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All differentiable structures involved in this paper are real structures af class C^{∞} or C^{ω}. Unless otherwise specified all propositions remain valid for both these classes of differentiability.

INTRODUCTION

The goal of this paper is to formulate in the modern way the theory of geometric invariants of submanifolds of a given manifold provided with a geometric structure. Our invariants are generalization of those known from the classical geometries. We start by a differentiable manifold M which becomes an object of differential geometry by giving a sheaf $\mathscr{F}$ of germs of vector fields on it. So that to study all imbeddings of a differentiable manifold B into M we take first the trivial fibered manifold $(M \times B, p, B)$ and then the fibered manifold $(\tilde{J}^l, \pi^l_{-1}, B)$ of l-jets of all local cross sections of $(M \times B, p, B)$. In fact our field of interest is much larger, for we consider all differentiable mappings of B into M.

Paragraph 1 is preliminary and is concerned with the basic definitions.

Paragraph 2 is devoted to the prolongation of $\mathscr{F}$. As the l-th prolongation of $\mathscr{F}$ we get a sheaf $\mathscr{F}^l$ of germs of vector fields on $\tilde{J}^l$. In this paragraph $\mathscr{F}$ is supposed to be a sheaf of vector spaces only.

Paragraph 3 treats the case when $\mathscr{F}$ is a sheaf of Lie algebras. The main result is that $\mathscr{F}^l$ is also a sheaf of Lie algebras. This result is based on Proposition 6 concerned with the prolongation of the bracket of two vector fields, which, as it seems to me, even if it is of a certain importance in the theory of differential equations (see [5], p. 48) has never been correctly proved.

In paragraph 4 we develop the theory of invariants. The sheaf of invariants of order l is defined here and it is shown that every fiber of this sheaf can be characterized by a finite number of its elements. We also prove here that there exists an integer $l_0 \geqq 0$ such that knowing all invariants of order l_0 we can get all invariants of any higher order $l \geqq l_0$ by a certain process of "prolongation of invariants".

Paragraph 5 introduces the pseudogroup $\Gamma(\mathscr{F})$ associated to the sheaf $\mathscr{F}$. As the main result we have proved here that a local 1-parameter group of transformations of M whose l_0-th prolongation preserves all invariants of order l_0 belongs to $\Gamma(\mathscr{F})$.

1. FIBERED MANIFOLDS

Definition 1. Let (E, p, B) be a *bundle* where E and B are differentiable manifolds and the map p is a submersion (i.e. p_* is of maximal rank at every point $x \in E$). Such bundle (E, p, B) will be called *fibered manifold*.

Let (E, p, B) be a fibered manifold, $\dim E = m$, $\dim B = n$. As p is a submersion we can find to every point $x \in E$ his open neighborhood U such that

(i) pU is an open neighborhood of $px \in B$;

(ii) there are a coordinate system $(y^1, \ldots, y^m)$ on U, and a coordinate system $(x^1, \ldots, x^n)$ on pU such that $y^1 = x^1 \circ p, \ldots, y^n = x^n \circ p$

(see [1], p. 80, Prop. 2). A coordinate system of type $(y^1, \ldots, y^m)$ we shall call *natural coordinate system* and write it in the form $(x^1, \ldots, x^n, y^{n+1}, \ldots, y^m)$.

Definition 2. *Local cross section* of the fibered manifold (E, p, B) is a differentiable mapping $\sigma : W \to E$, where $W \subset B$ is an open set, such that $p \circ \sigma = id$.

In the next let $\tilde{J}(E, p, B)$ (briefly $\tilde{J}$) denote the *sheaf of germs of all local cross sections*, $\tilde{J}^l(E, p, B)$ (briefly $\tilde{J}^l$) for any integer $l \geqq 0$ the *set of l-jets of all local cross sections* of the fibered manifold (E, p, B). For the sake of completeness we define $\tilde{J}^{-1} = B$. For $l_1 \geqq l_2 \geqq -1$ there exists the natural projection $\pi^{l_1}_{l_2} : \tilde{J}^{l_1} \to \tilde{J}^{l_2}$. Likewise for any $l \geqq -1$ there exists the natural projection $\pi_l : \tilde{J} \to \tilde{J}^l$. It is well known that $\tilde{J}^l(E, p, B)$ can be provided in the natural way with the structure of a differentiable manifold. At the same time $\pi^{l_1}_{l_2}$ is a differentiable mapping, π_l is a continuous mapping, and $(\tilde{J}^{l_1}, \pi^{l_1}_{l_2}, \tilde{J}^{l_2})$ is a fibered manifold. It can be easily shown that there is the natural diffeomorphism between $\tilde{J}^0$ and E.

