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ON THE WEAKLY NONLINEAR WAVE EQUATION INVOLVING A SMALL PARAMETER AT THE HIGHEST DERIVATIVE

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I. THE CAUCHY PROBLEM FOR A WEAKLY NONLINEAR WAVE EQUATION INVOLVING A SMALL PARAMETER AT THE HIGHEST DERIVATIVE

1. INTRODUCTION

We shall discuss the equation

$$(1^a) \qquad L_\varepsilon u \equiv \varepsilon u_{tt} - u_{xx} + 2au_t + cu = g(t, x) + \varepsilon f(t, x, u, u_x, u_t)$$

with the initial data

$$(1^b) \qquad u(0, x, \varepsilon) = \varphi(x, \varepsilon), \quad u_t(0, x, \varepsilon) = \psi(x, \varepsilon),$$

where $x \in E_1$, $u = u(t, x, \varepsilon)$; $[t, x] \in V = \{[t, x] \in E_2, t \in \langle 0, +\infty), x \in E_1\}$ and a, c are positive constants. As to a periodic solution of (1^a), the case $a < 0$ may be transferred to that of $a > 0$ by the substitution $\tau = -t$. We are interested in the behaviour of the solution $u = u(t, x, \varepsilon)$ as ε tends to zero and in the nonuniformity occurring at $t = 0$. Under certain conditions it will be proved that there exists $\varepsilon_0 > 0$ such that problem (1) possesses a unique solution $u = u(t, x, \varepsilon)$ for $\varepsilon \in \langle 0, \varepsilon_0 \rangle$. This solution is of the form

$$(2) \qquad u = u^0 + v + w$$

where $u^0 = u^0(t, x)$ is the solution of the linear parabolic equation

$$(3^a) \qquad 2au^0_t - u^0_{xx} + cu^0 = g$$

with the initial condition

$$(3^b) \qquad u^0(0, x) = \varphi(x, 0)$$

and $v = v(t, x, \varepsilon)$ or $w = w(t, x, \varepsilon)$ are the solutions of the nonlinear problems (4) or (5), respectively:

$$(4^a) \qquad L_\varepsilon v = \varepsilon f(t, x, u^0 + v, u^0_x + v_x, u^0_t + v_t) - \varepsilon u^0_{tt}$$

$$(4^b) \qquad v(0, x, \varepsilon) = \varphi(x, \varepsilon) - \varphi(x, 0)\,; \quad v_t(0, x, \varepsilon) = \psi(x, \varepsilon) - \psi(x, 0)$$

$$(5^a) \qquad L_\varepsilon w = \varepsilon f(t, x, u^0 + v + w, u^0_x + v_x + w_x, u^0_t + v_t + w_t) - \\ - \varepsilon f(t, x, u^0 + v, u^0_x + v_x, u^0_t + v_t)$$

$$(5^b) \qquad w(0, x, \varepsilon) = 0\,; \quad w_t(0, x, \varepsilon) = \psi(x, 0) - u^0_t(0, x)\,.$$

The problem will be solved in the Banach space $\mathfrak{C}_\varepsilon$ defined as follows: we put $V_t = \{[t, x], x \in E_1\}$, $t \geqq 0$ and

$$(6) \qquad \|h\|_{(t,\varepsilon)} = \|h\|_{1,V_t} + \|h_{xx}\|_{0,V_t} + \|h_{tx}\|_{0,V_t} + \varepsilon\|h_{tt}\|_{0,V_t}\,,$$

$$(7) \qquad \|h\|_\varepsilon \;\; = \|h\|_{1,V} + \|h_{xx}\|_{0,V} + \|h_{tx}\|_{0,V} + \varepsilon\|h_{tt}\|_{0,V}\,,$$

$$\|h\|^1_t \;\; = \|h\|_{0,V_t} + \|h_x\|_{0,V_t}\,; \quad \|h\|^1 = \|h\|_{0,V} + \|h_x\|_{0,V}$$

where

$$h = h(t, x)\,, \quad \|h\|_{k,M} = \sum_{0 \leqq i,j \leqq k} \sup_M \left| \frac{\partial^{i+j} h}{\partial t^i \, \partial x^j}(t, x) \right|.$$

Then for $\varepsilon > 0$ $\mathfrak{C}_\varepsilon$ is the Banach space of continuous functions with continuous derivatives up to the second order with the norm (7). Under some conditions we shall prove that $v = v(t, x, \varepsilon)$ tends to zero in $\mathfrak{C}_\varepsilon$, and $\|w\|_{(t,\varepsilon)} \to 0$ for $\varepsilon \to 0$ uniformly with respect to $t \geqq t_0$, $t_0 > 0$. w represents a "boundary layer" which is not negligible in the neighbourhood of $t = 0$.

2.

Proposition 1. *If* $0 < \varepsilon_1 < a^2/(2a + c)$ *and* $u = u(t, x, \varepsilon)$ *is a solution of* (1) *for* $\varepsilon \in (0, \varepsilon_1\rangle$ *then* u *is a solution of the integro-differential equation*

$$(8) \qquad u = \frac{1}{\varepsilon} P_\varepsilon[g(t, x) + \varepsilon f(t, x, u, u_x, u_t) + S_\varepsilon\varphi + Q_\varepsilon\psi$$

and conversely, every solution $u = u(t, x, \varepsilon)$ *of* (8), $u \in \mathfrak{C}_\varepsilon$ *is a solution of* (1). *The integral operators in* (8) *are defined by the following formulae*

$$(9) \qquad P_\varepsilon h(t, x) = \frac{\sqrt{\varepsilon}}{2} \int_0^t e^{-a(t-\tau)/\varepsilon} \int_{x-(t-\tau)/\sqrt{\varepsilon}}^{x+(t-\tau)/\sqrt{\varepsilon}} I_0\left(\frac{a\beta}{\sqrt{\varepsilon}}\,\zeta\right) h(\tau, \zeta)\, d\xi\, d\tau\,,$$

(10) $$S_\varepsilon\varphi(t,x) = \frac{1}{2}e^{-at/\varepsilon}\left\{\varphi\left(x+\frac{t}{\sqrt{\varepsilon}}\right) + \varphi\left(x-\frac{t}{\sqrt{\varepsilon}}\right) + \right.$$

$$\left. + \int_{x-t/\sqrt{\varepsilon}}^{x+t/\sqrt{\varepsilon}}\left[\frac{a}{\sqrt{\varepsilon}} I_0\left(\frac{a\beta}{\sqrt{\varepsilon}}\zeta_0\right) + \sqrt{\varepsilon}\frac{\partial I_0}{\partial t}\left(\frac{a\beta}{\sqrt{\varepsilon}}\zeta_0\right)\right]\varphi(\xi)\,\mathrm{d}\xi\right\},$$

(11) $$Q_\varepsilon\psi(t,x) = \frac{\sqrt{\varepsilon}}{2}e^{-at/\varepsilon}\int_{x-t/\sqrt{\varepsilon}}^{x+t/\sqrt{\varepsilon}} I_0\left(\frac{a\beta}{\sqrt{\varepsilon}}\zeta_0\right)\psi(\xi)\,\mathrm{d}\xi$$

where I_0 is the modified Bessel function of the first kind,

$$\beta = \sqrt{\left(1-\frac{c\varepsilon}{a^2}\right)}; \quad \zeta = \sqrt{\left(\frac{(t-\tau)^2}{\varepsilon} - (x-\xi)^2\right)}; \quad \zeta_0 = \sqrt{\left(\frac{t^2}{\varepsilon} - (x-\xi)^2\right)}.$$

This may be obtained by the substitution $t = \tau\sqrt{\varepsilon}$ in (1) and by using the statement A in [1]. It is known (see [1]) that $u_1 = S_\varepsilon\varphi$ is the solution of the equation

(12) $$L_\varepsilon u_1 = 0$$

with the initial data $u_1(0,x) = \varphi(x)$, $(u_1)_t(0,x) = 0$, $x \in E_1$, $u_2 = Q_\varepsilon\psi$ is the solution of (12) with the initial data $u_2(0,x) = 0$; $(u_2)_t(0,x) = \psi(x)$; $x \in E_1$ and $u_3 = P_\varepsilon h$ is the solution of equation

(13) $$L_\varepsilon u_3 = \varepsilon\, h(t,x)$$

with the homogeneous initial data.

3. THE LINEAR EQUATION

Firstly, we shall discuss the behaviour of the solution $u = u(t,x,\varepsilon)$ of the equation

(14^a) $$L_\varepsilon u = g(t,x) \quad \text{on} \quad V$$

with the initial data

(14^b) $$u(0,x,\varepsilon) = \varphi(x,\varepsilon), \quad u_t(0,x,\varepsilon) = \psi(x,\varepsilon).$$

The solution u will be sought in the form (2) where u^0 is the solution of (3), $v = v(t,x)$ is the solution of the equation

(15^a) $$L_\varepsilon v = -\varepsilon u^0_{tt}$$

with the initial data

(15^b) $$v(0,x) = \varphi(x,\varepsilon) - \varphi(x,0), \quad v_t(0,x) = \psi(x,\varepsilon) - \psi(x,0)$$

and $w = w(t, x)$ is the solution of the equation

(16^a) $$L_\varepsilon w = 0$$

with the initial data

(16^b) $$w(0, x) = 0\,; \quad w_t(0, x) = \psi(x, 0) - u_t^0(0, x)\,.$$

If $\varphi = \varphi(x, 0)$ is a function from $C^5(E_1)$ and g satisfies

(A) $$g \in C^{(1)}(V)\,, \quad \frac{\partial g}{\partial x} \in C^{(1)}(V)\,, \quad \frac{\partial^3 g}{\partial x^3} \in C(V)\,; \quad \frac{\partial^3 g}{\partial x^3}\,, \; \frac{\partial^2 g}{\partial t\, \partial x}$$

are Hölder-continuous of the order $\alpha \in (0, 1)$, then the solution $u^0(t, x)$ of (3) is of the form $u^0 = u_1^0 + u_2^0$ where

$$u_1^0(t, x) = \sqrt{\left(\frac{a}{2\pi t}\right)}\, e^{-ct/2a} \int_{-\infty}^{+\infty} \varphi(\xi, 0)\, e^{-(x-\xi)^2 a/2t}\, d\xi$$

$$u_2^0(t, x) = \sqrt{\left(\frac{a}{2\pi}\right)} \int_0^t \frac{1}{\sqrt{(t-\tau)}}\, e^{-c(t-\tau)/2a} \int_{-\infty}^{+\infty} g(\tau, \xi)\, e^{-a(x-\xi)^2/2(t-\tau)}\, d\xi\, d\tau$$

and one can easily show (see e.g. [6]) that there exists a constant C such that

(17) $$\|u_1^0\|_{1,V_t} \leqq C\|\varphi\|_{2,E_1}\, e^{-ct/2a}$$

$$\|u_1^0\|_t' \equiv \|u_1^0\|_{2,V_t} + \left\|\frac{\partial^3 u_1^0}{\partial t^2\, \partial x}\right\|_{0,V_t} + \left\|\frac{\partial^3 u_1^0}{\partial x^3}\right\|_t^1 + \left\|\frac{\partial^5 u_1^0}{\partial x^5}\right\|_{0,V_t} \leqq C\|\varphi\|_{5,E_1}\, e^{-ct/2a}$$

(18) $$\|u_2^0\|_{1,V} \leqq C\|g\|_{1,V}\,; \quad \|u_2^0\|' \equiv \sup_{t \in \langle 0, +\infty)} \|u_2^0\|_t' \leqq C(g)$$

where

$$C(g) = C\left(N + \|g\|_{1,V} + \|g_x\|_{1,V} + \left\|\frac{\partial^3 g}{\partial x^3}\right\|_{0,V}\right);$$

$$N = \sup_{\substack{x_1, x_2 \in E_1 \\ t \in \langle 0, +\infty)}} \left\{|x_1 - x_2|^{-\alpha}\left(\left|\frac{\partial^3 g}{\partial x^3}(t, x_1) - \frac{\partial^3 g}{\partial x^3}(t, x_2)\right| + \right.\right.$$
$$\left.\left. + \left|\frac{\partial^2 g}{\partial t\, \partial x}(t, x_1) - \frac{\partial^2 g}{\partial t\, \partial x}(t, x_2)\right|\right)\right\}.$$

The existence and uniqueness of the periodic solution of (3^a) is proved but here it will be given in the following form.

