, for a suitable non-zero integer m. Hence the factor-groups $\tilde{G}_\alpha / G_\alpha$ and K / G_α have the same type, such that the following isomorphisms are true:

$$\text{(7)} \qquad \tilde{G}_\alpha / G_\alpha \simeq K / G_\alpha \simeq J_\alpha .$$

If $x \in H$ is an arbitrary element then the factor-group $\{x\}_*^G / \{x\}_*^H$ is finite by Definition 3, hence

$$\text{(8)} \qquad \hat{\tau}^H(x) = \hat{\tau}^G(x) \quad \text{for an arbitrary} \quad x \in H .$$

The inclusion $\tilde{H}_\alpha \subseteq K \subseteq \tilde{G}_\alpha$ implies that to every $x \in K$ there exists a non-zero integer m such that $mx \in \tilde{H}_\alpha$. Then by (8) there holds $\hat{\tau}^G(x) = \hat{\tau}^{\tilde{G}_\alpha}(x) \geqq \hat{\tau}^K(x) = \hat{\tau}^K(mx) \geqq \hat{\tau}^{\tilde{H}_\alpha}(mx) = \hat{\tau}^H(mx) = \hat{\tau}^G(x)$ and hence

$$\text{(9)} \qquad \hat{\tau}^K(x) = \hat{\tau}^G(x) \quad \text{for all} \quad x \in K .$$

Now let $x \in \tilde{G}_\alpha \dot{-} G_\alpha$ be an arbitrary element. By (6) there exists a non-zero integer m such that $mx \in K$ and mx has a non-zero component in J_α (otherwise mx and consequently x belongs to G_α, contrary to our hypothesis). Then by (9) and (4) there is

$\hat{\tau}^G(x) = \hat{\tau}^G(mx) = \hat{\tau}^K(mx) = \hat{\tau}(J_\alpha)$. From this and from (7) there follows that all conditions of Baer's lemma (see Lemma 46.3 in [1]) are fulfilled, so that

(10) $$\tilde{G}_\alpha = G_\alpha \dotplus I_\alpha$$

for a suitable subgroup I_α of $\tilde{G}_\alpha$.

c) Now we shall prove that if p is an arbitrary prime and α_1, α_2 two incomparable elements from A then at least one of the groups $\tilde{H}_{\alpha_1}, \tilde{H}_{\alpha_2}$ is p-divisible. In fact, if $\tilde{H}_{\alpha_1}$ is not p-divisible, then the incomparability of the elements α_1, α_2 (in relation $\prec$) together with (3) implies the incomparability of the types $\hat{\tau}(J_{\alpha_1}), \hat{\tau}(J_{\alpha_2})$, so that $\tilde{H}_{\alpha_2}$ is p-divisible by hypothesis (β).

d) Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be elements from A incomparable to each other. If we put $A_i = \{\beta \in A; \beta \leqq \alpha_i\}$, $i = 1, 2, \ldots, k$ and $A^* = \bigcup_{i=1}^{k} A_i$ then

(11) $$\sum_{\beta \in A^*}{}_d J_\beta = \tilde{H}_{\alpha_1} \dotplus \tilde{H}_{\alpha_2} \dotplus \ldots \dotplus \tilde{H}_{\alpha_k},$$

and

(12) $$\{\tilde{H}_{\alpha_1} \dotplus \tilde{H}_{\alpha_2} \dotplus \ldots \dotplus \tilde{H}_{\alpha_k}\}_*^G = \tilde{G}_{\alpha_1} \dotplus \tilde{G}_{\alpha_2} \dotplus \ldots \dotplus \tilde{G}_{\alpha_k}.$$

(11) will be proved by showing that the sets $A_1, A_2, \ldots, A_k$ are mutually disjoint. Let us suppose that $\beta \in A_i \cap A_j$, $1 \leqq i, j \leqq k$, $i \neq j$. Then $\beta \prec \alpha_i$, $\beta \prec \alpha_j$ so that $\hat{\tau}(J_\beta) \geqq \hat{\tau}(J_{\alpha_i})$ and $\hat{\tau}(J_\beta) \geqq \hat{\tau}(J_{\alpha_j})$ by (4). From the incomparability of α_i, α_j (in $\prec$) the incomparability of the type $\hat{\tau}(J_{\alpha_i}), \hat{\tau}(J_{\alpha_j})$ follows by (3), so that $\hat{\tau}(J_\beta) = \hat{R}$ by Hypothesis (β). But H is assumed reduced, which is a contradiction.

We proceed now to prove (12). Evidently $\{\tilde{G}_{\alpha_1}, \tilde{G}_{\alpha_2}, \ldots, \tilde{G}_{\alpha_k}\} \subseteq \{\tilde{H}_{\alpha_1} \dotplus \tilde{H}_{\alpha_2} \dotplus \ldots \dotplus \tilde{H}_{\alpha_k}\}_*^G$. Further, if $\sum_{i=1}^{k} g_i = 0$, $g_i \in \tilde{G}_{\alpha_i}$, $i = 1, 2, \ldots, k$, then $mg_i \in \tilde{H}_{\alpha_i}$ for a suitably chosen non-zero integer m, and $\sum_{i=1}^{k} mg_i = 0$. From (11) we get $mg_i = 0$ and hence according to the torsion free character of G there is $g_i = 0$ $i = 1, 2, \ldots, k$. This shows the independence of $\tilde{G}_{\alpha_1}, \tilde{G}_{\alpha_2}, \ldots, \tilde{G}_{\alpha_k}$ in G. It remains to prove the inclusion $\{\tilde{H}_{\alpha_1} \dotplus \tilde{H}_{\alpha_2} \dotplus \ldots \dotplus \tilde{H}_{\alpha_k}\}_*^G \subseteq \tilde{G}_{\alpha_1} \dotplus \tilde{G}_{\alpha_2} \dotplus \ldots \dotplus \tilde{G}_{\alpha_k}$. If $g \in \{\tilde{H}_{\alpha_1} \dotplus \tilde{H}_{\alpha_2} \dotplus \ldots \dotplus \tilde{H}_{\alpha_k}\}_*^G$ is an arbitrary element then for a suitable non-zero integer m there is $mg \in \tilde{H}_{\alpha_1} \dotplus \tilde{H}_{\alpha_2} \dotplus \ldots \dotplus \tilde{H}_{\alpha_k}$, i.e.