Definition 3. Let M, B be differentiable manifolds. Let p and q be the natural projections of $E = M \times B$ onto B and M respectively. A fibered manifold (E, p, B) is called a *trivial fibered manifold*.

2. PROLONGATION OF SHEAVES

Let $\mathscr{S}(M)$ be the sheaf of germs of all differentiable vector fields on the manifold M. $\mathscr{S}(M)$ is a sheaf of Lie algebras. Let us introduce this notation: if X is a differentiable vector field defined on an open set $V \subset M$, $\xi \in V$, we denote by $g_\xi(X)$ the *germ* of X at ξ.

Let $V_1, V_2 \subset M$ be open sets and let $\varphi : V_1 \to V_2$ be a local diffeomorphism. Let us set $U_i = q^{-1}(V_i)$, $i = 1, 2$. First of all φ induces the local diffeomorphism

$\varphi^0 : U_1 \to U_2$. In fact, for $(\xi, a) \in U_1$, we set $\varphi^0(\xi, a) = (\varphi\xi, a)$. Further φ induces the local homeomorphism $\tilde{\varphi} : (\pi_0)^{-1} U_1 \to (\pi_0)^{-1} U_2$, and for every $l \geqq 1$ the local diffeomorphism $\varphi^l : (\pi_0^l)^{-1} U_1 \to (\pi_0^l)^{-1} U_2$. For $g_a(\sigma) \in (\pi_0)^{-1} U_1$ and $j_a^l(\sigma) \in \in (\pi_0^l)^{-1} U_1$ we set $\tilde{\varphi}\, g_a(\sigma) = g_a(\varphi^0\sigma)$ and $\varphi^l j_a^l(\sigma) = j_a^l(\varphi^0\sigma)$ respectively. Let X be a differentiable vector field defined on an open set $V \subset M$ and generated by a local 1-parameter group $h_t : V \times (-\varepsilon, \varepsilon) \to M$. For every $l \geqq 0$ the local 1-parameter group h_t^l generates a differentiable vector field on $(q\pi_0^l)^{-1} V$, which we shall denote by X^l.

Definition 4. Vector field X^l is called the *l-th prolongation of X*.

Definition 5. Let (E, p, B) be a fibered manifold. Let X be a vector field defined on $U \subset E$. X is called *vertical* if $p_* X = 0$.

Proposition 1. *X^l defined on $(q\pi_0^l)^{-1} V$ is a vertical vector field on the fibered manifold $(\tilde{J}^l, \pi_{-1}^l, B)$.*

It is well known that if local 1-parameter groups h_s and $\hat{h}_t$ generate vector fields X and $\hat{X}$ respectively, then a 1-parameter system of local transformations $h_t \circ \hat{h}_t$ generates a vector field $X + \hat{X}$. Considering this fact and using a natural coordinate system we can prove by the direct calculation the following proposition.

Proposition 2. *Let X and $\hat{X}$ be two differentiable vector fields defined on an open set $V \subset M$. For any $l \geqq 0$ we have $(X + \hat{X})^l = X^l + \hat{X}^l$.*

Let us keep the notation from the preceding proposition. We have an obvious

Proposition 3. *For any $\alpha \in \mathbf{R}$ there is $(\alpha X)^l = \alpha X^l$.*

Now let us consider a subsheaf $\mathscr{F} \subset \mathscr{S}(M)$ of vector spaces. For any $l \geqq 0$ we shall attach to $\mathscr{F}$ a subsheaf $\mathscr{F}^l \subset \mathscr{S}(\tilde{J}^l)$. Let $g_x(Y) \in \mathscr{S}(\tilde{J}^l)$, where $x \in \tilde{J}^l$ and Y is a differentiable vector field defined on an open neighborhood of x. $g_x(Y) \in \mathscr{F}^l$ if and only if there is a differentiable vector field X defined on an open neighborhood of $\xi = q\pi_0^l(x)$ such that $g_\xi(X) \in \mathscr{F}$ and $g_x(Y) = g_x(X^l)$. Propositions 2 and 3 imply that $\mathscr{F}^l \subset \mathscr{S}(\tilde{J}^l)$ is a sheaf of vector spaces.