Theorem 1. *Let g satisfy (A), g be an ω-periodic (in t) function. Then there exists a unique ω-periodic solution U^0 of (3) and $\|U^0\|' \leqq C(g)$.*

Proof. Let u^0 be the solution of (3^a) and $u^0(0, x) = 0$. By (18) we have $\|u^0\|' \leqq \leqq C(g)$. If we put $u_n^0(t, x) = u^0(t + n\omega, x)$, $n = 1, 2, \ldots$, then u_n^0 solves equation (3^a) and $u_n^0 - u^0$ solves the equation

$$2au_t - u_{xx} + cu = 0$$

with the initial conditions $u(0, x) = u^0(n\omega, x) - u^0(0, x) = u^0(n\omega, x)$. From (17) we obtain for $n \geqq m$

$$\|u_n - u_m\|_t' = \|u_{n-m} - u\|_{t+m\omega}' \leqq C\|u^0((n-m)\omega, x)\|_{5,E_1} e^{-c(t+m\omega)/2a} \leqq C(g)\, e^{-c\omega m/2a} .$$

Thus $\{u_n\}$ is a fundamental sequence in the norm $\|\ldots\|'$; therefore there exists the function $U^0 = U^0(t, x)$ such that $\|U^0\|' \leqq \|u^0\|' \leqq C(g)$,

$$U^0(t, x) = \lim_{n\to+\infty} u_{n+1}^0(t, x) = \lim_{n\to+\infty} u_n^0(t + \omega, x) = U^0(t + \omega, x)$$

and U^0 solves equation (3^a). If U_1^0, U_2^0 are two ω-periodic solutions of (3^a) then

$$\|U_1^0 - U_2^0\|_t' = \|U_1^0 - U_2^0\|_{t+n\omega}' \leqq C\|U_1^0(0, x) - U_2^0(0, x)\|_{5,E_1} e^{-c(t+n\omega)/2a} ;$$
$$n = 1, 2, \ldots$$

This implies $U_1^0(t, x) = U_2^0(t, x)$ for $t \in \langle 0, +\infty)$, $x \in E_1$.

Now, let us prove the fundamental lemma.

Lemma 1. *The integral operators P_ε, Q_ε, S_ε defined by formulae (9)–(11) map the functions $h = h(t, x)$, $h \in C(V)$, $\partial h/\partial x \in C(V)$, $q = q(x)$, $q \in C^1(E_1)$, $s = s(x)$, $s \in C^2(E_1)$, respectively into $\mathfrak{C}_\varepsilon$ for every $\varepsilon \in (0, \varepsilon_1\rangle$, $\varepsilon_1 \in (0, a^2/(2a + c))$ and the following estimates hold*

(19) $$\|P_\varepsilon h\|_{(t,\varepsilon)} \leqq C \int_0^t \left(\sqrt{(\varepsilon)}\, e^{-a(1-\beta)(t-\tau)/\varepsilon} + e^{-a(t-\tau)/\varepsilon}\right) \|h\|_\tau^1\, d\tau + \varepsilon\gamma(t, \varepsilon)\, C \int_0^{t-2\varepsilon/a\beta^2} \frac{1}{(t-\tau)}\, e^{-c(t-\tau)/a} \|h\|_\tau^1\, d\tau + \varepsilon\|h\|_t ,$$

(20) $$\|P_\varepsilon h\|_\varepsilon \leqq A_4(\varepsilon)\, \|h\|^1 ,$$

(21) $$\|Q_\varepsilon q\|_{(t,\varepsilon)} \leqq A_2(t, \varepsilon)\, \|q\|_{1,E_1} , \quad t > 0 ,$$

(22) $$\|Q_\varepsilon q\|_\varepsilon \leqq C\|q\|_{1,E_1} ,$$

(23) $$\|S_\varepsilon s\|_{(t,\varepsilon)} \leqq A_3(t, \varepsilon)\, \|s\|_{2,E_1} ,$$

(24) $$\|S_\varepsilon s\|_\varepsilon \leqq C\varepsilon^{-1/2}\|s\|_{2,E_1}$$

where the constant C depends on a, c, ε_1,

(25) $$\gamma(t, \varepsilon) = \begin{cases} 0 & \text{for} \quad t \leqq \dfrac{2\varepsilon}{a\beta^2} \\ 1 & \text{for} \quad t > \dfrac{2\varepsilon}{a\beta^2} \end{cases}$$

$$\lim_{\varepsilon \to 0+} \varepsilon^{-1/2} A_4(\varepsilon) = 0\,,$$

$$A_2(t, \varepsilon) = C\left(\sqrt{(\varepsilon)}\, e^{-a(1-\beta)t/\varepsilon} + e^{-at/\varepsilon} + \varepsilon \frac{1}{t} e^{-ct/a}\right), \quad t > 0\,;$$

(26) $\lim_{\varepsilon \to 0+} A_2(t, \varepsilon) = 0$ *uniformly with respect to* $t \in \langle t_0, +\infty), \; t_0 > 0\,;$

(27) $$\varepsilon^{1/2} A_3(t, \varepsilon) = C e^{-ct/2a}\,.$$

First, we shall recall some properties of the modified Bessel functions of the first kind $I_\nu(z)$ on $\langle 0, +\infty)$. These functions (see e.g. [5]) are defined by the formula

$$I_\nu(z) = i^{-\nu} J_\nu(iz)\,,$$

where $J_\nu(z)$ is the Bessel function of the order ν. I_ν may be written as a series

(28) $$I_\nu(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n + \nu + 1)\,\Gamma(n + 1)} \left(\frac{z}{2}\right)^{2n+\nu},$$

solves the equation

(29) $$y'' + \frac{1}{z} y' - \left(1 + \frac{\nu^2}{z^2}\right) y = 0$$

and for z sufficiently large there holds

(30) $$I_\nu(z) = \sqrt{\left(\frac{1}{2\pi z}\right)} e^z \left(1 + O\left(\frac{1}{z}\right)\right).$$

From (30) and (28) there follows

(31) $$\lim_{z \to +\infty} \frac{I_1(z)}{I_0(z)} = 1\,; \quad \lim_{z \to 0+} \frac{I_1(z)}{z\, I_0(z)} = \frac{1}{2}\,;$$

more over $I_0'(z) = I_1(z)$.

In the following we shall need to know that $H(z) \equiv I_0(z) - I_1(z) > 0$ for $z \in \langle 0, +\infty)$. In fact, as I_0, I_1 are continuous and $H(0) = I_0(0) - I_1(0) = 1$, the function $H(z) > 0$ in a neighbourhood of $z = 0$. Let us suppose that there exists

$\bar{z} \in \langle 0, +\infty)$ so that $H(\bar{z}) = 0$. If we denote $z_1 = \min \bar{z}$, then $z_1 > 0$ and $H(z) > 0$ for $z \in \langle 0, z_1)$, $H(z_1) = 0$ and $H'(z_1) \leqq 0$. By (29) we have $H'(z_1) = I_0'(z_1) - I_0''(z_1) = (1/z_1) I_1(z_1) > 0$ which is a contradiction. Now, we shall prove the following proposition.

Proposition 2. *Let $F_\alpha(z) \equiv \alpha(I_1(z)/I_0(z)) - z$ for $z \in \langle 0, +\infty)$, $\alpha \geqq 0$. Then for every $\alpha \in \langle 0, +\infty)$ there exists a unique number $z_0 = z_0(\alpha)$, $z_0 \in \langle 0, \alpha\rangle$ such that*

$$(32) \qquad F_\alpha(z) \geqq 0 \quad for \quad z \in \langle 0, z_0), \qquad F_\alpha(z) \leqq 0 \quad for \quad z \in (z_0, +\infty).$$

Furthermore, $z_0(\alpha) = 0$ for $\alpha \in \langle 0, 2\rangle$, $z_0(\alpha)$ is nondecreasing and

$$\lim_{\alpha \to +\infty} z_0(\alpha) = +\infty, \quad \lim_{\alpha \to +\infty} \frac{z_0(\alpha)}{\alpha} = 1.$$

Proof. The case $\alpha = 0$ is simple; thus let us suppose $\alpha > 0$. If $z \geqq \alpha$ then

$$F_\alpha(z) = \alpha \frac{I_1(z)}{I_0(z)} - z \leqq \alpha[I_1(z) - I_0(z)] \frac{1}{I_0(z)} < 0;$$

this implies that $z_0 \in \langle 0, \alpha\rangle$. For $\alpha \leqq 2$ there holds

$$F_\alpha(z) = I_0^{-1}(z)\,[\alpha I_1(z) - z I_0(z)] \leqq I_0^{-1}(z)\,[2 I_1(z) - z I_0(z)] =$$
$$= -I_0^{-1}(z) \sum_{n=0}^{\infty} \frac{1}{(n+1)!\,(n-1)!} \left(\frac{z}{2}\right)^{2n} < 0, \quad z > 0.$$

Thus we have $z_0(\alpha) = 0$ for $\alpha \in \langle 0, 2\rangle$. Now, by (29), (31)

$$F_\alpha'(z) = \alpha\left[1 - \frac{1}{z}\frac{I_1(z)}{I_0(z)} - \frac{I_1^2(z)}{I_0^2(z)}\right] - 1, \quad z > 0; \quad F'(0) = \lim_{z \to 0+} F_\alpha'(z) = \frac{\alpha}{2} - 1.$$

If $F_\alpha(z) = 0$ then $F_\alpha'(z) = \alpha - 2 - (z^2/\alpha)$, thus the function $F_\alpha'(z)$ is decreasing on the set of the points z: $F_\alpha(z) = 0$. As the function $F_\alpha(z)$ may be written in the form of a power series the number of the points z: $F_\alpha(z) = 0$ is finite on $\langle 0, \alpha\rangle$. If $\alpha > 2$then there exists at least one point z_0 where $F_\alpha(z)$ changes its sign. Let us suppose that there exist two such points z_1, z_2. Let $z_1 > 0$ be the least point with this property and $z_2 > z_1$ be the next one. Since $F_\alpha(z) \geqq 0$ for $z \in \langle 0, z_1\rangle$, $F_\alpha'(z_1) \leqq 0$. But $F_\alpha'(z_2) < F_\alpha'(z_1) \leqq 0$. As $F_\alpha(z) \leqq 0$ for $z \in \langle z_1, z_2\rangle$ there must be $F_\alpha'(z_2) \geqq 0$ which is a contradiction.