(13) $$mg = h_1 + h_2 + \ldots + h_k, \quad h_i \in \tilde{H}_{\alpha_i}, \quad i = 1, 2, \ldots, k.$$

Let $m = p_1^{l_1} \cdot p_2^{l_2} \cdot \ldots \cdot p_s^{l_s}$ be the canonical decomposition of m. Using the induction by s we shall prove that $g \in \tilde{G}_{\alpha_1} \dotplus \tilde{G}_{\alpha_2} \dotplus \ldots \dotplus \tilde{G}_{\alpha_k}$. For $s = 1$ we can assume, without loss of generality, that $\tilde{H}_{\alpha_1}$ is not p_1-divisible. By c) there exist elements $g_i = \tilde{H}_{\alpha_i}$, $i = 2, 3, \ldots, k$ such that $p_1^{l_1} g_i = h_i$, $i = 2, 3, \ldots, k$. If we put $g_1 = g - g_2 - \ldots - g_k$ then $g_1 \in \tilde{G}_{\alpha_1}$, because $p_1^{l_1} g_1 = h_1$. The proof of this case is complete.

Now let s be a positive integer, $s > 1$ and let us assume that for all $t < s$ our assertion is true. For the sake of simplicity we may assume again that $p_1, p_2, \ldots, p_r$, $r > 0$, are all primes dividing m and such that $\tilde{H}_{\alpha_1}$ is not p_i-divisible, $i = 1, 2, \ldots, r$. If we put $m_1 = p_1^{l_1} \cdot p_2^{l_2} \cdot \ldots \cdot p_r^{l_r}$, $m_2 = p_{r+1}^{l_{r+1}} \cdot \ldots \cdot p_s^{l_s}$, then $m = m_1 \cdot m_2$ and $(m_1, m_2) = 1$. From the part c), the existence of the elements $h'_i \in \tilde{H}_{\alpha_i}$, $i = 2, 3, \ldots, k$, such that $m_1 h'_i = h_i$, $i = 2, 3, \ldots, k$ follows easily. If we put

$$(14) \qquad g'_1 = m_2 g - h'_2 - \ldots - h'_k$$

then $m_1 g'_1 = h_1$ so that $g'_1 \in \tilde{G}_{\alpha_1}$. The choice of the primes $p_1, p_2, \ldots, p_r$ implies the m_2-divisibility of $\tilde{H}_{\alpha_1}$. Hence there exists an element $h'_1 \in \tilde{H}_{\alpha_1}$, $m_2 h'_1 = h_1$. The integers m_1, m_2 are relatively prime so that there exist integers u, v satisfying the relation $m_1 u + m_2 v = 1$. If we put $g_1 = u h'_1 + v g'_1$ then $g_1 \in \tilde{G}_{\alpha_1}$ and $m_2 g_1 = m_2 u h'_1 + m_2 v g'_1 = u h_1 + m_2 v g'_1 = m_1 u g'_1 + m_2 v g'_1 = g'_1$. Hence by (14) there is

$$(15) \qquad m_2(g - g_1) = h'_2 + h'_3 + \ldots + h'_k\,, \quad h'_i \in \tilde{H}_{\alpha_i} \in \tilde{H}_{\alpha_i}\,, \quad i = 2, 3, \ldots, k$$

and formula (12) follows now by the induction hypothesis.

e) Now we shall prove that the subgroups I_α, $\alpha \in A$ are in G independent. Let us suppose that

$$(16) \quad i_1 + i_2 + \ldots + i_k = 0\,, \quad i_1 \in I_{\alpha_1}, i_2 \in I_{\alpha_2}, \ldots, i_k \in I_{\alpha_k}\,, \quad \alpha_i \neq \alpha_j \quad \text{for} \quad i \neq j\,.$$

Let (without loss of generality) $\alpha_1, \alpha_2, \ldots, \alpha_l$, $l \leqq k$ be all the maximal elements of the set $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ (in the sense of partial order $\prec$ of the set A). Denoting by g_i, $i = 1, 2, \ldots, l$ the sum of all i_j from (16) belonging to $\tilde{G}_{\alpha_i}$, $i = 1, 2, \ldots, l$, we can rewrite (16) in the form

$$(17) \qquad g_1 + g_2 + \ldots + g_l = 0\,, \quad g_i \in \tilde{G}_{\alpha_i}\,, \quad i = 1, 2, \ldots, l\,.$$

Then by (12) $g_i = 0$, $i = 1, 2, \ldots, l$. Thus we can assume that (16) holds and $\alpha_1 \succ \alpha_i$, $i = 2, 3, \ldots, k$ and $i_1 \neq 0$. Now by (5) there is $(i_2 + i_3 + \ldots + i_k) \in G_{\alpha_1}$ and hence (16) contradicts to (8).

f) It remains to prove that G is generated by I_α, $\alpha \in A$ and that $G \simeq H$. Let us denote $G^* = \sum_{d,\, \alpha \in A} I_\alpha$ and let us assume that $G \dot{-} G^* \neq \emptyset$. Let $g \in G \dot{-} G^*$ be an arbitrary element. From the periodicity of the factor-group G/H there follows the existence of a non-zero integer m such that $mg \in H$. In view of the form of H it is clear that mg may be written in the form (13). Hence by (12) we can write

$$(18) \qquad g = g_1 + g_2 + \ldots + g_k\,, \quad g_i \in \tilde{G}_{\alpha_i}\,, \quad i = 1, 2, \ldots, k\,.$$

Now it is clear that the set M of all elements $g \in G \dot{-} G^*$ lying in some $\tilde{G}_\alpha$ is non-void. Due to the fact that the set A (with $\prec$) satisfies the minimum condition, we may associate with any $g \in M$ an element $\alpha_g \in A$ such that $g \in \tilde{G}_{\alpha_g}$ but $g \notin \tilde{G}_\beta$ for $\beta \prec \alpha_g$.