Definition 6. $\mathscr{F}^l$ is called the *l-th prolongation of $\mathscr{F}$.*

Definition 7. Let $\mathscr{F} \subset \mathscr{S}(M)$ be a subsheaf of vector spaces. We say that $\mathscr{F}$ is locally finitely generated if we can find to every point $\xi \in M$ its open neighborhood V and a finite number of differentiable vector fields $X_1, \ldots, X_k$ on V such that for any $\eta \in V$ the germs $g_\eta(X_1), \ldots, g_\eta(X_k)$ generate the fiber $\mathscr{F}_\eta$ of $\mathscr{F}$.

It is clear that if $\mathscr{F}$ is locally finitely generated then $\mathscr{F}^l$ is also locally finitely generated. In § 4 we shall study a locally finitely generated sheaf $\mathscr{F} \subset \mathscr{S}(M)$ such that dim $\mathscr{F}_\xi$ is constant on M. Likewise it is clear that if dim $\mathscr{F}_\xi = k$ on M then dim $\mathscr{F}^l_x = k$ on $\tilde{J}^l$. We shall end this paragraph by proving two propositions concerning such sheaves.

Proposition 4. *Let M be a connected analytic manifold, let $\mathscr{S}(M)$ be the sheaf of germs of all analytic vector fields on M, and let $\mathscr{F} \subset \mathscr{S}(M)$ be a locally finitely generated subsheaf of vector spaces. Then* dim $\mathscr{F}_\xi$ *is constant on M.*

Proof. We shall prove that the function dim $\mathscr{F}_\xi$ is locally constant. Let $\xi \in M$, let V be its open neighborhood, and let $X_1, \ldots, X_k$ be analytic vector fields defined on V such that for any $\eta \in V$ the germs $g_\eta(X_1), \ldots, g_\eta(X_k)$ generate the fiber $\mathscr{F}_\eta$ of $\mathscr{F}$. Let dim $\mathscr{F}_\xi = r$. Then we can choose $X_{i_1}, \ldots, X_{i_r}$ such that $(g_\xi(X_{i_1}), \ldots, g_\xi(X_{i_r}))$ is a basis of $\mathscr{F}_\xi$. In other words there exists a connected neighborhood $V_1 \subset V$ of ξ such that for any $i \neq i_1, \ldots, i_r$ the field X_i is equal to a linear combination of the fields $X_{i_1}, \ldots, X_{i_r}$ on V_1, and on the other hand the fields $X_{i_1}, \ldots, X_{i_r}$ are not linearly dependent on any open neighborhood of ξ. Hence it is clear that for any $\eta \in V_1$ the germs $g_\eta(X_{i_1}), \ldots, g_\eta(X_{i_r})$ generate the fiber $\mathscr{F}_\eta$. Let us suppose that for some $\eta \in V_1$, $\eta \neq \xi$ there is dim $\mathscr{F}_\eta < r$. Then there exists a non-zero vector $(\alpha_1, \ldots, \alpha_r) \in \mathbf{R}^r$ such that $\sum_{j=1}^{r} \alpha_j g_\eta(X_{i_j}) = 0$. In other words there is a neighborhood $W \subset V_1$ of η such that we have $\sum_{j=1}^{r} \alpha_j X_{i_j} = 0$ on W. As V_1 is connected and the vector fields $X_{i_1}, \ldots, X_{i_r}$ are analytic, it follows by the standard argument that $\sum_{j=1}^{r} \alpha_j X_{i_j} = 0$ on V_1, and this is the contradiction. Thus the function dim $\mathscr{F}_\xi$ is locally constant and therefore constant on the connected manifold M.