$F_{\alpha 1}(z) < F_{\alpha_2}(z)$ for $z > 0$, $\alpha_2 > \alpha_1 \geqq 0$, therefore $z_0(\alpha_1) < z_0(\alpha_2)$ for $2 < \alpha_1 < \alpha_2$. Prove $\lim_{\alpha \to +\infty} z_0(\alpha) = +\infty$. Let $z_0(\alpha_n) \leqq K$, $\alpha_n \in \langle 0, +\infty)$, $\alpha_n \to +\infty$. If $\alpha_n \geqq k > 2$ then $z_0(\alpha_n) \geqq z_0(k) > 0$. As $F_\alpha(z_0(\alpha)) = 0$ there must be

$$\alpha_n = z_0(\alpha_n) \frac{I_0(z_0(\alpha_n))}{I_1(z_0(\alpha_n))} \leqq \frac{I_0(K)}{I_1(z_0(k))} K$$

and this is a contradiction to the assumption $\alpha_n \to +\infty$. Finally, as $F_\alpha(z_0(\alpha)) = 0$ we get

$$\lim_{\alpha \to +\infty} \frac{z_0(\alpha)}{\alpha} = \lim_{\alpha \to +\infty} \frac{I_1(z_0(\alpha))}{I_0(z_0(\alpha))} = \lim_{z \to +\infty} \frac{I_1(z)}{I_0(z)} = 1 .$$

Proposition 2 is proved completely.

In the following we must prove

$$p \equiv \sup_{\alpha \in \langle 0, +\infty)} [\alpha - z_0(\alpha)] < +\infty . \tag{33}$$

If α and hence also $z_0(\alpha)$ is large enough we may use (30) and thus obtain that the function $\alpha - z_0(\alpha) = z_0(I_0(z_0) - I_1(z_0))/(I_1(z_0))$ is bounded for α, z_0 large enough which yields $p < +\infty$. From equation (29) and the relations

$$\varepsilon \left(\frac{\partial \zeta}{\partial t}\right)^2 - \left(\frac{\partial \zeta}{\partial x}\right)^2 = 1 ; \quad \varepsilon \frac{\partial^2 \zeta}{\partial t^2} - \frac{\partial^2 \zeta}{\partial x^2} = \frac{1}{\zeta}$$

there follows

$$\varepsilon \frac{\partial^2 I_0}{\partial t^2} \left(\frac{a\beta\zeta}{\sqrt{\varepsilon}}\right) - \frac{\partial^2 I_0}{\partial x^2} \left(\frac{a\beta\zeta}{\sqrt{\varepsilon}}\right) = \left(\frac{a^2}{\varepsilon} - c\right) I_0 \left(\frac{a\beta\zeta}{\sqrt{\varepsilon}}\right). \tag{34}$$

From (28) we have

$$\sqrt{(\varepsilon)} \lim_{\xi \to x \pm (t-\tau)/\sqrt{\varepsilon}} \frac{\partial I_0}{\partial t} \left(\frac{\alpha\beta\zeta}{\sqrt{\varepsilon}}\right) = \pm \lim_{\xi \to x \pm (t-\tau)/\sqrt{\varepsilon}} \frac{\partial I_0}{\partial x} \left(\frac{\alpha\beta\zeta}{\sqrt{\varepsilon}}\right) = \frac{a^2\beta^2}{2\varepsilon\sqrt{\varepsilon}} (t - \tau) . \tag{35}$$

The integrals

$$M_1(t, \varepsilon) = \frac{\sqrt{\varepsilon}}{2} \int_{x - t/\sqrt{\varepsilon}}^{x + t/\sqrt{\varepsilon}} I_0 \left(\frac{a\beta}{\sqrt{\varepsilon}} \zeta_0\right) d\xi , \quad M_2(t, \varepsilon) = \frac{\sqrt{\varepsilon}}{2} \int_{x - t/\sqrt{\varepsilon}}^{x + t/\sqrt{\varepsilon}} \left|\frac{\partial I_0(a\beta\zeta_0/\sqrt{\varepsilon})}{\partial x}\right| d\xi ,$$

$$M_3(t, \varepsilon) = \frac{\sqrt{\varepsilon}}{2} \int_{x - t/\sqrt{\varepsilon}}^{x + t/\sqrt{\varepsilon}} \left|\frac{\partial}{\partial t} I_0 \left(\frac{a\beta}{\sqrt{\varepsilon}} \zeta_0\right) - \frac{a}{\varepsilon} I_0 \left(\frac{a\beta}{\sqrt{\varepsilon}} \zeta_0\right)\right| d\xi ,$$

$$M_4(t, \varepsilon) = \frac{\sqrt{\varepsilon}}{2} \int_{x - t/\sqrt{\varepsilon}}^{x + t/\sqrt{\varepsilon}} \left|\frac{\partial}{\partial t} I_0 \left(\frac{a\beta}{\sqrt{\varepsilon}} \zeta_0\right)\right| d\xi .$$

may be estimated for $t \geqq 0$, $\varepsilon \in \langle 0, \varepsilon_1 \rangle$, $\varepsilon_1 < a^2/(2a + c)$ in the following way. Using the substitutions $x - \xi = z$, $\sqrt{(1 - z^2\varepsilon/t^2)} = y$, $y^2 = s$ and formula (83), p. 198 [3] we get

$$M_1(t, \varepsilon) = \sqrt{(\varepsilon)} \int_0^{t/\sqrt{\varepsilon}} I_0 \left(\frac{a\beta}{\sqrt{\varepsilon}} \sqrt{\left(\frac{t^2}{\varepsilon} - z^2\right)}\right) dz = t \int_0^1 I_0 \left(\frac{a\beta t}{\varepsilon} y\right) \frac{y}{\sqrt{(1 - y^2)}} dy =$$

$$= \frac{t}{2} \int_0^1 I_0 \left(\frac{a\beta t}{\varepsilon} \sqrt{s}\right) \frac{1}{\sqrt{(1 - s)}} ds = \frac{t}{2} \sqrt{\left(\frac{2\pi\varepsilon}{a\beta t}\right)} I_{1/2} \left(\frac{a\beta t}{\varepsilon}\right) = \frac{\varepsilon}{a\beta} \operatorname{sh} \left(\frac{a\beta t}{\varepsilon}\right).$$

Since

$$\frac{\partial}{\partial x} I_0\left(\frac{a\beta}{\sqrt{\varepsilon}}\zeta_0\right) = -\frac{\partial}{\partial \xi} I_0\left(\frac{a\beta}{\sqrt{\varepsilon}}\zeta_0\right) = \frac{a\beta}{\sqrt{\varepsilon}}\frac{x-\xi}{\zeta_0} I_1\left(\frac{a\beta}{\sqrt{\varepsilon}}\zeta_0\right)$$

is positive for $\xi < x$, we obtain

$$M_2(t, \varepsilon) = -\sqrt{\varepsilon}\int_x^{x+t/\sqrt{\varepsilon}} \frac{\partial I_0}{\partial \xi}\left(\frac{a\beta}{\sqrt{\varepsilon}}\zeta_0\right) \mathrm{d}\xi = \sqrt{\varepsilon}\left[I_0\left(\frac{a\beta t}{\varepsilon}\right) - 1\right].$$

Using the same substitutions as before, formula (79), p. 197, [3] and the relation

$$\frac{\partial}{\partial t} I_0\left(\frac{a\beta}{\sqrt{\varepsilon}}\zeta_0\right) = \frac{a\beta t}{\sqrt{(\varepsilon)}\,\varepsilon\zeta_0} I_1\left(\frac{a\beta}{\sqrt{\varepsilon}}\zeta_0\right)$$

we have

$$M_4(t, \varepsilon) = \sqrt{\varepsilon}\int_0^{t/\sqrt{\varepsilon}} \frac{a\beta t}{\varepsilon^{3/2}\zeta_0} I_1\left(\frac{a\beta}{\sqrt{\varepsilon}}\sqrt{\left(\frac{t^2}{\varepsilon} - z^2\right)}\right) \mathrm{d}z =$$

$$= \frac{a\beta t}{\varepsilon}\int_0^1 I_1\left(\frac{a\beta t}{\varepsilon} y\right)\frac{1}{\sqrt{(1-y^2)}}\,\mathrm{d}y = \frac{a\beta t\pi}{2\varepsilon} I_{1/2}^2\left(\frac{a\beta t}{2\varepsilon}\right) = \operatorname{ch}\left(\frac{a\beta t}{\varepsilon}\right) - 1\,.$$

Substituting $y = [1 - (x-\xi)^2 \,.\, \varepsilon/t^2]^{1/2}$ we obtain

$$M_3(t, \varepsilon) = \frac{at}{\varepsilon}\int_0^1 \left|\beta I_1\left(\frac{a\beta t}{\varepsilon} y\right) - y I_0\left(\frac{a\beta t}{\varepsilon} y\right)\right| \frac{1}{\sqrt{(1-y^2)}}\,\mathrm{d}y\,.$$

As

$$\beta I_1\left(\frac{a\beta t}{\varepsilon} y\right) - y I_0\left(\frac{a\beta t}{\varepsilon} y\right) = \frac{\varepsilon}{a\beta t} I_1\left(\frac{a\beta t}{\varepsilon} y\right) F_\alpha\left(\frac{a\beta t}{\varepsilon} y\right)$$

for $t > 0$ where $\alpha = a\beta^2 t/\varepsilon$ we get

$$\beta I_1\left(\frac{a\beta t}{\varepsilon} y\right) - y I_0\left(\frac{a\beta t}{\varepsilon} y\right) \begin{cases} \geqq 0 & \text{for} \quad y \in \langle 0, y_0\rangle\,, \\ \leqq 0 & \text{for} \quad y \in (y_0, 1\rangle\,, \end{cases}$$

where $y_0 = \beta(a\beta^2 t/\varepsilon)^{-1}\, z_0(a\beta^2 t/\varepsilon)$ and $y_0 \leqq \beta \leqq 1$. Therefore

(36)

$$M_3(t, \varepsilon) = \frac{at}{\varepsilon}\left\{M(1, t, \varepsilon) - 2\int_0^{y_0}\left[y I_0\left(\frac{a\beta t}{\varepsilon} y\right) - \beta I_1\left(\frac{a\beta t}{\varepsilon} y\right)\right]\frac{1}{\sqrt{(1-y^2)}}\,\mathrm{d}y\right\},$$

where

$$M(r, t, \varepsilon) = \int_0^r \left[y I_0\left(\frac{a\beta t}{\varepsilon} y\right) - \beta I_1\left(\frac{a\beta t}{\varepsilon} y\right)\right]\frac{r}{\sqrt{(r^2 - y^2)}}\,\mathrm{d}y\,.$$

Applying the inequality $(1 - y^2)^{-1/2} \leqq y_0(y_0^2 - y^2)^{-1/2}$ to the second integral from (36) we have

$$M_3(t, \varepsilon) \leqq \frac{at}{\varepsilon} \left[M(1, t, \varepsilon) - 2\gamma(t, \varepsilon)\, M(y_0, t, \varepsilon)\right],$$

where $\gamma(t, \varepsilon)$ is defined by (25). From formulae (79), p. 197 and (83), p. 198 [3] we obtain

$$M(r, t, \varepsilon) = -\frac{\varepsilon}{2a\beta t} \{(\beta - r)\, e^{+a\beta tr/\varepsilon} + (\beta + r)\, e^{-a\beta tr/\varepsilon} - 2\beta\}$$

and

$$M_3(t, \varepsilon) \leqq \frac{1}{2\beta} \{(1 - \beta)\, e^{a\beta t/\varepsilon} - (1 + \beta)\, e^{-a\beta t/\varepsilon} + 2\beta\} +$$

$$+ \frac{\gamma(t, \varepsilon)}{\beta} \{(\beta - y_0)\, e^{a\beta t y_0/\varepsilon} + (\beta + y_0)\, e^{-a\beta t y_0/\varepsilon} - 2\beta\}.$$