The set $\{\alpha_g, g \in M\}$ contains a minimal element γ. Let g be that element from M for which $\alpha_g = \gamma$ (if those g are several, we take any one of them). By (10) we have $g = g_\gamma + i_\gamma$, $g_\gamma \in G_\gamma$, $i_\gamma \in I_\gamma$. For a suitable non-zero integer m there holds

$$(19) \qquad mg_\gamma = h_1 + h_2 + \ldots + h_k\,, \quad h_i \in \tilde{H}_{\alpha_i}\,, \quad \alpha_i \prec \gamma\,, \quad i = 1, 2, \ldots, k$$

so that by (12) there holds

$$(20) \qquad g_\gamma = g_1 + g_2 + \ldots + g_k\,, \quad g_i \in \tilde{G}_{\alpha_i}\,, \quad i = 1, 2, \ldots, k\,.$$

In view of the choice of γ there is $g_i \in G^*$, $i = 1, 2, \ldots, k$ and hence $g_\gamma \in G^*$ and $g \in G^*$, which is a contradiction proving that $G = G^*$.

By (7) and (10) there is $I_\alpha \simeq J_\alpha$ for any $\alpha \in A$ so that $G \simeq H$ and the proof is now complete.

Now we shall proceed to the "dual" theorem.

Lemma 2. *Let $G = G_1 \dotplus G_2$ be a torsion free group, G_2 being divisible. If H is regular in G then*

$$(21) \qquad H = (G_1 \cap H) \dotplus G_2\,.$$

Proof. Let $g \in G_2$ be an arbitrary element. In view of the regularity of H in G there is $mg \in H$ for a suitable non-zero integer m. The factor-group $\{mg\}_*^G/\{mg\}_*^H$ is finite by Definition 1 so that $\{mg\}_*^G \simeq \{mg\}_*^G$. Then $\{mg\}_*^H$ is divisible, $g \in H$ and hence $G_2 \subseteq H$. Our lemma now easily follows.

Theorem 2. *Let G be a completely decomposable torsion free group such that*

(α) $T(G)$ satisfies the maximum condition,

(β) for any two incomparable types τ_1, τ_2 from $T(G)$ there is $\hat{\tau}_1 \vee \hat{\tau}_2 = \hat{R}$.

If H is regular in G, then $G \simeq H$.

Proof. In view of Lemma 2 we may restrict ourselves to a reduced group G.

a) Let $G = \sum_{d\, \alpha\in A} J_\alpha$. Similarly as in the proof of the preceding theorem we shall define a partial order $\prec$ on the set A. Especially we shall use the formulae (3) and (4). For an arbitrary $\alpha \in A$ we shall introduce the following notation: $G_\alpha = \sum_{d\, \beta\prec\alpha} J_\beta$, $\tilde{G}_\alpha = G_\alpha \dotplus J_\alpha$, $H_\alpha = G_\alpha \cap H$, $\tilde{H}_\alpha = \tilde{G}_\alpha \cap H$. Clearly

$$(22) \qquad \alpha \prec \beta \Leftrightarrow \tilde{H}_\alpha \subseteq H_\beta\,.$$

b) We shall prove now that $H_\alpha \subsetneqq \tilde{H}_\alpha$ for any $\alpha \in A$. In fact, for $H_\alpha = \tilde{H}_\alpha$ there is $J_\alpha \simeq \tilde{G}_\alpha/G_\alpha \simeq \tilde{G}_\alpha/\tilde{H}_\alpha/G_\alpha/\tilde{H}_\alpha$, where on the right hand there is a periodical group, because $\tilde{G}_\alpha/\tilde{H}_\alpha = \tilde{G}_\alpha/\tilde{G}_\alpha \cap H \simeq \{\tilde{G}_\alpha, H\}/H \subseteq G/H$. This contradicts to the torsion free character of J_α.

c) We proceed to show that H_α is a direct summand of $\tilde{H}_\alpha$ for any $\alpha \in A$. The factor-group $\{x\}^G_*/\{x\}^H_*$ is finite for any $x \in H$, hence $\hat{\tau}^H(x) = \hat{\tau}^G(x)$. For an arbitrary element $g \in \tilde{H}_\alpha \dot{-} H_\alpha$ there is $g = g_1 + g_2$, $g_1 \in G_\alpha$, $g_2 \in J_\alpha$, $g_2 \neq 0$ and $\hat{\tau}^H(g) = \hat{\tau}^G(g) = \hat{\tau}^G(g_1) \cap \hat{\tau}^G(g_2) = \hat{\tau}^G(g_2) = \hat{\tau}(J_\alpha) = \hat{\tau}$ by (4). Further, $\tilde{H}_\alpha/H_\alpha = \tilde{G}_\alpha \cap H/G_\alpha \cap \cap H = \tilde{G}_\alpha \cap H/G_\alpha \cap \tilde{G}_\alpha \cap H \simeq \{G_\alpha, \tilde{G}_\alpha \cap H\}/G_\alpha \subseteq \tilde{G}_\alpha/G_\alpha \simeq J_\alpha$, hence $\tilde{H}_\alpha/H_\alpha$ is a group of rank one and $\hat{\tau}(\tilde{H}_\alpha/H_\alpha) \leqq \hat{\tau}$. On the other hand, $\tilde{H}_\alpha/H_\alpha$ is torsion free, hence H_α is pure in $\tilde{H}_\alpha$ and $\hat{\tau}(\tilde{H}_\alpha/H_\alpha) \geqq \hat{\tau}$. Using Baer's lemma (46.3 in [1]) we get

$$\tilde{H}_\alpha = H_\alpha \dotplus I_\alpha \quad \text{for any} \quad \alpha \in A\,. \tag{23}$$

d) If p is an arbitrary prime and α_1, α_2 two incomparable elements from A, then at least one of the groups $\tilde{G}_{\alpha_1}, \tilde{G}_{\alpha_2}$ is p-divisible. The proof of this fact runs on the same lines as the proof of c) in Theorem 1.

e) If p is an arbitrary prime and $\alpha \in A$ an arbitrary element, then the p-divisibility of $\tilde{G}_\alpha$ implies the p-divisibility of $\tilde{H}_\alpha$. Evidently, for an arbitrary $h \in \tilde{H}_\alpha$ the factor group $\{h\}^G_*/\{h\}^H_*$ is finite, so that $\{h\}^G_* \simeq \{h\}^G_*$ and the assertion follows.

f) Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be elements from A incomparable to each other. Let us put $A_i = \{\beta \in A;\ \beta \preceq \alpha_i\}$, $i = 1, 2, \ldots, k$ and $A^* = \bigcup_{i=1}^{k} A_i$. Then

$$\sum_{\beta \in A^*}{}_d\, J_\beta = \tilde{G}_{\alpha_1} \dotplus \tilde{G}_{\alpha_2} \dotplus \ldots \dotplus \tilde{G}_{\alpha_k}\,, \tag{24}$$

and

$$(\tilde{G}_{\alpha_1} \dotplus \tilde{G}_{\alpha_2} \dotplus \ldots \dotplus \tilde{G}_{\alpha_k}) \cap H = \tilde{H}_{\alpha_1} \dotplus \tilde{H}_{\alpha_2} \dotplus \ldots \dotplus \tilde{H}_{\alpha_k}\,. \tag{25}$$

The proof of (24) is the same as the proof of (11). Proceeding to (25) let us note that $\tilde{H}_{\alpha_i} = \tilde{G}_{\alpha_i} \cap H \subseteq (\tilde{G}_{\alpha_1} \dotplus \tilde{G}_{\alpha_2} \dotplus \ldots \dotplus \tilde{G}_{\alpha_k}) \cap H$ implies the inclusion $\{\tilde{H}_{\alpha_1}, \tilde{H}_{\alpha_2}, \ldots \ldots, \tilde{H}_{\alpha_k}\} \subseteq (\tilde{G}_{\alpha_1} \dotplus \tilde{G}_{\alpha_2} \dotplus \ldots \dotplus \tilde{G}_{\alpha_k}) \cap H$. Further, the subgroups $\tilde{H}_{\alpha_i}$, $i = 1, 2, \ldots, k$ are obviously independent in H (due to $\tilde{H}_{\alpha_i} \subseteq \tilde{G}_{\alpha_i}$ and (24)). Now if $h \in (\tilde{G}_{\alpha_1} \dotplus \dotplus \tilde{G}_{\alpha_2} \dotplus \ldots \dotplus \tilde{G}_{\alpha_k}) \cap H$ is an arbitrary element, then we may write

$$h = g_1 + g_2 + \ldots + g_k\,, \quad g_i \in \tilde{G}_{\alpha_i}\,, \quad i = 1, 2, \ldots, k\,. \tag{26}$$

From the periodicity of G/H the existence of a non-zero integer m such that $mg_i \in \in \tilde{H}_{\alpha_i}$, $i = 1, 2, \ldots, k$ follows. Let $m = p_1^{l_1} \cdot p_2^{l_2} \cdot \ldots \cdot p_s^{l_s}$ be the canonical decomposition of m. Using the induction by s we shall prove that $h \in \tilde{H}_{\alpha_1} \dotplus \tilde{H}_{\alpha_2} \dotplus \ldots \dotplus \tilde{H}_{\alpha_k}$. For $s = 1$ we can assume (without loss of generality) that $\tilde{G}_{\alpha_1}$ is not p_1-divisible (if it is then the assertion immediately follows from e)). Then by d) the groups $\tilde{G}_{\alpha_i}$, $i = 2, 3, \ldots, k$ are p_1-divisible, so that $g_i \in \tilde{H}_{\alpha_i}$, $i = 2, 3, \ldots, k$ by e). Further, $g_1 = h - g_2 - \ldots - g_k \in \tilde{G}_{\alpha_1} \cap H = \tilde{H}_{\alpha_1}$ and we are ready. For $s > 1$ we may assume that $p_1, p_2, \ldots, p_r$, $r > 0$ are all primes dividing m and such that $\tilde{G}_{\alpha_1}$ is not p_i-divisible, $i = 1, 2, \ldots, r$. If we denote $m_1 = p_1^{l_1} \cdot p_2^{l_2} \cdot \ldots \cdot p_r^{l_r}$, $m_2 = p_{r+1}^{l_{r+1}} \cdot \ldots \cdot p_s^{l_s}$, then it is easy to see that $m_2 g_i \in \tilde{H}_{\alpha_i}$, $i = 2, 3, \ldots, k$ (by using d) and e) several times). But $m_2 h = m_2 g_1 + m_2 g_2 + \ldots + m_2 g_k$ and $m_2 g_1 = m_2 h - m_2 g_2 - \ldots - m_2 g_k \in$ $\in \tilde{G}_{\alpha_1} \cap H \subseteq \tilde{H}_{\alpha_1}$. Hence the assertion follows by induction hypothesis.

g) We shall prove now that the subgroups I_α, $\alpha \in A$ are independent in H. The proof of this fact runs on the same lines as the proof of the part e) in Theorem 1, and hence it may be left to the reader.

h) The proof that H is generated by I_α, $\alpha \in A$ is very similar to the part f) from the proof of the preceding theorem and may be left to the reader.

Finally, $\tilde{H}_\alpha / H_\alpha \simeq I_\alpha$ by (23) so that $\hat{\tau}(I_\alpha) = \hat{\tau}(J_\alpha)$ for any $\alpha \in A$ and therefore $G \simeq H$, which completes the proof of the theorem.

Corollary (Kovács [8], Theorems B1, B2). *If G is a completely decomposable torsion free group such that $T(G)$ is inversely well-ordered (in the natural order of the types), then G is isomorphic to any its subgroup H or extension K such that $nG \subseteq H$ or $nK \subseteq G$, respectively, for a suitable positive integer n.*

Proof. It may be easily shown that all the conditions of Theorem 2 (Theorem 1 resp.) are satisfied.

Theorem 3. *Let G be a completely decomposable torsion free group such that $T(G)$ satisfies conditions (α), (β) from the preceding theorem. Then every its subgroup H such that $\hat{\tau}^H(x) = \hat{\tau}^G(x)$ for any $x \in H$ and $\{H\}_*^G$ is a direct summand of G is again completely decomposable.*

Proof. From the theorems proved by Kulikov [11] and Kaplansky [12] there follows (see also Fuchs [2], § 1) that a direct summand of a completely decomposable group is again completely decomposable. Hence we may restrict ourselves to the case that G/H is periodical. Now it is easy to see that the condition $\hat{\tau}^H(x) = \hat{\tau}^G(x)$ for any $x \in H$ is equivalent to the regularity of H in G and it suffices to use Theorem 2.