Proposition 5. *Let M be a connected analytic manifold and let V be a vector space of analytic vector fields defined on M,* dim $V = k$. *Let us denote by $\mathscr{F}$ the sheaf of germs of all vector fields from V. $\mathscr{F}$ is a sheaf of vector spaces and* dim $\mathscr{F}_\xi = k$ *for every $\xi \in M$.*

The proposition follows easily from the preceding one.

3. SHEAVES OF LIE ALGEBRAS

Let us denote by $\mathbf{N}^*$ the set of all positive integers. k-tuple $(i_1, \ldots, i_k)$ of elements of $\mathbf{N}^*$ is called admissible if $i_1 \leqq \ldots \leqq i_k \leqq n$, where $n = \dim B$. Let $x \in E$ and let V and W be coordinate neighborhoods of points $q(x)$ and $p(x)$ with coordinates $(y^1, \ldots, y^m)$ and $(x^1, \ldots, x^n)$ respectively. Let us set $U = p^{-1}(W) \cap q^{-1}(V)$. On $(\pi^l_0)^{-1} U$ we have the associated coordinate system $(x^i, y^\alpha, y^\alpha_{i_1}, \ldots, y^\alpha_{i_1 \ldots i_q})$ (see [2], p. 3), where $i = 1, \ldots, n$; $\alpha = 1, \ldots, m$ and the k-tuple $(i_1, \ldots, i_k)$ runs through all admissible k-tuples. Let X be a differentiable vector field on V. Under the usual summation convention we may write $X = a^\eta(\partial/\partial y^\eta)$. In terms of the associated coordinate system we can find

$$X^0 = a^\eta \frac{\partial}{\partial y^\eta},$$

$$X^1 = X^0 + \frac{\partial a^\eta}{\partial y^{\alpha_1}} y^{\alpha_1}_{i_1} \frac{\partial}{\partial y^\eta_{i_1}},$$

$$X^2 = X^1 + \left[\frac{\partial^2 a^\eta}{\partial y^{\alpha_1}\,\partial y^{\alpha_2}} y^{\alpha_1}_{i_1} y^{\alpha_2}_{i_2} + \frac{\partial a^\eta}{\partial y^{\alpha_1}} y^{\alpha_1}_{i_1 i_2}\right] \frac{\partial}{\partial y^\eta_{i_1 i_2}},$$

$$X^3 = X^2 + \left[\frac{\partial^3 a^\eta}{\partial y^{\alpha_1}\,\partial y^{\alpha_2}\,\partial y^{\alpha_3}} y^{\alpha_1}_{i_1} y^{\alpha_2}_{i_2} y^{\alpha_3}_{i_3} + \frac{\partial^2 a^\eta}{\partial y^{\alpha_1}\,\partial y^{\alpha_2}} (y^{\alpha_1}_{i_1} y^{\alpha_2}_{i_2 i_3} + y^{\alpha_1}_{i_2} y^{\alpha_2}_{i_1 i_3} + y^{\alpha_1}_{i_1 i_2} y^{\alpha_2}_{i_3}) + \right.$$
$$\left. + \frac{\partial a^\eta}{\partial y^{\alpha_1}} y^{\alpha_1}_{i_1 i_2 i_3}\right] \frac{\partial}{\partial y^\eta_{i_1 i_2 i_3}}.$$

We recall once more that the sums are taken over all admissible k-tuples $(i_1, \ldots, i_k)$, $k = 1, 2, 3$. Now we are going to express X^l in terms of the associated coordinate system for any $l \geqq 0$. An ordered partition D of order r of an admissible k-tuple $(i_1, \ldots, i_k)$ is an ordered r-tuple $(D_1, \ldots, D_r)$ of sets, where $D_\alpha = (i_{j_1{}^d}, \ldots, i_{j_{h(d)}{}^d})$, $d = 1, \ldots, r$, such that

(i) $\bigcup_{d=1}^{r} D_\alpha = (i_1, \ldots, i_k)$,

(ii) $D_{d_1} \cap D_{d_2} = 0$ for $d_1 \neq d_2$,

(iii) $j^d_1 \leqq \ldots \leqq j^d_{h(d)}$ for all $d = 1, \ldots, r$.