As $y_0 \leqq \beta \leqq 1$, M_3 may be estimated as follows:

$$M_3(t, \varepsilon) \leqq \frac{1 - \beta}{2\beta}\, e^{a\beta t/\varepsilon} + \frac{\gamma(t, \varepsilon)}{\beta} (\beta - y_0)\, e^{a\beta^2 t/\varepsilon} + 1.$$

Now, we are able to estimate $\|P_\varepsilon h\|_{(t,\varepsilon)}$. By C we always denote a constant depending on a, c, ε_1 only. Supposing $h = h(t, x) \in C(V)$ we obtain

$$\|P_\varepsilon h\|_{0,V_t} \leqq \int_0^t e^{-(a/\varepsilon)(t-\tau)}\, M_1(t - \tau, \varepsilon)\, \|h\|_{0,V_\tau}\, d\tau \leqq \frac{\varepsilon}{2a\beta} \int_0^t e^{-(a/\varepsilon)(1-\beta)(t-\tau)} \|h\|_{0,V_\tau}\, d\tau,$$

$$\left\| \frac{\partial}{\partial x} P_\varepsilon h \right\|_{0,V_t} \leqq \int_0^t e^{-(a/\varepsilon))t-\tau)} [M_2(t - \tau, \varepsilon) + \sqrt{\varepsilon}]\, \|h\|_{0,V_\tau}\, d\tau =$$

$$= \sqrt{\varepsilon} \int_0^t e^{-(a/\varepsilon)(t-\tau)}\, I_0 \left(\frac{a\beta}{\varepsilon} (t - \tau) \right) \|h\|_{0,V_\tau}\, d\tau,$$

$$\left\| \frac{\partial}{\partial t} (P_\varepsilon h) \right\|_{0,V_t} \leqq \int_0^t e^{-(a/\varepsilon)(t-\tau)} (M_3(t - \tau, \varepsilon) + 1)\, \|h\|_{0,V_\tau}\, d\tau \leqq$$

$$\leqq \int_0^t \left[\frac{1 - \beta}{2\beta}\, e^{-(a/\varepsilon)(1-\beta)(t-\tau)} + 2e^{-(a/\varepsilon)(t-\tau)} \right] \|h\|_{0,V_\tau}\, d\tau +$$

$$+ \int_0^{t - 2\varepsilon/a\beta^2} \frac{\beta - y_0}{\beta}\, e^{-(c/a)/(t-\tau)}\, \|h\|_{0,V_\tau}\, d\tau, \quad t \geqq 0.$$

If $h_x \in C(V)$ then $(\partial/\partial x)(P_\varepsilon h) = P_\varepsilon h_x$ and

$$\left\| \frac{\partial^2}{\partial x^2}(P_\varepsilon h) \right\|_{0,V_t} \leqq \sqrt{(\varepsilon)} \int_0^t e^{-(a/\varepsilon)(t-\tau)} I_0\left(\frac{a\beta}{\sqrt{\varepsilon}}(t-\tau)\right) \|h_x\|_{0,V_\tau} \,d\tau ,$$

$$\left\| \frac{\partial^2}{\partial t\, \partial x}(P_\varepsilon h) \right\|_{0,V_t} \leqq \int_0^t \left[\frac{1-\beta}{2\beta} e^{-(a/\varepsilon)(1-\beta)(t-\tau)} + 2e^{-(a/\varepsilon)(t-\tau)} \right] \|h_x\|_{0,V_\tau} +$$

$$+ \frac{1}{\beta} \int_0^{t-2\varepsilon/a\beta^2} (\beta - y_0)\, e^{-(c/a)(t-\tau)} \|h_x\|_{0,V_\tau} \,d\tau .$$

As the function $P_\varepsilon h$ satisfies equation (13), the expression $\varepsilon(\partial^2/\partial t^2)(P_\varepsilon h)$ may be estimated by

$$\varepsilon \left\| \frac{\partial^2}{\partial t^2}(P_\varepsilon h) \right\|_{0,V_t} \leqq c\|P_\varepsilon h\|_{0,V_t} + 2a \left\| \frac{\partial}{\partial t}(P_\varepsilon h) \right\|_{0,V_t} + \left\| \frac{\partial^2}{\partial x^2}(P_\varepsilon h) \right\|_{0,V_t} + \varepsilon\|h\|_{0,V_t} .$$

Finally, we may write

(37)

$$\|P_\varepsilon h\|_{(t,\varepsilon)} \leqq \int_0^t \left[\frac{\varepsilon}{2a\beta}(c+1)\, e^{-(a/\varepsilon)(1-\beta)(t-\tau)} + 2\sqrt{(\varepsilon)}\, e^{-(a/\varepsilon)(t-\tau)} I_0\left(\frac{a\beta}{\varepsilon}(t-\tau)\right) + \right.$$

$$\left. + (2a+1)\left(\frac{\varepsilon c}{2a^2\beta(1+\beta)} e^{-(a/\varepsilon)(1-\beta)(t-\tau)} + 2e^{-(a/\varepsilon)(t-\tau)} \right) \right] \|h\|_\tau^1 \,d\tau +$$

$$+ \frac{2a+1}{\beta} \int_0^{t-2\varepsilon/a\beta^2} (\beta - y_0)\, e^{-(c/a)(t-\tau)} \|h\|_\tau^1 \,d\tau + \varepsilon\|h\|_t^1$$

because

$$1 - \beta = \frac{1-\beta^2}{1+\beta} = \frac{c\varepsilon}{a^2(1+\beta)} .$$

Since

$$I_0(z) \leqq e^z , \quad \beta - y_0 = \beta\left(1 - \frac{z_0(\alpha)}{\alpha}\right) = \beta \frac{\alpha - z_0(\alpha)}{\alpha} \leqq \frac{\varepsilon p}{a\beta t} , \quad \alpha = \frac{a\beta^2}{\varepsilon} t ,$$

we get (19). Furthermore, using formula 6.611, [4] in the second integral of (37) and estimating $\|h\|_t' \leqq \|h\|'$ we get

$$\|P_\varepsilon h\|_\varepsilon \leqq \varepsilon \left[C + \frac{p(2a+1)}{a\beta^2} \left| \log \frac{2\varepsilon}{a\beta^2} \right| \right] \|h\|^1 .$$

If we denote $A_4(\varepsilon) = \varepsilon C(1 + |\log \varepsilon|)$ we obtain (20) and

$$\lim_{\varepsilon \to 0+} \varepsilon^{-1/2} A_4(\varepsilon) = 0 .$$

If $q \in C^1(E_1)$ we obtain

$$\|Q_\varepsilon q\|_{0,V_t} \leqq \|q\|_{0,E_1} e^{-at/\varepsilon} M_1(t,\varepsilon) \leqq \frac{\varepsilon}{a\beta} \|q\|_{0,E_1} e^{-at/\varepsilon} \operatorname{sh}\left(\frac{a\beta t}{\varepsilon}\right)$$

$$\left\|\frac{\partial}{\partial x}(Q_\varepsilon q)\right\|_{0,V_t} + \left\|\frac{\partial^2}{\partial x^2}(Q_\varepsilon q)\right\|_{0,V_t} \leqq \sqrt{(\varepsilon)}\, \|q\|_{1,V_t} e^{-at/\varepsilon} I_0\left(\frac{a\beta t}{\varepsilon}\right)$$

$$\left\|\frac{\partial}{\partial t}(Q_\varepsilon q)\right\|_t^1 \leqq \|q\|_{1,E_1} e^{-at/\varepsilon}(M_3(t,\varepsilon)+1) \leqq$$

$$\leqq \|q\|_{1,E_1}\left(\frac{1-\beta}{2\beta} e^{-(a/\varepsilon)(1-\beta)t} + 2e^{-at/\varepsilon} + \frac{\gamma(t,\varepsilon)}{\beta}(\beta - y_0)\, e^{-ct/a}\right).$$

As $Q_\varepsilon q$ satisfies equation (12) we get

$$\varepsilon\left\|\frac{\partial^2}{\partial t^2}(Q_\varepsilon q)\right\|_{0,V_t} \leqq c\|Q_\varepsilon q\|_{0,V_t} + 2a\left\|\frac{\partial}{\partial t}(Q_\varepsilon q)\right\|_{0,V_t} + \left\|\frac{\partial^2}{\partial x^2}(Q_\varepsilon q)\right\|_{0,V_t}$$

and finally

$$(38) \quad \|Q_\varepsilon q\|_{(t,\varepsilon)} \leqq \left(\varepsilon c e^{-(a/\varepsilon)(1-\beta)t} + 2(2a+1)\, e^{-at/\varepsilon} + 2\sqrt{(\varepsilon)}\, e^{-at/\varepsilon} I_0\left(\frac{a\beta t}{\varepsilon}\right) + \right.$$

$$\left. + \frac{\gamma(t,\varepsilon)}{\beta}(\beta - y_0)\, e^{-ct/a}\right) \|q\|_{C^1(E_1)}$$

which implies (21) and (22). As $\beta - y_0 \leqq \varepsilon p/a\beta t$ we may write

$$\|Q_\varepsilon q\|_{(t,\varepsilon)} \leqq A_2(t,\varepsilon)\, \|q\|_{C^1(E_1)}$$

where

$$A_2(t,\varepsilon) = C\left(\sqrt{(\varepsilon)}\, e^{-(a/\varepsilon)(1-\beta)t} + e^{-at/\varepsilon} + \gamma(t,\varepsilon)\frac{\varepsilon}{t} e^{-ct/a}\right).$$

Now, $\|S_\varepsilon s\|_{(t,\varepsilon)}$ must be estimated. Using (34), (35) we get

$$\frac{\partial}{\partial t}(S_\varepsilon s)(t,x) = \frac{1}{2\sqrt{\varepsilon}} e^{-at/\varepsilon}\left\{s'\left(x + \frac{t}{\sqrt{\varepsilon}}\right) - s'\left(x - \frac{t}{\sqrt{\varepsilon}}\right) + \right.$$

$$\left. + \int_{x-t/\sqrt{\varepsilon}}^{x+t/\sqrt{\varepsilon}} \left[\frac{\partial I_0}{\partial x}\left(\frac{a\beta}{\sqrt{\varepsilon}}\zeta_0\right) s'(\xi) - cI_0\left(\frac{a\beta}{\sqrt{\varepsilon}}\zeta_0\right) s(\xi)\right] d\xi\right\}$$

and we may write the following estimates

$$\|S_\varepsilon s\|_{0,V_t} + \left\|\frac{\partial}{\partial x}(S_\varepsilon s)\right\|_t^1 \leqq e^{-at/\varepsilon}\left(1 + \frac{a}{\varepsilon} M_1 + M_4\right) \|s\|_{2,E_1} \leqq$$

$$\leqq \frac{1}{2\beta}\left[(1+\beta)\,e^{-(a/\varepsilon)(1-\beta)t} + (\beta - 1)\,e^{-(a/\varepsilon)(1+\beta)t}\right]\|s\|_{2,E_1}$$

$$\left\|\frac{\partial}{\partial t}(S_\varepsilon s)\right\|_t^1 \leqq \frac{1}{\varepsilon}\,e^{-at/\varepsilon}\left(\sqrt{\varepsilon} + M_2 + cM_1\right)\|s\|_{2,E_1} \leqq$$

$$\leqq \frac{1}{\sqrt{\varepsilon}}\,e^{-at/\varepsilon}\left[I_0\left(\frac{a\beta t}{\varepsilon}\right) + \frac{\sqrt{(\varepsilon)}\,c}{a\beta}\,\mathrm{sh}\left(\frac{a\beta t}{\varepsilon}\right)\right].$$