Theorem 4. *Let G be a torsion free group containing a homogeneous completely decomposable subgroup H such that G/H is a periodical group. Then $G \simeq H$ if and only if*

(α) *H is strongly regular in G,*
(β) *$G \in \Gamma_\alpha$.*[1])

Proof. First let us assume that $G \simeq H$. Then G is completely decomposable and hence $G \in \Gamma_1$ or $G \in \Gamma_2$. If S is an arbitrary subgroup of finite rank r pure in G, then evidently $S \cap H$ is pure in H and hence by Theorem 46.6 from [1] the group $S \cap H$ is completely decomposable, $S \cap H = \sum_d^r J_k$. Let us denote $I_k = \{J_k\}_*^G$, $k = 1, 2, \ldots, r$

[1]) Γ_α denotes Baer's classes of torsion free groups (see [1] § 48, p. 174).

and $B = \sum_{k=1}^{r} {}_d I_k$. There is $\hat{\tau}(I_k) = \hat{\tau}(J_k)$ so that the factor-group I_k/J_k is finite, $k = 1, 2, \ldots, r$. Then $B/S \cap H \simeq \sum_{k=1}^{r} {}_d I_k/J_k$ is finite. Further, the group S is completely decomposable by Theorem 46.6 from [1], so that S/B is finite by Theorem 48.1 from [1]. Finally, the index $[S : S \cap H] = [S : B] . [B : S \cap H]$ is finite and this implies the necessity of the condition (α).

In order to prove sufficiency, let us note that G is homogeneous of the same type as H. In fact, from the periodicity of the factor-group G/H there follows that to every $g \in G$ there exists a non-zero integer m such that $mg \in H$. The homogeneity of G now easily follows from the fact that $\{mg\}_*^G/\{mg\}_*^H$ is finite (condition (α)).

Now let S be an arbitrary rank finite subgroup pure in G and T/S a rank one subgroup pure in G/S. It is not too difficult to prove that T is a rank finite subgroup pure in G. Clearly, $T \cap H$ is pure in H so that $T \cap H$ is completely decomposable by Theorem 46.6 from [1]. Hence the corollary to Theorem 2 implies $T \simeq T \cap H$. S being a pure subgroup in a homogeneous completely decomposable group T of finite rank, it is a direct summand of T by Theorem 46.8 from [1]. All the conditions of Theorem 48.2 from [1] are satisfied now, hence G is a completely decomposable group which is homogeneous of the same type as H. From the periodicity of the factor-group G/H there follows that $r(H) = r(G)$ (the equality of ranks) and hence $G \simeq H$.

Corollary 1. *Let G be a torsion free group containing a countable homogeneous completely decomposable subgroup H such that G/H is periodical. Then $G \simeq H$ if and only if H is strongly regular in G.*

Proof. From the periodicity of the factor-group G/H there follows that $r(G) = r(H)$ and then it suffices to use Theorem 4.

Corollary 2. *Let G be a torsion free group containing a homogeneous completely decomposable subgroup H. If $G \in \Gamma_\alpha$, the factor-group G/H is periodical Π-primary where Π is a finite set of primes, and if for every $p \in \Pi$ the subgroup H is p^∞-independent in G then $G \simeq H$.*

Remark. In the case $\alpha = 1$ we get Theorem 1 from [6]. For the definition of p^∞-independence see [6] and [7].

Proof. Let S be a subgroup of a finite rank n pure in G. Then $S/S \cap H$ is periodical Π-primary and for every $p \in \Pi$ the p-primary component of $S/S \cap H$ is of rank at most n. By Lemma 7 from [7] $S \cap H$ is p^∞-independent in G (because $S \cap H$ is pure in H). By Theorem 46.6 from [1] $S \cap H$ is completely decomposable and by Lemma 4 from [6] there is $r_p(S) = 0$ [2]). Hence by Theorem 5 from [5],

[2]) $r_p(G)$ denotes the p-rank of the torsion free group G and $r_p^*(P)$ denotes the rank of a maximal divisible subgroup of the p-primary component of a periodical group P. For the definitions see e.g. [4].

$r_p^*(S/S \cap H) = 0^*)$ holds for every $p \in \Pi$ so that the factor-group $S/S \cap H$ is reduced. Now it is easy to see that $S/S \cap H$ is finite (e.g. by using the notion of the basic subgroup – see [1]) and it suffices to apply Theorem 4.

Now we shall give an example showing that for countable groups Corollary 1 is more general than Corollary 2.

Example. Let $H = \sum_{n=1}^{\infty}{}_d \{y_n\}$ be a free group of countable rank. Let us define a group G as follows: $G = \{H, x_n, n = 1, 2, \ldots\}$ where $p_n x_n = y_n + y_{n+1}$ and p_n, $n = 1, 2, \ldots$ are different primes. It is easy to see that G is torsion free again (e.g. by considering a suitable subgroup in a divisible closure of H). If S is an arbitrary rank finite subgroup pure in G then $S \cap H \subseteq H_n = \sum_{i=1}^{n}{}_d \{y_i\}$ for a suitable positive integer n. Then the x_k for $k \geqq n$ do not belong to S, and $p_1 \cdot p_2 \cdot \ldots \cdot p_{n-1} S \subseteq H_n \subseteq H$ so that $S/S \cap H$ is a finite group. By Corollary 1 there is $G \simeq H$, but G/H is infinite Π-primary $\Pi = \{p_1, p_2, \ldots\}$, so that Corollary 2 may not be applied.

Theorem 5. *Let G be a homogeneous completely decomposable torsion free group and H its subgroup such that G/H is periodical. Then $G \cong H$ if and only if H is regular in G.*

Proof. The condition is sufficient as follows from Theorem 2.

If S is an arbitrary rank one subgroup pure in G and $g \in S$ an arbitrary element, then the periodicty of G/H implies the existence of a non-zero integer m, $mg \in H$. In view of the isomorphism $G \cong H$ there holds $\hat{\tau}(S) = \hat{\tau}^G(g) = \hat{\tau}^G(mg) = \hat{\tau}(S \cap H)$. Now it is easy to see that the factor-group $S/S \cap H$ is finite, which proves the necessity of the condition stated in our theorem.