Let us denote by $\Xi(r)$ the set of all ordered partitions of order r of the admissible k-tuple $(i_1, \ldots, i_k)$. Let $r \in \mathbf{N}^*$, $r \leqq k$ and let $\alpha_1, \ldots, \alpha_r \in \mathbf{N}^*$; $\alpha_1, \ldots, \alpha_r \leqq m$, where $m = \dim M$. We define a polynomial

$$H^{\alpha_1 \ldots \alpha_r}_{i_1 \ldots i_k} = \frac{1}{r!} \sum_{D \in \Xi(r)} y^{\alpha_1}_{D_1} \ldots y^{\alpha_r}_{D_r}.$$

Obviously for any permutation π of r elements there is $H^{\alpha_{\pi(1)} \ldots \alpha_{\pi(r)}}_{i_1 \ldots i_k} = H^{\alpha_1 \ldots \alpha_r}_{i_1 \ldots i_k}$. Now for every $0 \leqq k \leqq l$ let us assign to the vector field X a vector field $p_k(X)$ on $(\pi^l_0)^{-1}\, U$ defined by

$$p_0(X) = a^\eta \frac{\partial}{\partial y^\eta},$$

$$p_k(X) = \sum_{r=1}^{k} \frac{\partial^r a^\eta}{\partial y^{\alpha_1} \ldots \partial y^{\alpha_r}} H^{\alpha_1 \ldots \alpha_r}_{i_1 \ldots i_k} \frac{\partial}{\partial y^\eta_{i_1 \ldots i_k}} \quad \text{for} \quad k \geqq 1,$$

where the sums are taken again over all admissible k-tuples $(i_1, \ldots, i_k)$. Now it can be easily seen that there is $X^l = \sum_{k=0}^{l} p_k(X)$ on $(\pi_0^l)^{-1}\, U$. In addition let us set $\tilde{H}^{\alpha_1\ldots\alpha_r}_{i_1\ldots i_k} = r!\, H^{\alpha_1\ldots\alpha_r}_{i_1\ldots i_k}$. Now we shall discuss the properties of H and $\tilde{H}$.

Lemma 1. *For $r \leqq v \leqq l$ there is*

$$\sum_{k=r}^{l} \tilde{H}^{\alpha_1\ldots\alpha_r}_{i_1\ldots i_k} \frac{\partial \tilde{H}^{\gamma_1}_{j_1\ldots j_v}}{\partial y^\eta_{i_1\ldots i_k}} = \delta^{\gamma_1}_\eta \tilde{H}^{\alpha_1\ldots\alpha_r}_{j_1\ldots j_v}\,.$$

Proof. Obviously $\tilde{H}^{\gamma_1}_{j_1\ldots j_v} = y^{\gamma_1}_{j_1\ldots j_v}$ and we have

$$\sum_{k=r}^{l} \tilde{H}^{\alpha_1\ldots\alpha_r}_{i_1\ldots i_k} \frac{\partial \tilde{H}^{\gamma_1}_{j_1\ldots j_v}}{\partial y^\eta_{i_1\ldots i_k}} = \tilde{H}^{\alpha_1\ldots\alpha_r}_{i_1\ldots i_v} \frac{\partial y^{\gamma_1}_{j_1\ldots j_v}}{\partial y^\eta_{i_1\ldots i_v}} = \tilde{H}^{\alpha_1\ldots\alpha_r}_{i_1\ldots i_v} \delta^{\gamma_1}_\eta \delta^{i_1}_{j_1} \ldots \delta^{i_v}_{j_v} = \delta^{\gamma_1}_\eta \tilde{H}^{\alpha_1\ldots\alpha_r}_{j_1\ldots j_v}$$

Lemma 2. *Let $r + s - 1 \leqq z \leqq l$. Then there is*

$$\sum_{k=r}^{z} \tilde{H}^{\alpha_1\ldots\alpha_r}_{i_1\ldots i_k} \frac{\partial \tilde{H}^{\gamma_1\ldots\gamma_s}_{j_1\ldots j_z}}{\partial y^\eta_{i_1\ldots i_k}} = \sum_{p=1}^{s} \delta^{\gamma_p}_\eta \tilde{H}^{\alpha_1\ldots\alpha_r\gamma_1\ldots\hat{\gamma}_p\ldots\gamma_s}_{j_1\ldots j_z}$$

where "$\wedge$" denotes as usual the omission of the corresponding index.