Since $S_\varepsilon s$ satisfies equation (12), $\varepsilon(\partial^2/\partial t^2)(S_\varepsilon s)$ may be estimated as

$$\varepsilon\left\|\frac{\partial^2}{\partial t^2}(S_\varepsilon s)\right\|_{0,V_t} \leqq C\|S_\varepsilon s\|_{0,V_t} + 2a\left\|\frac{\partial}{\partial t}(S_\varepsilon s)\right\|_{0,V_t} + \left\|\frac{\partial^2}{\partial x^2}(S_\varepsilon s)\right\|_{0,V_t}$$

and finally we obtain

$$\|S_\varepsilon s\|_{(t,\varepsilon)} \leqq A_3(t,\varepsilon)\,\|s\|_{2,E_1}\,; \tag{39}$$

$$A_3(t,\varepsilon) = C\left[e^{-(a/\varepsilon)(1-\beta)t} + e^{-(a/\varepsilon)(1+\beta)t} + \frac{1}{\sqrt{\varepsilon}}\,e^{-at/\varepsilon}\,I_0\left(\frac{a\beta t}{\varepsilon}\right)\right],$$

which implies (23), (24) and the lemma is completely proved. Let the function $g = g(t, x)$ satisfy (A) and let $\varphi(x, \varepsilon), \psi(x, \varepsilon)$ satisfy the following assumptions:

(B) $\varphi(x, \varepsilon), \psi(x, \varepsilon)$ have bounded and continuous derivatives up to the second or the first order, respectively, with respect to $x \in E_1$ for every $\varepsilon \in \langle 0, \varepsilon_1 \rangle$, $\varepsilon_1 \in (0, a^2/(2a + c))$, $\varphi(x, 0) \in C^{(5)}(E_1)$ and

$$\sup_{\varepsilon\in\langle 0,\varepsilon_1\rangle} \varepsilon^{-1/2}\|\varphi(x,\varepsilon) - \varphi(x,0)\|_{2,E_1} \leqq \sigma < +\infty\,, \quad \sup_{\varepsilon\in\langle 0,\varepsilon_1\rangle} \|\psi(x,\varepsilon)\|_{1,E_1} \leqq \sigma < +\infty\,.$$

(C) $\lim_{\varepsilon\to 0+} \varepsilon^{-1/2}\|\varphi(x,\varepsilon) - \varphi(x,0)\|_{2,E_1} = 0\,, \quad \lim_{\varepsilon\to 0+} \|\psi(x,\varepsilon) - \psi(x,0)\|_{1,E_1} = 0\,.$

Then the functions v, w defined by the formulae

$$v = -P_\varepsilon u_{tt}^0 + Q_\varepsilon(\psi_\varepsilon - \psi_0) + S_\varepsilon(\varphi_\varepsilon - \varphi_0)\,,$$

$$w = Q_\varepsilon(\psi_0(x) - u_t^0(0, x))\,, \quad \psi_\varepsilon(x) = \psi(x, \varepsilon)\,, \quad \varphi_\varepsilon(x) = \varphi(x, \varepsilon)$$

are the solutions of (15) or (16), respectively; $v, w \in \mathfrak{C}_\varepsilon$. From this and from Lemma 1 the following theorem may be obtained.

Theorem 2. *If $g = g(t, x)$ satisfies (A) and $\varphi(x, \varepsilon), \psi(x, \varepsilon)$ satisfy (B), (C) then there exists the unique solution $u = u(t, x, \varepsilon)$ of (14) for every $\varepsilon \in \langle 0, \varepsilon_1 \rangle$, u being of the form (2) where u^0 is the solution of (3), $\lim_{\varepsilon\to 0+} \|v\|_\varepsilon = 0$ and $\lim_{\varepsilon\to 0} \|w\|_{(t,\varepsilon)} = 0$ uniformly with respect to $t \geqq t_0$, $t_0 > 0$.*

We say that $f = f(t, x, p_1, p_2, p_3)$ satisfies assumption (D) if

(D) f and its derivatives f_x, f_{p_i}, $i = 1, 2, 3$ are Lipschitz continuous and bounded, i.e., there exist functions $K_1(\varrho)$, $K_2(\varrho)$ on $\langle 0, +\infty)$ so that for $[t, x] \in V$, $|p_i|, |\tilde{p}_i| \leqq \varrho$ there holds

$$|f(t, x, p_1, p_2, p_3)| \leqq K_1(\varrho),$$

$$|f_x(t, x, p_1, p_2, p_3)| \leqq K_1(\varrho),$$

$$|f_{p_i}(t, x, p_1, p_2, p_3)| \leqq K_1(\varrho), \quad i = 1, 2, 3,$$

$$|f(t, x, p_1, p_2, p_3) - f(t, x, \tilde{p}_1, \tilde{p}_2, \tilde{p}_3)| \leqq K_2(\varrho) \sum_{i=1}^{3} |p_i - \tilde{p}_i|,$$

$$|f_x(t, x, p_1, p_2, p_3) - f_x(t, x, \tilde{p}_1, \tilde{p}_2, \tilde{p}_3)| \leqq K_2(\varrho) \sum_{i=1}^{3} |p_i - \tilde{p}_i|,$$

$$|f_{p_i}(t, x, p_1, p_2, p_3) - f_{p_i}(t, x, \tilde{p}_1, \tilde{p}_2, \tilde{p}_3)| \leqq K_2(\varrho) \sum_{i=1}^{3} |p_i - \tilde{p}_i|.$$

Now, we shall prove the following theorem.

Theorem 3. *Let the functions g, φ, ψ, f satisfy assumptions (A), (B), (D), respectively. Then there exists $\varepsilon_0 > 0$ so that for each $\varepsilon \in \langle 0, \varepsilon_0\rangle$ there exists a unique solution $u = u(t, x, \varepsilon)$ of (1), $u \in \mathfrak{C}_\varepsilon$, $\|u\|_\varepsilon \leqq \varrho$ where ϱ does not depend on ε. Furthermore, if φ, ψ satisfy (C) then the solution u is of the form (2), where u^0 is the solution of (3), $\lim_{\varepsilon\to 0+} \|v\|_\varepsilon = 0$, $\lim_{\varepsilon\to 0+} \|w\|_{(t,\varepsilon)} = 0$ uniformly with respect to $t \geqq t_0$, $t_0 > 0$, and $\|w\|_\varepsilon \leqq \text{const} \|\psi(x, 0) - u_t^0(0, x)\|_{1,E_1}$.*

Proof. Let u^0 solves equation (3^a) and satisfies the initial data (3^b). The solution $u = u(t, x, \varepsilon)$ of (1) will be found in the form $u = u^0 + y$ where $y = y(t, x, \varepsilon)$ satisfies the equation

$$L_\varepsilon y = \varepsilon f(t, x, u^0 + y, u_x^0 + y_x, u_t^0 + y_t) - \varepsilon u_{tt}^0 \tag{39}$$

and the initial data

$$y(0, x, \varepsilon) = \varphi(x, \varepsilon) - \varphi(x, 0), \quad y_t(0, x, \varepsilon) = \psi(x, \varepsilon) - u_t^0(0, x). \tag{40}$$

When $y \in \mathfrak{C}_\varepsilon$ and assumptions (A), (B), (D) are fulfilled, then this is equivalent to the integro-differential equation

$$y = P_\varepsilon[f(t, x, u^0 + y, u_x^0 + y_x, u_t^0 + y_t) - u_{tt}^0] + \\ + Q_\varepsilon[\psi(x, \varepsilon) - u_t^0(0, x)] + S_\varepsilon[\varphi(x, \varepsilon) - \varphi(x, 0)]. \tag{41}$$

Equation (41) will be solved by the method of successive approximations. Denoting the right hand side of (41) by $T_\varepsilon y$, we obtain for $y_i \in \mathfrak{C}_\varepsilon$, $\|y_i\| \leqq \varrho$, $i = 1, 2$ by lemma 1 and by (17), (18)

$$\|T_\varepsilon y_1 - T_\varepsilon y_2\|_\varepsilon \leqq A_4(\varepsilon) \|f(t, x, u^0 + y_1, u_x^0 + (y_1)_x, u_t^0 + (y_1)_t) - \\ - f(t, x, u^0 + y_2, u_x^0 + (y_2)_x, u_t^0 + (y_2)_t)\|^1 \leqq \\ \leqq A_4(\varepsilon) \{K_2(\tilde{\varrho}) (2 + \varrho + C\|\varphi\|_{5,E_1} + C(g)) + K_1(\tilde{\varrho})\} \|y_1 - y_2\|_\varepsilon$$

and

$$\|T_\varepsilon y_i\| \leqq A_4(\varepsilon) \|f(t, x, u^0 + y_i, u_x^0 + (y_i)_x, u_t^0 + (y_i)_t\|^1 + A_4(\varepsilon) \|u_{tt}^0\|^1 + \\ + C(\|\psi(x, \varepsilon) - u_t^0(0, x)\|_{1,E_1} + \varepsilon^{-1/2}\|\varphi(x, \varepsilon) - \varphi(x, 0)\|_{2,E_1}) \leqq \\ \leqq A_4(\varepsilon) K_1(\tilde{\varrho}) (2 + \varrho + C\|\varphi\|_{5,E_1} + C(g)) + C(\|\varphi\|_{5,E_1} + C(g)) + 2C\sigma,$$

where $\tilde{\varrho} = \varrho + C\|\varphi\|_{2,E_1} + C\|g\|_{1,V}$.
If

$$A_4(\varepsilon) \{K_2(\tilde{\varrho}) (2 + \varrho + C\|\varphi\|_{5,E_1} + C(g)) + K_1(\tilde{\varrho})\} < 1$$

$$A_4(\varepsilon) K_1(\tilde{\varrho}) (2 + \varrho + C\|\varphi\|_{5,E_1} + C(g)) + C(\|\varphi\|_{5,E_1} + C(g)) + 2C\sigma \leqq \varrho$$

then the mapping T_ε will map the sphere $\|y\|_\varepsilon \leqq \varrho$ into itself and will be contracting in it. Thus, let us choose $q \in (0, 1)$ and then let us find ϱ satisfying

$$q(2 + \varrho + C\|\varphi\|_{5,E_1} + C(g)) + C(\|\varphi\|_{5,E_1} + C(g)) + 2C\sigma \leqq \varrho \tag{43}$$

and, finally, let us find ε_0, $\varepsilon_0 \in (0, \varepsilon_1\rangle$ so that for this ϱ there holds

$$A_4(\varepsilon) \{K_2(\tilde{\varrho}) (2 + \varrho + C\|\varphi\|_{5,E_1} + C(g)) + K_1(\tilde{\varrho})\} \leqq q. \tag{44}$$