Corollary ([6], Theorem 11). *Let G be a homogeneous completely decomposable torsion free group and H its subgroup such that G/H is periodical Π-primary where Π is a finite set of primes. If for any $p \in \Pi$ every non-zero element from G has in G a finite p-height, then $G \cong H$.*

Proof. Let S be an arbitrary rank one subgroup pure in G. Due to the fact that the set Π is finite, we may choose an element g in S for which

$$h_p^G(g) = 0 \quad \text{for all} \quad p \in \Pi\,. \tag{27}$$

Since G/H is Π-primary, there is $mg \in H$ for some integer m divisible by primes from Π only. If $g' \in S$ is an arbitrary element then there exist integers r, s, $rg' = sg$. We may assume, without loss of generality, that $(r, s) = 1$. From (27) we get that $(r, m) = 1$. Clearly $rmg' = smg \in H$. If we assume that $mg' \notin H$ then the element $mg' + H$ of the factor-group G/H has an order $r' \mid r$ with $(r', m) = 1$, which contradicts to Π-primarity of G/H. Hence $mS \subseteq S \cap H$ and by using Theorem 5 the proof is completed.

Theorem 6. *A homogeneous group G belonging to some class* Γ_α *is completely decomposable if and only if every its pure subgroup of finite rank is completely decomposable.*

Proof. In fact, the condition is necessary by Theorem 46.6 from [1].

On the other hand, if S is a finite rank subgroup pure in G and T/S a pure subgroup of G/S of rank one, then, obviously, T is a pure subgroup of G of finite rank. T is completely decomposable by hypothesis and by Theorem 46.8 from [1] S is a direct summand of T. Hence G/S is homogeneous of the same type as G and Theorem 48.2 from [1] completes the proof.

Corollary. *A torsion free group G belonging to some Baer's class* Γ_α *is free if and only if every its rankfinite subgroup is free.*

Proof. It suffices to prove that the condition is sufficient. Because, in particular every rank finite subgroup pure in G is free, G is homogeneous of the type $\hat{\tau}$ where $\tau = (0, 0, \ldots, 0, \ldots)$, and the assertion follows from Theorem 6.

Remark. This corollary generalizes Pontrjagin's criterion of the freeness of a countable group (see [1], Theorem 14.2).

Theorem 7. *Let G be a torsion free group containing a homogeneous completely decomposable subgroup H. If G contains a subgroup* G_1 *such that* $H \subseteq G_1 \subseteq G$, *H is fully regular in* G_1, G_1 *is strongly regular in G and* G/G_1 *is countable, then* $G \simeq H$.

Proof. By Theorem 1 there is $G_1 \cong H$. If $g \in G \dot{-} G_1$ then $mg \in G_1$ for a suitably chosen non-zero integer m and mg has a non-zero-component in a finitely many direct summands of a given complete decomposition of $G_1 = \sum_{d\, \alpha \in A} J_\alpha$. In any coset of G/G_1 we choose one representant and the set of all these representants we denote by M. Let us denote by A_1 the set of all indices $\alpha \in A$ with the property that J_α contains a (non-zero) component of at least one element mg, $g \in M$ [3]). Obviously A_1 is countable (because M is countable). If we put $A_2 = A \dot{-} A_1$, $G' = \{\sum_{d\, \alpha \in A_1} J_\alpha; M\}$, $G'' = \sum_{d\, \alpha \in A_2} J_\alpha$ then it is easy to see that $G' \cap G'' = 0$. On the other hand, there is $G = \{G_1, M\} = \{G', G''\}$ so that $G = G' \dotplus G''$. From Definition 2 it may be easily derived that $\sum_{d\, \alpha \in A_1} J_\alpha$ is strongly regular in G' so that $G' \simeq \sum_{d\, \alpha \in A_1} J_\alpha$ by Corollary 1 to Theorem 4, and the assertion now easily follows.

Lemma 3. *Let G be a torsion free group containing a subgroup* $H = H_1 \dotplus H_2$,

[3]) The set of those J_α, $\alpha \in A$ in which mg has a non-zero component does not depend on the choice of the non-zero integer m for which $mg \in G_1$. In fact, let t be the least positive integer such that $tg \in G_1$. Then $m = tq + r$, $0 \leqq r < t$. For $r \neq 0$ there is $rg = mg - qtg \in G$ which is a contradiction with the minimality of t. Hence $r = 0$ and the assertion follows.

where H_1 is reduced and H_2 divisible. If H is strongly regular in G then

$$G = \{H_1\}_*^G \dotplus H_2 \tag{28}$$

and

$$G \simeq H \Leftrightarrow \{H_1\}_*^G \simeq H_1 \,. \tag{29}$$

Proof. Using Definition 2 we get that G/H is a periodical group so that (28) holds by Lemma 1. If $\{H_1\}_*^G \simeq H_1$ then evidently $G \simeq H$. Conversely, from Definition 2 there easily follows that $\{H_1\}_*^G$ is reduced and then it is not too difficult to show that $\{H_1\}_*^G \simeq G/H_2 \simeq G\varphi/H_2\varphi = H/H_2 \simeq H_1$, φ being an isomorphism between G and H.

Lemma 4. *Let $H = \sum_{n=1}^{\infty}{}_d J_n$ be a reduced completely decomposable torsion free group. Then either there exists a prime p and indices $n_1, n_2, \ldots$ different to each other so that J_{n_k} are p-reduced, $k = 1, 2, \ldots$, or there exist different primes p_1, $p_2, \ldots$ and different indices $n_1, n_2, \ldots$ such that J_{n_k} is p_k-reduced, $k = 1, 2, \ldots$*

Proof. Assume that for any prime p only a finite number of J_n is p-reduced. The group H is reduced so that there exists a prime p_1 for which H is not p_1-divisible. Let us choose an index n_1 so that J_{n_1} is p_1-reduced and for any $s > n_1$ the group J_s is p_1-divisible.