Proof. We shall proceed by induction on s. For $s = 1$ and any z such that $r \leqq z \leqq l$ our assertion is nothing else than the assertion of lemma 1. Thus let us suppose that our assertion holds for any $1 \leqq t \leqq s - 1$ and for any z such that $r + t - 1 \leqq z \leqq l$. First we must realize that for $a + b \leqq z \leqq l$ we have

$$\tilde{H}^{\alpha_1\ldots\alpha_a\beta_1\ldots\beta_b}_{j_1\ldots j_z} = \sum_{v=a}^{z-b} \sum_{D_v} \tilde{H}^{\alpha_1\ldots\alpha_a}_{j_{h(1)}\ldots j_{h(v)}} \tilde{H}^{\beta_1\ldots\beta_b}_{j_{h(v+1)}\ldots j_{h(z)}}$$

where $\sum_{D_v}$ is taken over all ordered partitions (D_1, D_2) of order 2 of the admissible z-tuple $(j_1, \ldots, j_z)$ such that D_1 consists of v elements. Using this formula for $a = 1$ we easily get

$$\tilde{H}^{\gamma_1\ldots\gamma_s}_{j_1\ldots j_z} = \frac{1}{s} \sum_{u=1}^{s} \sum_{v=1}^{z-s+1} \sum_{D_v} \tilde{H}^{\gamma_u}_{j_{h(1)}\ldots j_{h(v)}} \tilde{H}^{\gamma_1\ldots\hat{\gamma}_u\ldots\gamma_s}_{j_{h(v+1)}\ldots j_{h(z)}}\,.$$

On the basis of the last two formulas and the induction hypothesis we have

$$\sum_{k=r}^{z} \tilde{H}^{\alpha_1\ldots\alpha_r}_{i_1\ldots i_k} \frac{\partial \tilde{H}^{\gamma_1\ldots\gamma_s}_{j_1\ldots j_z}}{\partial y^\eta_{i_1\ldots i_k}} =$$

$$= \sum_{k=r}^{z} \tilde{H}^{\alpha_1\ldots\alpha_r}_{i_1\ldots i_k} \frac{1}{s} \sum_{u=1}^{s} \sum_{v=1}^{z-s+1} \sum_{D_v} \left(\frac{\partial \tilde{H}^{\gamma_u}_{j_{h(1)}\ldots j_{h(v)}}}{\partial y^\eta_{i_1\ldots i_k}} \tilde{H}^{\gamma_1\ldots\hat{\gamma}_u\ldots\gamma_s}_{j_{h(v+1)}\ldots j_{h(z)}} + H^{\gamma_u}_{j_{h(1)}\ldots j_{h(v)}} \frac{\partial \tilde{H}^{\gamma_1\ldots\hat{\gamma}_u\ldots\gamma_s}_{j_{h(v+1)}\ldots j_{h(z)}}}{\partial y^\eta_{i_1\ldots i_k}} \right) =$$

$$= \frac{1}{s} \left[\sum_{u=1}^{s} \sum_{v=r}^{z-s+1} \sum_{D_v} \sum_{k=r}^{z} \tilde{H}^{\alpha_1\ldots\alpha_r}_{i_1\ldots i_k} \frac{\partial \tilde{H}^{\gamma_u}_{j_{h(1)}\ldots j_{h(v)}}}{\partial y^\eta_{i_1\ldots i_k}} \tilde{H}^{\gamma_1\ldots\hat{\gamma}_u\ldots\gamma_s}_{j_{h(v+1)}\ldots j_{h(z)}} + \right.$$