Such a number $\varepsilon_0 > 0$ exists because $\lim_{\varepsilon \to 0+} A_4(\varepsilon) = 0$. By the theorem on contracting mapping (see [7]) there exists a unique $y \in \mathfrak{C}_\varepsilon$ such that $y = T_\varepsilon y$ and $\|y\|_\varepsilon \leqq \varrho$ for $\varepsilon \in (0, \varepsilon_0\rangle$. By Proposition 1 this function solves (39), (40) and hence $u = u_0 + y$ solves (1). The above mentioned function y can be obtained as $\lim_{n \to +\infty} y_n$, where $y_0 \equiv 0$, $y_{n+1} = T_\varepsilon y_n$, $\|y_n\|_\varepsilon \leqq \varrho$, $n = 1, 2, \ldots$, and satisfies the inequalities

$$\|y_n\|_\varepsilon = \sum_{k=1}^{n} \|(y_k - y_{k-1})\|_\varepsilon \leqq \sum_{k=1}^{n} \|y_k - y_{k-1}\|_\varepsilon$$

and

$$\|y\|_\varepsilon \leqq \sum_{k=1}^{\infty} \|y_k - y_{k-1}\|_\varepsilon \leqq \sum_{k=0}^{+\infty} q^k \|y_1\|_\varepsilon \leqq 1/(1 - q) \|y_1\|_\varepsilon$$

and

$$\|y_1\|_\varepsilon = \|T_\varepsilon y_0\|_\varepsilon \leqq A_4(\varepsilon)\, K_1(C\|\varphi\|_{2,E_1} + C\|g\|_{1,V})\,(2 + C\|\varphi\|_{5,E_1} + C(g)) + \\ + A_4(\varepsilon)\,(C\|\varphi\|_{5,E_1} + C(g)) + C\|\psi(x,\varepsilon) - \psi(x,0)\|_{1,E_1} + \\ + C\varepsilon^{-1/2}\|\varphi(x,\varepsilon) - \varphi(x,0)\|_{2,E_1}\,.$$

Thus we obtain

$$\|y\|_\varepsilon \leqq \frac{1}{1-q}\Big\{A_4(\varepsilon)\, K_1(C\|\varphi\|_{2,E_1} + C\|g\|_{1,V})\,(2 + C\|\varphi\|_{5,E_1} + C(g)) + \\ + A_4(\varepsilon)\,(C\|\varphi\|_{5,E_1} + C(g)) + C\|\psi(x,\varepsilon) - \psi(x,0)\|_{1,E_1} + \\ + C\varepsilon^{-1/2}\|\varphi(x,\varepsilon) - \varphi(x,0)\|_{2,E_1}\Big\}.$$

Now, let assumption (C) be fulfilled. If $v, w \in \mathfrak{C}_\varepsilon$ are the solutions of

$$(46^{a})\qquad L_\varepsilon v = \varepsilon f(t, x, u^0 + v, u^0_x + v_x, u^0_t + v_t) - \varepsilon u^0_{tt}$$

$$(46^{b})\qquad v(0,x) = \varphi(x,\varepsilon) - \varphi(x,0)\,, \quad v_t(0,x) = \psi(x,\varepsilon) - \psi(x,0)$$

and

$$(47^{a})\qquad L_\varepsilon w = \varepsilon f(t, x, u^0 + v + w, u^0_x + v_x + w_x, u^0_t + v_t + w_t) - \\ - \varepsilon f(t, x, u^0 + v, u^0_x + v_x, u^0_t + v_t)\,,$$

$$(47^{b})\qquad w(0,x) = 0\,, \\ w_t(0,x) = \psi(x,0) - u^0_t(0,x)\,,$$

respectively, then the function $u = u^0 + v + w$ solves (1). Problems (46) and (47) are equivalent (by Proposition 1) to the integro-differential equations

$$(48)\qquad v = P_\varepsilon[f(t, x, u^0 + v, u^0_x + v_x, u^0_t + v_t) - u^0_{tt}] + \\ + Q_\varepsilon[\psi(x,\varepsilon) - \psi(x,0)] + S_\varepsilon[\varphi(x,\varepsilon) - \varphi(x,0)]\,,$$

$$(49)\qquad w = P_\varepsilon[f(t, x, u^0 + v + w, u^0_x + v_x + w_x, u^0_t + v_t + w_t) - \\ - f(t, x, u^0 + v, u^0_x + v_x, u^0_t + v_t)] + Q_\varepsilon[\psi(x,0) - u^0_t(0,x)]\,,$$

respectively, for $v, w \in \mathfrak{C}_\varepsilon$. These equations will be solved again by the method of successive approximations. Denoting the right hand side of (48), (49) by $\tilde{T}_\varepsilon v$ or $\tilde{\tilde{T}}_\varepsilon w$, respectively we have for $v_i \in \mathfrak{C}_\varepsilon$, $\|v_i\|_\varepsilon \leqq \varrho_1$, $i = 1, 2$

$$\|\tilde{T}_\varepsilon v_1 - \tilde{T}_\varepsilon v_2\|_\varepsilon \leqq A_4(\varepsilon)\,[K_2(\tilde{\varrho}_1)\,(2 + \varrho_1 + C\|\varphi\|_{5,E_1} + C(g)) + \\ + K_1(\tilde{\varrho}_1)]\,\|v_1 - v_2\|_\varepsilon\,,$$

$$\|\tilde{T}_\varepsilon v_i\|_\varepsilon \leqq A_4(\varepsilon)\, K_1(\tilde{\varrho}_1)\,(2 + \varrho_1 + C\|\varphi\|_{5,E_1} + C(g)) + \\ + A_4(\varepsilon)\,(C\|\varphi\|_{5,E_1} + C(g)) + 3C\sigma\,.$$

As before, we choose $q \in (0, 1)$ and ϱ_1 such that

$$q(2 + C\|\varphi\|_{5,E_1} + C(g)) + \max_{\varepsilon\in\langle 0,\varepsilon_1\rangle} A_4(\varepsilon)\,(C\|\varphi\|_{5,E_1} + C(g)) + 3C\sigma \leqq \varrho_1(1 - q)$$

and $\varepsilon_0 = \varepsilon_0(\varrho_1)$ such that for $\varepsilon \in \langle 0, \tilde{\varepsilon}_0\rangle$,

$$A_4(\varepsilon)\,[K_2(\tilde{\varrho}_1)\,(2 + \varrho_1 + C\|\varphi\|_{5,E_1} + C(g)) + K_1(\tilde{\varrho}_1)] \leqq q$$

hold. There exists a unique $v \in \mathfrak{C}_\varepsilon$ such that $T_\varepsilon v = v$, $\|v\|_\varepsilon \leqq \varrho_1$. This function can be obtained as the limit of v_n, $u = 0, 1, \ldots$ where $v^0 \equiv 0$, $v^{n+1} = \tilde{T}_\varepsilon v_n$ and we have

$$\|v\|_\varepsilon \leqq \frac{1}{1-q}\,\|v^1\|_\varepsilon = \frac{1}{1-q}\,\|\tilde{T}_\varepsilon v^0\|_\varepsilon \leqq A_4(\varepsilon)\,K_1(C\|\varphi\|_{2,E_1} +$$
$$+ C\|g\|_{1,V})\,(2 + C\|\varphi\|_{5,E_1} + C(g)) + A_4(\varepsilon)\,(C\|\varphi\|_{5,E_1} + C(g)) +$$
$$+ C\|\psi(x, \varepsilon) - \psi(x, 0)\|_{1,E_1} + C\varepsilon^{-1/2}\|\varphi(x, \varepsilon) - \varphi(x, 0)\|_{2,E_1}$$

for $\varepsilon \in \langle 0, \tilde{\varepsilon}_0\rangle$. Thus by (C) we get $\lim_{\varepsilon\to 0+} \|v^1\|_\varepsilon = 0$ and hence $\lim_{\varepsilon\to 0+} \|v\|_\varepsilon = 0$. Now, for $w_i \in \mathfrak{C}_\varepsilon$, $\|w_i\| \leqq \varrho_2$, $i = 1, 2$, we obtain from (49)

$$\|\tilde{\tilde{T}}_\varepsilon w_1 - \tilde{\tilde{T}}_\varepsilon w_2\|_\varepsilon \leqq A_4(\varepsilon)\,\{K_2(\tilde{\varrho}_1 + \varrho_2)\,[2 + \tilde{\varrho}_1 + \varrho_2 + C\|\varphi\|_{5,E_1} + C(g)] +$$
$$+ K_1(\tilde{\varrho}_1 + \varrho_2)\}\,\|w_1 - w_2\|_\varepsilon$$
$$\|\tilde{\tilde{T}}_\varepsilon w_i\|_\varepsilon \leqq A_4(\varepsilon)\,K_1(\tilde{\varrho}_1 + \varrho_2)\,[\varrho_2 + 2(2 + \tilde{\varrho}_1 + C\|\varphi\|_{5,E_1} + C(g)] +$$
$$+ C\|\psi(x, 0) - u_t^0(0, x)\|_{1,E_1}$$

and again, if we choose ϱ_2 such that

$$2q(2 + \tilde{\varrho}_1 + C\|\varphi\|_{5,E_1} + C(g)) + C\|\psi(x, 0)\|_{1,E_1} + C\|\varphi\|_{2,E_1} +$$
$$+ C\|g\|_{1,V} \leqq \varrho_2(1 - q)$$

and $\tilde{\tilde{\varepsilon}}_0 > 0, \tilde{\tilde{\varepsilon}}_0 \leqq \varepsilon_1$ such that

$$A_4(\tilde{\tilde{\varepsilon}}_0)\,\{K_2(\tilde{\varrho}_1 + \varrho_2)\,[(2 + \tilde{\varrho}_1 + C\|\varphi\|_{5,E_1} + C(g))\,2 + \varrho_2] + K_1(\tilde{\varrho}_1 + \varrho_2)\} \leqq q$$

then there exists a unique solution w of the equation $w = \tilde{\tilde{T}}_\varepsilon w$, such that $\|w\| \leqq \varrho_2$ and $w = \lim_{n\to+\infty} w^n$ in $\mathfrak{C}_\varepsilon$ where $w^0 \equiv 0$, $w^{n+1} = \tilde{\tilde{T}}_\varepsilon w^n$, $n = 0, 1, 2, \ldots$ As

$$\|w^{n+1}\|_\varepsilon \leqq \sum_{k=0}^{n} \|w^{k+1} - w^k\|_\varepsilon \leqq \frac{1}{1-q}\,\|w^1\|_\varepsilon$$

and

$$\|w^1\|_\varepsilon \leqq C\|\psi(x, 0) - u_t^0(0, x)\|_{1,E_1}$$

we obtain $\|w\|_\varepsilon \leqq (c/(1-q))\,\|\psi(x, 0) - u_t^0(0, x\|_{1,E_1}$. By (49) and by estimates (20), (21) there follows that $\|w\|_{(t,\varepsilon)} \to 0$ as $\varepsilon \to 0+$ uniformly with respect to $t \geqq t_0$, $t_0 > 0$.

Remark 1. From (43) and (44) there follows that ϱ depends only on $C(g)$, $\|\varphi\|_{5,E_1}$ and σ, and ε_0 depends on ϱ, $C(g)$ and $\|\varphi\|_{5,E_1}$. For every $\varrho > 0$, ϱ satisfying (43) there exists $\varepsilon_0(\varrho)$ such that for every $\varepsilon \in (0, \varepsilon_0(\varrho))$ the solution u of (1^a) satisfying $\|u\|_\varepsilon \leqq \varrho$ is unique. Furthermore, one can prove (using for example the method of energy estimates), that for every $\varepsilon \in (0, \varepsilon_1\rangle$ there exists at most one solution u of (1), $u \in \mathfrak{C}_\varepsilon$. The theorem is completely proved.