Proceeding by the induction we shall assume that we have chosen the primes $p_1, p_2, \ldots, p_k$ and the groups $J_{n_1}, J_{n_2}, \ldots, J_{n_k}$ such that $n_1 < n_2 < \ldots < n_k$, J_{n_i} is p_i-reduced and for any $s > n_i$ the group J_s is p_i-divisible. Then we can write $H = \sum_{i=1}^{n_k}{}_d J_i \dotplus \sum_{i=n_k+1}^{\infty}{}_d J_i = H_1 \dotplus H_2$ where H_2 is p_i-divisible for all $i = 1, 2, \ldots, k$. On the other hand, the group H_2 is reduced by hypothesis, so that there exists a prime p_{k+1} different from all $p_1, p_2, \ldots, p_k$ and such that H_2 is not p_{k+1}-divisible. By hypothesis at the beginning of this proof there exists an index n_{k+1} such that Jn_{k+1} is p_{k+1}-reduced and for any $s > n_{k+1}$ the group J_s is p_{k+1}-divisible. Obviously $n_{k+1} > n_k$ and our lemma now easily follows.

Lemma 5. *Let $H = \sum_{\alpha \in A}{}_d J_\alpha$ be a reduced completely decomposable torsion free group such that there exists a type $\hat{\tau} \in T(H)$ for which $H^*(\hat{\tau})$ is of infinite rank. Then there exists a torsion free extension G of H such that H is strongly regular in G, but $G \not\simeq H$.*

Proof. The group $H^*(\hat{\tau})$ obviously contains a reduced completely decomposable subgroup of countable rank. In view of preceding Lemma two cases are to be considered:

1. There exists a prime p and infinitely many different direct summands $J_{\alpha_1}, J_{\alpha_2}, \ldots$ of $H^*(\hat{\tau})$ such that $J_{\alpha_1}, J_{\alpha_2}, \ldots$ are p-reduced.

2. There exist different primes $p_1, p_2, \ldots$ and different direct summands $J_{\alpha_1}, J_{\alpha_2}, \ldots$ of $H^*(\hat{\tau})$ such that J_{α_n} is p_n-reduced.

Moreover, let us denote by $J_{\alpha 0}$ that direct summand of a given complete decomposition of H for which $\hat{\tau}(J_{\alpha 0}) = \hat{\tau}$. In the first case we choose an element $x_n \in J_{\alpha_n}$ such that $h_p^H(x_n) = 0$, $n = 0, 1, 2, \ldots$ and in the divisible closure of H we form the subgroup

(30) $$G = \{H; y_n; p^n y_n = x_0 + x_n, \ n = 1, 2, \ldots\}.$$

In the second case we choose an arbitrary element $x_0 \in J_{\alpha_0}$ and we denote $h_n = h_{p_n}^H(x_0)$. We choose elements $x_n \in J_{\alpha_n}$ such that $h_{p_n}^H(x_n) = h_n$, $n = 1, 2, \ldots$ and in the divisible closure of H we form the subgroup

(31) $$G = \{H; y_n; p_n^{h_n+1} y_n = x_0 + x_n, \ n = 1, 2, \ldots\}.$$

Now, if S is a pure subgroup of G of finite rank, then $S \cap H \subseteq \sum_{i=0}^{n}{}_d J_{\alpha_i} \dotplus H'$ where H' is the direct sum of a finite number of J_α, $\alpha \neq \alpha_i$, $i = 0, 1, 2, \ldots$ Then clearly $S \subseteq \{\sum_{i=0}^{n}{}_d J_{\alpha_i} \dotplus H'\}_*^G = \{\sum_{i=0}^{n}{}_d J_{\alpha_i} \dotplus H'; y_1, y_2, \ldots, y_n\}$ so that in the first case we have $p^n S \subseteq H$ and in the second case we have $(p_1 . p_2 . \ldots . p_n) S \subseteq H$. Hence in both cases H is strongly regular in G.

In both cases there holds $\hat{\tau}^G(x_i) \geqq \hat{\tau}^H(x_i) > \hat{\tau}$ for all $i = 1, 2, \ldots$ and hence

(32) $$x_i \in G^*(\hat{\tau}), \quad i = 1, 2, \ldots$$

Assume that $G \simeq H$. Then $G(\hat{\tau})/G^*(\hat{\tau})$ is a homogeneous group of the type $\hat{\tau}$. Since H is strongly regular in G there is $\hat{\tau}^G(x_0) = \hat{\tau}^H(x_0) = \hat{\tau}$ so that $\hat{\tau}^{G(\hat{\tau})/G^*(\hat{\tau})}(\bar{x}_0) = \hat{\tau}$ holds for $\bar{x}_0 = x_0 + G^*(\hat{\tau})$. However, in the first case the element $\bar{x}_0$ is p-divisible according to (30) and (32), and in the second case $h_{p_n}^{G(\hat{\tau})/G^*(\hat{\tau})}(\bar{x}_0) = h_n + 1$ holds by (31) and (32). This contradiction completes the proof of the lemma.

Theorem 8. *Let H be the direct sum of a divisible and a countable completely decomposable torsion free groups. Then any torsion free extension G of H such that H is strongly regular in G is isomorphic to H if and only if*

(α) *for any $\hat{\tau} \in T(H)$ the group $H^*(\hat{\tau})$ is the direct sum of a divisible group and a reduced group of finite rank,*

(β) *for any two incomparable types $\hat{\tau}_1, \hat{\tau}_2$ from $T(H)$ there is $\hat{\tau}_1 \vee \hat{\tau}_2 = \hat{R}$.*

Proof. The condition (α) is necessary according to Lemmas 8 and 10. If $U \subseteq H$ is such a subgroup that $pH \subseteq U$ for a suitable prime p, then pH is strongly regular in U. In view of the isomorphism relation $pH \simeq H$, there follows from Definition 2 and Lemma 1 [14] that the group H is an IQ-group; hence the condition (β) is necessary following Theorem 2 [14].

In the proof of the sufficiency we may restrict ourselves to a reduced group $H = \sum_{\alpha \in A}^{d} J_\alpha$, due to Lemma 8. We may define a partial order in g on the set A as in the proof of Theorem 1, with the following difference only: If $A(\hat{\tau})$ is infinite (necessarily countable according to the hypothesis) then $<_{\hat{\tau}}$ denotes the well-ordering of the type ω. Since this case may occur only if $\hat{\tau}$ is a minimal element of $T(H)$ (condition (α)), we may follow now, word by word, the proof of Theorem 1, as it is evidently sufficient to assume the strong regularity at all places where the full regularity is assumed in the proof of Theorem.