$$\left. + \sum_{u=1}^{s} \sum_{v=1}^{z-r-s+2} \sum_{D_v} \sum_{k=r}^{z} \tilde{H}^{\gamma_u}_{j_{h(1)}\ldots j_{h(v)}} \tilde{H}^{\alpha_1\ldots\alpha_r}_{i_1\ldots i_k} \frac{\partial \tilde{H}^{\gamma_1\ldots\hat{\gamma}_u\ldots\gamma_s}_{j_{h(v+1)}\ldots j_{h(z)}}}{\partial y^\eta_{i_1\ldots i_k}} \right] =$$

$$= \frac{1}{s}\Big[\sum_{u=1}^{s} \sum_{v=r}^{z-s+1} \sum_{D_v} \delta_\eta^{\gamma_u} \tilde{H}^{\alpha_1 \ldots \alpha_r}_{j_{h(1)} \ldots j_{h(v)}} \tilde{H}^{\gamma_1 \ldots \hat{\gamma}_u \ldots \gamma_s}_{j_{h(v+1)} \ldots j_{h(z)}} +$$

$$+ \sum_{u=1}^{s} \sum_{v=1}^{z-r-s+2} \sum_{D_v} \sum_{\substack{w=1 \\ w \neq u}} \delta_\eta^{\gamma_w} \tilde{H}^{\gamma_u}_{j_{h(1)} \ldots j_{h(v)}} \tilde{H}^{\alpha_1 \ldots \alpha_r \gamma_1 \ldots \hat{\gamma}_u \ldots \hat{\gamma}_w \ldots \gamma_s}_{j_{h(v+1)} \ldots j_{h(z)}}\Big] =$$

$$= \frac{1}{s}\Big[\sum_{u=1}^{s} \delta_\eta^{\gamma_u} \tilde{H}^{\alpha_1 \ldots \alpha_r \gamma_1 \ldots \hat{\gamma}_u \ldots \gamma_s}_{j_1 \ldots j_z} + \sum_{w=1}^{s} \sum_{\substack{u=1 \\ u \neq w}}^{s} \sum_{v=1}^{z-r-s+2} \sum_{D_v} = \delta_\eta^{\gamma_w} \tilde{H}^{\gamma_u}_{j_{h(1)} \ldots j_{h(v)}} \tilde{H}^{\alpha_1 \ldots \alpha_r \gamma_1 \ldots \hat{\gamma}_u \ldots \hat{\gamma}_w \ldots \gamma_s}_{j_{h(v+1)} \ldots j_{h(z)}} =$$

$$= \frac{1}{s}\Big[\sum_{u=1}^{s} \delta_\eta^{\gamma_u} \tilde{H}^{\alpha_1 \ldots \alpha_r \gamma_1 \ldots \hat{\gamma}_u \ldots \gamma_s}_{j_1 \ldots j_z} + (s-1) \sum_{w=1}^{s} \delta_\eta^{\gamma_w} \tilde{H}^{\alpha_1 \ldots \alpha_r \gamma_1 \ldots \hat{\gamma}_w \ldots \gamma_s}_{j_1 \ldots j_z}\Big] = \sum_{p=1}^{s} \delta_\eta^{\gamma_p} \tilde{H}^{\alpha_1 \ldots \alpha_r \gamma_1 \ldots \hat{\gamma}_p \ldots \gamma_s}_{j_1 \ldots j_z}$$

and this finishes the proof.

From lemma 2 we get easily

Lemma 3. *Let $r + s - 1 \leqq z \leqq l$. Then there is*

$$\sum_{k=r}^{z} H^{\alpha_1 \ldots \alpha_r}_{i_1 \ldots i_k} \frac{\partial H^{\gamma_1 \ldots \gamma_s}_{j_1 \ldots j_z}}{\partial y^\eta_{i_1 \ldots i_k}} = \frac{1}{s} \binom{r+s-1}{r} \sum_{p=1}^{s} \delta_\eta^{\gamma_p} H^{\alpha_1 \ldots \alpha_r \gamma_1 \ldots \hat{\gamma}_p \ldots \gamma_s}_{j_1 \ldots j_z}.$$

Proposition 6. *Let X, Y be two differentiable vector fields defined on V. There is $[X, Y]^l = [X^l, Y^l]$ on $(\pi^l_0)^{-1} U$.*