Remark 2. If we have $\psi(x, 0) \equiv u_t^0(0, x)$ in condition (C) then $\|u - u^0\|_\varepsilon \to 0$, as $\varepsilon \to 0$, and if furthermore $\psi(x, \varepsilon) \equiv u_t^0(0, x)$, $\varphi(x, \varepsilon) \equiv u^0(0, x)$ then $\lim_{\varepsilon \to 0+} \varepsilon^{-1/2}\|u - u^0\|_\varepsilon = 0$.

II. THE PERIODIC SOLUTION OF THE EQUATION (1^a) IN V

In order to find the periodic solution of (1^a) in V we need the following lemma.

Lemma 2. *Let $g = g(t, x)$, $f = f(t, x, p_1, p_2, p_3)$ satisfy* (A), (D), *$u_i = u_i(t, x, \varepsilon)$, $u_i \in \mathfrak{C}_\varepsilon$, $\|u_i\|_\varepsilon \leqq \varrho$, $i = 1, 2$ be the solutions of* (1^a) *with the initial data $\varphi_i(x, \varepsilon)$, $\psi_i(x, \varepsilon)$ for $\varepsilon \in (0, \varepsilon_1\rangle$, $\varepsilon_1 < a^2/(2a + c)$ where φ_i, ψ_i satisfy* (B) *and $\|\varphi_i(x, 0)\|_{5,E_1} \leqq \sigma_1$, $i = 1, 2$. Then there exists $\varepsilon_0 \in (0, \varepsilon_1\rangle$, $\varepsilon_0 = \varepsilon_0(\sigma_1, \varrho)$ such that $\lim_{t \to +\infty} \|u_1 - u_2\|_{(t,\varepsilon)} = 0$ uniformly with respect to $\varepsilon \in \langle 0, \varepsilon_0\rangle$ and φ_i, ψ_i satisfying* (B) *with the same constant σ on $\langle 0, \varepsilon_1\rangle$ and $\|\varphi_i(x, 0)\|_{5,E_1} \leqq \sigma_1$, $i = 1, 2$.*

Proof. Denoting u_i^0 the solution of (3^a) with the initial conditions $u_i^0(0, x) = \varphi_i(x, 0)$ we get from Proposition 1 that $y_i = u_i - u_i^0$ satisfy (41) and then we can write for $y = y_1 - y_2$

$$(50) \qquad y = P_\varepsilon[f(t, x, u_1, (u_1)_x, (u_1)_t) - f(t, x, u_2, (u_2)_x, (u_2)_t)] + \\ + Q_\varepsilon[\psi_1(x, \varepsilon) - \psi_2(x, \varepsilon) - u_t^0] + S_\varepsilon[\varphi_1(x, \varepsilon) - \varphi_1(x, 0)] - S_\varepsilon[\varphi_2(x, \varepsilon) - \varphi_2(x, 0)];$$

as the function $u^0 = u_1^0 - u_2^0$ satisfies the equation $2au_t^0 - u_{xx}^0 + cu^0 = 0$ and $u^0(0, x) = \varphi_1(x, 0) - \varphi_2(x, 0)$, we obtain by (17) and the assumptions of Lemma 2 the estimate

$$(51) \qquad \|u^0\|_{(t,\varepsilon)} \leqq \max(1, \varepsilon_1)\left\{\|u^0\|_{2,V_t} + \left\|\frac{\partial^3 u^0}{\partial t^2 \partial x}\right\|_{0,V_t}\right\} \leqq 2C\sigma e^{-ct/2a},$$

where C depends only on a, c, ε_1.

By Lemma 1 we obtain from (50)

$$\|y\|_{(t,\varepsilon)} \leqq c\mathfrak{M}(\|f(t, x, u_1, (u_1)_x (u_1)_t) - f(t, x, u_2, (u_2)_x, (u_2)_t)\|^1)(t) + \\ + A_2(t, \varepsilon)[\|\psi_1(x, \varepsilon) - \psi_2(x, \varepsilon)\|_{1,E_1} + \|u_t^0(0, x)\|_{1,E_1}] + \\ + A_3(t, \varepsilon)[\|\varphi_1(x, \varepsilon) - \varphi_1(x, 0)\|_{2,E_1} + \|\varphi_2(x, \varepsilon) - \varphi_2(x, 0)\|_{2,E_2}]$$

where

$$(52)\qquad (\mathfrak{M}r)(t) = \int_0^t [\sqrt{(\varepsilon)}\, e^{-(a/\varepsilon)(1-\beta)(t-\tau)} + e^{-(a/\varepsilon)(t-\tau)}]\, r(\tau)\, d\tau + $$
$$+ \int_0^{t-2\varepsilon/a\beta^2} \frac{1}{t-\tau} e^{-(c/a)(t-\tau)}\, r(\tau)\, d\tau + \varepsilon\, r(t)\,.$$

By (26), (27) the following estimates hold

$$A_2(t, \varepsilon) \leqq C e^{ct/2a}\,; \quad A_3(t, \varepsilon) \leqq \varepsilon^{-1/2} C e^{-ct/2a}\,.$$

Using assumptions (B), (D) we can write

$$(53)\qquad \|y\|_{(t,\varepsilon)} \leqq C[K_2(\varrho)(2 + \varrho) + K_1(\varrho)]\, \mathfrak{M}(\|y\|_{(t,\varepsilon)} + \|u^0\|_{(t,\varepsilon)}) + $$
$$+ C(\sigma + \sigma_1)\, e^{-ct/2a}\,.$$

To prove Lemma 2 we use the following

Proposition 3. *If* $0 \leqq r_1(\tau) \leqq r_2(\tau)$, $\tau \in \langle 0, t\rangle$ *then* $\mathfrak{M}(r_1)(t) \leqq \mathfrak{M}(r_2)(t)$, $t \geqq 0$. *Furthermore, if* $0 \leqq r(t) \leqq C_1 e^{-ct/2a}$ *then*

$$(54)\qquad (\mathfrak{M}^n r)(t) \leqq C_1 [A_5(\varepsilon)(t + 1)]^n\, e^{-ct/2a}\,,$$

where

$$A_5(\varepsilon) = \sqrt{(\varepsilon)} \max \left\{ [1 + \sqrt{(\varepsilon)}],\ \frac{2\sqrt{\varepsilon}}{a(1 + \beta^2)} + \sqrt{(\varepsilon)} \left|\log \frac{2\varepsilon}{a\beta^2}\right| + \sqrt{\varepsilon} \right\}.$$

The first statement is evident, the latter one will be proved by the mathematical induction. From (52) we obtain

$$(\mathfrak{M}r)(t) \leqq \left\{ \sqrt{(\varepsilon)}\, e^{-ct/a(1+\beta)} \int_0^t e^{-c\tau(1-\beta)/2a(1+\beta)}\, d\tau + e^{-at/\varepsilon} \int_0^t e^{(a/2\varepsilon)(1+\beta^2)\tau}\, d\tau + \right.$$
$$\left. + \varepsilon e^{-ct/a} \int_0^{t-2\varepsilon/a\beta^2} \frac{1}{t-\tau} e^{c\tau/2a}\, d\tau + \varepsilon e^{-ct/2a} \right\} C_1 \leqq$$
$$\leqq C_1 \left\{ \sqrt{(\varepsilon)}\, t e^{-ct/2a} + \frac{2\varepsilon}{a(1 + \beta^2)} e^{-at/\varepsilon} (e^{(a/2\varepsilon)(1+\beta^2)t} - 1) + \right.$$
$$\left. + \varepsilon e^{-ct/2a} \left|\log t - \log \frac{2\varepsilon}{a\beta^2}\right| + \varepsilon e^{-ct/2a} \right\} \leqq C_1 A_5(\varepsilon)(t + 1)\, e^{-ct/2a}\,;$$

thus Proposition 3 holds for $n = 1$. Let it hold for n ($n \geqq 1$); using the first statement of Proposition 3 we have

$$(\mathfrak{M}^{n+1} r)(t) = \mathfrak{M}(\mathfrak{M}^n r)(t) \leqq \mathfrak{M}(C_1 A_5^n(\varepsilon)(t + 1)^n e^{-ct/2a}) \leqq$$
$$\leqq C_1 A_5^n(\varepsilon)(t + 1)^n\, \mathfrak{M} e^{-ct/2a} \leqq C_1 A_5^{n+1}(\varepsilon)(t + 1)^{n+1}\, e^{-ct/2a}$$

and Proposition 3 is completely proved.

Now, using (51), (20) and $\|y\|_{(t,\varepsilon)} \leqq \|u_1 - u_2\|_{(t,\varepsilon)} + \|u^0\|_{(t,\varepsilon)} \leqq 2(\varrho + C\sigma_1)$ we obtain from (53)

$$\|y\|_{(t,\varepsilon)} \leqq 2(\varrho + C\sigma_1)\, C\, K(\varrho)\, A_4(\varepsilon) + C\, K(\varrho)\, \mathfrak{M}(\sigma_1 e^{-ct/2a}) + C(\sigma + \sigma_1)\, e^{-ct/2a},$$

where $K(\varrho) = K_2(\varrho)(2 + \varrho) + K_1(\varrho)$. Using this inequality in (53) and repeating this process n-times we get

$$\|y\|_{(t,\varepsilon)} \leqq 2(\varrho + C\sigma_1)\,[C\, K(\varrho)\, A_4(\varepsilon)]^n + \sum_{k=1}^{n} [C\, K(\varrho)]^k\, \mathfrak{M}^k(\sigma_1 e^{-ct/2a}) + $$
$$+ C(\sigma + \sigma_1) \sum_{k=0}^{n-1} \mathfrak{M}^k(e^{-ct/2a})\, [C\, K(\varrho)]^k\,; \quad n = 1, 2, \ldots$$

If we choose $\varepsilon_0 \in (0, \varepsilon_1\rangle$ so that

$$C\, K(\varrho)\, A_4(\varepsilon) \leqq q < 1\,, \quad C\, K(\varrho)\, A_5(\varepsilon) \leqq q < 1$$

for every ε, $\varepsilon \in \langle 0, \varepsilon_0\rangle$ we obtain by Proposition 3

$$\|y\|_{(t,\varepsilon)} \leqq 2\varrho q^n + C(\sigma_1 + \sigma) \frac{1}{1 - q} (t + 1)^n\, e^{-ct/2a}\,.$$

Now, to every $\eta > 0$ we can find n_0 natural, $t_0 > 0$ so that $2(\varrho + C\sigma_1)\, q^{n_0} < \eta/2$ and

$$C(\sigma + \sigma_1) \frac{1}{1 - q} (t + 1)^{n_0}\, e^{-ct/2a} < \frac{\eta}{2}\,, \quad t \geqq t_0$$

which means $\lim_{t \to +\infty} \|y_1 - y_2\|_{(t,\varepsilon)} = 0$ uniformly with respect to $\varepsilon \in \langle 0, \varepsilon_0\rangle$ and φ_i, ψ_i, $i = 1, 2$ satisfying the assumption of Lemma 2 with the constants σ_1, σ.

Theorem 4. *Let the functions $g = g(t, x)$, $f = f(t, x, p_1, p_2, p_3)$ be ω-periodic (in t) on V and satisfy assumptions* (A) *or* (D), *respectively. Then there exists $\varepsilon_0 > 0$ so that for every $\varepsilon \in \langle 0, \varepsilon_0\rangle$ equation* (1^a) *has an ω-periodic solution $U = U(t, x, \varepsilon)$ which is of the form $U = U^0 + V$ where U^0 is the ω-periodic (in t) solution of* (3^a), $\lim_{\varepsilon \to 0+} \|V\|_\varepsilon = 0$.