Theorem 9. *Let G be a torsion free group containing a completely decomposable subgroup* $H = \sum_{\alpha \in A}^{d} J_\alpha$ *such that* $T(H)$ *is ordered. Suppose that*

(α) *H is strongly regular in G and*

(β) *for any* $\hat{\tau} \in T(H)$ *the group* $H^*(\hat{\tau})$ *is the direct sum of a divisible group and a reduced group of finite rank. Further, if* $T(H)$ *contains the least element* $\hat{\tau}$, *then* $G/\{H^*(\hat{\tau})\}_*^G$ *belongs to some* Γ_α. *Then* $G \simeq H$.

Proof. In view of Lemma 3 we may restrict ourselves to a reduced group H. If $T(H)$ does not contain the least element $\hat{\tau}$ then H is necessarily countable and it suffices to use Theorem 8.

Suppose now that $T(H)$ contains the least element $\hat{\tau}$. By Theorem 8 it holds $\{H^*(\hat{\tau})\}_*^G \simeq H^*(\hat{\tau})$. For the sake of simplicity let us denote $U = \{H^*(\hat{\tau})\}_*^G$ and let V be the direct sum of those J_α of a given complete decomposition of H for which $\hat{\tau}(J_\alpha) = \hat{\tau}$. Then G/U contains a homogeneous completely decomposable subgroup $U \dotplus V/U$. If S/U is a pure subgroup of G/H of finite rank, then S is a rank finite pure subgroup of G by hypothesis (U is of finite rank pure in G). Hence there exists a non-zero integer m, $mS \subseteq S \cap H \subseteq S \cap (U \dotplus V)$, so that $U \dotplus V/U$ is strongly regular in G/U. Now according to Theorem 4 there is $G/U \simeq U \dotplus V/U$ so that $G/U = \sum_{\beta \in B}^{d} \bar{J}_\beta = \sum_{\beta \in B}^{d} J_\beta/U$ is a completely decomposable group homogeneous of the type $\hat{\tau}$. For $g \in G \dot{-} U$ there is $mg \in H \dot{-} H^*(\hat{\tau})$ for a suitable integer $m \neq 0$ such that $\hat{\tau}^G(g) = \hat{\tau}^G(mg) = \hat{\tau}^H(mg) = \hat{\tau}$. By using Baer's lemma 46.3 from [1] we get that U is a direct summand of any J_β, $\beta \in B$, so that Lemma 2.2 from [1] completes the proof.

Theorem 10. *Let a torsion free group G be the direct sum of a divisible group and a reduced countable group. Suppose that* $\mathfrak{T}(G)$ *is ordered and for any* $\hat{\tau} \in \mathfrak{T}(G)$ *the group* $G^*(\hat{\tau})$ *is a direct sum of a divisible and a reduced rank finite groups. Then G is completely decomposable if and only if any subgroup regular in G is strongly regular in G.*

Proof. First let G be completely decomposable, H regular in G and S a pure subgroup of G of a finite rank n. Then S is contained in a completely decomposable direct summand of finite rank such that S is completely decomposable, $S = J_1 \dotplus J_2 \dotplus \ldots \dotplus J_n$ by hypothesis ($\mathfrak{T}(G)$ is ordered) and by Theorem proved in [5]. By hypothesis the factor-group $J_i/J_i \cap H$ is finite for all $i = 1, 2, \ldots, n$ so that the factor-group $S/\sum_{i=1}^{n} {}_d(J_i \cap H) \cong \sum_{i=1}^{n} {}_d J_i/J_i \cap H$ is finite, too. The necessity of the condition stated in Theorem 10 now immediately follows from the inclusion $\sum_{i=1}^{n} {}_d J_i \cap H \subseteq S \cap H$.

Proving the sufficiency, we may assume, without loss of generality, that G is reduced (hence countable). First we shall suppose that G is of countable rank. By Lemma 42.1 from [1] the set $\mathfrak{T}(G)$ is either a finite or an infinite decreasing sequence, i.e., either $\mathfrak{T}(G) = \{\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_n;\ \hat{t}_1 > \hat{t}_2 > \ldots > \hat{t}_n\}$, or $\mathfrak{T}(G) = \{\hat{t}_1, \hat{t}_2, \ldots;\ \hat{t}_i > \hat{t}_j \text{ for } i < j\}$. In both cases we choose an arbitrary maximal independent set of elements of $G(\hat{t}_1)$. This set is finite provided $G(\hat{t}_1) \neq G$. In this case we proceed to $G(\hat{t}_2)$ by extending this set to a maximal independent set of $G(\hat{t}_2)$, etc. It is clear that in both cases we obtain finally the set $x_1, x_2, \ldots$, which is maximal independent in G. We shall show that the subgroup $H = \sum_{n=1}^{\infty} {}_d \{x_n\}_*^G$ is regular in G. In fact, if $x \in H$ is an arbitrary element, $x = \sum_{i=1}^{n} y_i,\ y_i \in \{x_i\}_*^G,\ y_n \neq 0$ then from the choice of the elements $x_1, x_2, \ldots$ there follows easily that $\hat{t}^H(x) = \hat{t}^H(y_n) = \hat{t}^G(y_n) = \hat{t}^G(x)$. By hypothesis H is strongly regular in G, hence Theorem 8 shows that $G \simeq H$.

If G is of finite rank, then the proof runs along the same lines. The only difference is that the maximal independent set $x_1, x_2, \ldots$ is finite.

A special case of this Theorem is a theorem of Wang (see [13], Theorem 2), which states:

Corollary. *Let G be a torsion free group of finite rank, the type set of which is ordered. Then G is completely decomposable if and only if the factor-group G/H is finite whenever H is regular in G.*

Theorem 11. *A homogeneous torsion free group G is completely decomposable if and only if*

(α) *$G \in \Gamma_\alpha$ and*

(β) *any subgroup regular in G is strongly regular in G.*

Proof. The necessity of the condition (α) is evident, condition (β) is proved in the the same way as in the preceding theorem. Conversely, if $\{x_\alpha, \alpha \in A\}$ is an arbitrary maximal independent set of elements of G then the subgroup $H = \sum_{\alpha \in A} {}_d \{x_\alpha\}_*^G$ is regular in G, hence it is strongly regular in G by hypothesis and Theorem 4 completes the proof.

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Author's address: Praha 8 - Karlín, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta UK).