Proof. If g satisfies assumption (A) then the ω-periodic (in t) solution U^0 of (3^a) exists and is unique, $U^0 \in C^2(V)$, $(\partial^3 U^0/\partial t^2\, \partial x) \in C(V)$, $(\partial^5 V^0/\partial x^5) \in C(V)$. Now, as $\varphi(x, \varepsilon) \equiv U^0(0, x)$, $\psi(x, \varepsilon) \equiv U^0_t(0, x)$ satisfy assumptions (B), (C) there exist $\varepsilon_2 > 0$, $\varrho \geqq 0$ so that (1^a) has the solution $u = u(t, x, \varepsilon)$, $\|u\|_\varepsilon \leqq \varrho$, $0 \leqq \varepsilon \leqq \varepsilon_2$ with the initial data $u(0, x, \varepsilon) = U^0(0, x)$, $u_t(0, x, \varepsilon) = U^0_t(0, x)$. u is of the form (2) where $w \equiv 0$ (from the proof of Theorem 3)

$$\|v\|_\varepsilon \leqq C\, C(g)\, [K_1(C\|U^0(t, x)\|_{1,V} + C\|g\|_{1,V}) + 1]\, A_4(\varepsilon)\,. \tag{55}$$

If we put $u_n(t, x) = u(t + n\omega, x)$, $n = 1, 2, \ldots$ for $[t, x] \in V$ then $u_n = u_n(t, x, \varepsilon)$ are the solutions of (1^a) with the initial data

$$u_n(0, x, \varepsilon) = u(n\omega, x) \equiv \varphi_n(x, \varepsilon)\,, \quad (u_n)_t\,(0, x, \varepsilon) = u_t(n\omega, x) \equiv \psi_n(x, \varepsilon)$$

and $u_n \in \mathfrak{C}_\varepsilon$, $\|u_n\|_\varepsilon \leqq \|u\|_\varepsilon$. By (55) we have

$$\|\varphi_n(x, \varepsilon) - U^0(0, x)\|_{2,E_1} = \|u(n\omega, x, \varepsilon) - U^0(n\omega, x)\|_{2,E_1} =$$
$$= \|v(n\omega, x, \varepsilon)\|_{2,E_1} \leqq \|v\|_\varepsilon \leqq A_4(\varepsilon)\, C\, C(g)\, K_1(C\, C(g))\,.$$

As $\varepsilon^{-1/2} A_4(\varepsilon) \xrightarrow[\varepsilon \to 0+]{} 0$, $\varepsilon^{-1/2}\|\varphi_n(x, \varepsilon) - U^0(0, x)\|_{2,E_1} \to 0$. Furthermore, we get from (55)

$$\|\psi_n(x, \varepsilon) - U_t^0(0, x)\|_{1,E_1} = \|v_t(n\omega, x, \varepsilon)\|_{1,E_1} \leqq \|v\|_\varepsilon\,.$$

Thus φ_n, ψ_n satisfy the assumptions of Lemma 2 and by this Lemma there exists $\varepsilon_0 \in (0, \varepsilon_2\rangle$ such that to any $\eta > 0$ there exists $t_\eta > 0$ so that $\|u_k - u\|_{(t,\varepsilon)} < \eta$ for every $t \geqq t_\eta$, $k = 1, 2, \ldots$ Thus for every $t \geqq 0$, $n, m \geqq m_\eta = t_\eta/\omega$, this implies the inequality $\|u_n - u_m\|_{(t,\varepsilon)} < \eta$; it means that $\{v_n\}$, $v_n = u_n - V^0$ is a fundamental sequence. Thus there exists a function $V \in \mathfrak{C}_\varepsilon$ such that $\lim_{n \to +\infty} v_n = V$ in $\mathfrak{C}_\varepsilon$. Further, we can write $V(t, x) = \lim_{n \to +\infty} v_{n+1}(t, x) = \lim_{n \to +\infty} v_n(t + \omega, x) = V(t + \omega, x)$. $U = U^0 + V$ is ω-periodic (in t) solution of (1^a). As $\|v_n\|_\varepsilon \leqq \|v\|_\varepsilon$, the function V is bounded by the same number as v: $\|V\|_\varepsilon \leqq \|v\|_\varepsilon$, therefore $\lim_{\varepsilon \to 0+} \|V\|_\varepsilon = 0$.

Theorem 5. *Let $u_i \in \mathfrak{C}_\varepsilon$, $\|u_i\|_\varepsilon \leqq R$, u_i be the ω-periodic solutions of (1^a) for $\varepsilon \in (0, \varepsilon_3\rangle$. Then there exists $\varepsilon_0 > 0$ such that $u_1(t, x, \varepsilon) = u_2(t, x, \varepsilon)$ for $[t, x] \in V$ and $\varepsilon \in (0, \varepsilon_0\rangle$, $\lim_{\varepsilon \to 0+} \|u_i - U^0\|_\varepsilon = 0$, $i = 1, 2$.*

Proof. By (8) $u = u_1 - u_2$ satisfies the equation

$$u(t, x) = P_\varepsilon[f(t, x, u_1, (u_1)_x, (u_1)_t) - f(t, x, u_2, (u_2)_x, (u_2)_t)] +$$
$$+ Q_\varepsilon[u_t(0, x)] + S_\varepsilon[u(0, x)]\,;$$

from Lemma 1 and (52) we obtain

$$\|u\|_{(t,\varepsilon)} \leqq C\{K(R)\, \mathfrak{M}(\|u\|_{(t,\varepsilon)}) + 2R\left(1 + \frac{1}{\sqrt{\varepsilon}}\right) e^{-ct/2a}\,.$$

Now, we shall use the same process as that in Lemma 3 to obtain

$$\|u\|_{(t,\varepsilon)} \leqq 2R[CK(R)\, A_4(\varepsilon)]^n + 2RC\left(1 + \frac{1}{\sqrt{\varepsilon}}\right) e^{-ct/2a} \sum_{k=0}^{n} (t + 1)^k\, [C\, K(R)\, A_5(\varepsilon)]^k\,,$$
$$n = 1, 2, \ldots$$

Choosing $\varepsilon_0 > 0$ so that for $\varepsilon \in (0, \varepsilon_0\rangle$ there holds $C K(R) A_4(\varepsilon) \leqq q < 1$ and $C K(R) A_5(\varepsilon) \leqq q$ we can find to every $\varepsilon > 0$, $\varepsilon \in \langle 0, \varepsilon_0\rangle$ and $\eta > 0$ a number $t_0 > 0$ such that $\|u\|_{(t,\varepsilon)} < \eta$ for $t \geqq t_0$. It means: $0 \leqq \|u\|_{(t,\varepsilon)} = \|u\|_{(t+n\omega,\varepsilon)}$ for $n \to +\infty$ and $t \geqq 0$; from this there follows $u_1(t, x, \varepsilon) \equiv u_2(t, x, \varepsilon)$. The last statement of Theorem 5 may be obtained easily from Theorem 4.

Remark: The mixed problem (1) on $\langle 0, +\infty) \times \langle 0, \pi\rangle$ with the boundary data

$$u(t, 0) = u(t, \pi) = 0$$

may be solved by the same procedure. Let g, φ, ψ, f satisfy (A), (B) or (C) and (D), respectively for $t \geqq 0$, $x \in \langle 0, \pi\rangle$ and, moreover, let

$$(56) \qquad g(t, 0) = g(t, \pi) = 0\,, \quad \frac{\partial^2 g}{\partial x^2}(t, 0) = \frac{\partial^2 g}{\partial x^2}(t, \pi) = 0\,, \quad t \geqq 0\,,$$

$$\varphi^{(2k)}(0, \varepsilon) = \varphi^{(2k)}(\pi, \varepsilon) = 0\,, \quad k = 0, 1\,; \quad \varphi^{(4)}(0, 0) = \varphi^{(4)}(\pi, 0) = 0\,, \varepsilon \geqq 0\,,$$

$$\psi(0, \varepsilon) = \psi(\pi, \varepsilon) = 0\,,$$

$$(57) \qquad f(t, 0, 0, p_2, 0) = f(t, \pi, 0, p_2, 0)\,.$$

Then we define the functions $\tilde{f}, \tilde{g}, \tilde{\varphi}, \tilde{\psi}$ for $t \geqq 0$, $x \in E_1$, $p_i \in E_1$, $i = 1, 2, 3$, as follows: $\tilde{g}, \tilde{\varphi}, \tilde{\psi}$ are odd and 2π-periodic in x and equal to g, φ, ψ for $x \in \langle 0, \pi\rangle$, respectively, $\tilde{f}(t, -x, -p_1, p_2, -p_3) = -\tilde{f}(t, x + 2\pi, p_1, p_2, p_3) = -f(t, x, p_1, p_2, p_3)$ for $x \in (0, \pi)$, $\tilde{f}(t, n\pi, 0, p_2, 0) = f(t, 0, 0, p_2, 0)$, for any integer n, $t \geqq 0$, $p_2 \in E_1$. Denoting $\mathfrak{C}^1_\varepsilon$ as a set of the odd and 2π-periodic in x functions from $\mathfrak{C}_\varepsilon$, it may be easily shown that $\mathfrak{C}^1_\varepsilon$ with the norm $\|\ldots\|_\varepsilon$ is a Banach space and the operators $T_\varepsilon, \tilde{T}_\varepsilon, \tilde{\tilde{T}}_\varepsilon$ map $\mathfrak{C}^1_\varepsilon$ into itself. By the same way as before (with $\mathfrak{C}^1_\varepsilon$ instead of $\mathfrak{C}_\varepsilon$) we can prove that there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in \langle 0, \varepsilon_0\rangle$ there exists a unique solution $\tilde{u} = \tilde{u}(t, x, \varepsilon)$ of problem (1) with the functions $\tilde{g}, \tilde{\varphi}, \tilde{\psi}, \tilde{f}$, $\tilde{u} \in \mathfrak{C}^1_\varepsilon$ and $\tilde{u} = \tilde{u}^0 + \tilde{y}$ (under assumption (B)), $\tilde{u} = \tilde{u}^0 + \tilde{v} + \tilde{w}$, $\|\tilde{v}\|_\varepsilon \underset{\varepsilon \to 0}{\to} 0$, $\|\tilde{w}\|_{(t,\varepsilon)} \underset{\varepsilon \to 0+}{\to} 0$, $t > 0$ (under assumption (C)), where $\tilde{u}^0$ is the solution of (3) with $\tilde{g}, \tilde{\varphi}$.

If we define the functions $u = \tilde{u}$, $u^0 = \tilde{u}^0$, $y = \tilde{y}$, $v = \tilde{v}$, $w = \tilde{w}$ for $x \in \langle 0, \pi\rangle$, $t \geqq 0$ then u^0 solves the linear mixed problem given by (3) and by the boundary conditions $u^0(t, \pi) = u^0(t, 0) = 0$ on $\langle 0, +\infty) \times \langle 0, \pi\rangle$ and $u = u(t, x, \varepsilon)$ solves the mixed problem formulated above. Similarly, the existence of the periodic solution u of (1^a), $u(t, 0) = u(t, \pi) = 0$ on $\langle 0, +\infty) \times \langle 0, \pi\rangle$ under assumptions (A), (D), (56), (57) and g, f being ω-periodic in t may be treated.

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