Proof. Let $X = a^\eta \, \partial/\partial y^\eta$, $Y = b^\xi \, (\partial/\partial y^\xi)$. We shall proceed by induction on l. The assertion is obvious for $l = 0$. Let us suppose that it holds for $l - 1$. In order to prove that it holds for l it is clearly sufficient to prove the equality

$$\Big[\sum_{k=0}^{l-1} p_k(X), p_l(Y)\Big] + \Big[p_l(X), \sum_{q=0}^{l-1} p_q(Y)\Big] + [p_l(X), p_l(Y)] = p_l([X, Y]).$$

Let us calculate the left and the right-hand side respectively.

$$L = \left[a^\eta \frac{\partial}{\partial y^\eta} + \sum_{k=1}^{l-1} \sum_{r=1}^{k} \frac{\partial^r a^\eta}{\partial y^{\alpha_1} \ldots \partial y^{\alpha_r}} H^{\alpha_1 \ldots \alpha_r}_{i_1 \ldots i_k} \frac{\partial}{\partial y^\eta_{i_1 \ldots i_k}}, \sum_{s=1}^{l} \frac{\partial^s b^\xi}{\partial y^{\gamma_1} \ldots \partial y^{\gamma_s}} H^{\gamma_1 \ldots \gamma_s}_{j_1 \ldots j_l} \frac{\partial}{\partial y^\xi_{j_1 \ldots j_l}}\right] +$$

$$+ \left[\sum_{r=1}^{l} \frac{\partial^r a^\eta}{\partial y^{\alpha_1} \ldots \partial y^{\alpha_r}} H^{\alpha_1 \ldots \alpha_r}_{i_1 \ldots i_l} \frac{\partial}{\partial y^\eta_{i_1 \ldots i_l}}, b^\xi \frac{\partial}{\partial y^\xi} + \sum_{q=1}^{l-1} \sum_{s=1}^{q} \frac{\partial^s b^\xi}{\partial y^{\gamma_1} \ldots \partial y^{\gamma_s}} H^{\gamma_1 \ldots \gamma_s}_{j_1 \ldots j_q} \frac{\partial}{\partial y^\xi_{j_1 \ldots j_q}}\right] +$$

$$+ \left[\sum_{r=1}^{l} \frac{\partial^r a^\eta}{\partial y^{\alpha_1} \ldots \partial y^{\alpha_r}} H^{\alpha_1 \ldots \alpha_r}_{i_1 \ldots i_l} \frac{\partial}{\partial y^\eta_{i_1 \ldots i_l}}, \sum_{s=1}^{l} \frac{\partial^s b^\xi}{\partial y^{\gamma_1} \ldots \partial y^{\gamma_s}} H^{\gamma_1 \ldots \gamma_s}_{j_1 \ldots j_l} \frac{\partial}{\partial y^\xi_{j_1 \ldots j_l}}\right] =$$

$$= \sum_{k=1}^{l} \sum_{s=1}^{l} \sum_{r=1}^{k} \frac{\partial^r a^\eta}{\partial y^{\alpha_1} \ldots \partial y^{\alpha_r}} \frac{\partial^s b^\xi}{\partial y^{\gamma_1} \ldots \partial y^{\gamma_s}} H^{\alpha_1 \ldots \alpha_r}_{i_1 \ldots i_k} \frac{\partial H^{\gamma_1 \ldots \gamma_s}_{j_1 \ldots j_l}}{\partial y^\eta_{i_1 \ldots i_k}} \frac{\partial}{\partial y^\xi_{j_1 \ldots j_l}} -$$

$$- \sum_{q=1}^{l} \sum_{r=1}^{l} \sum_{s=1}^{q} \frac{\partial^s b^\xi}{\partial y^{\gamma_1} \ldots \partial y^{\gamma_s}} \frac{\partial^r a^\eta}{\partial y^{\alpha_1} \ldots \partial y^{\alpha_r}} H^{\gamma_1 \ldots \gamma_s}_{j_1 \ldots j_q} \frac{\partial H^{\alpha_1 \ldots \alpha_r}_{i_1 \ldots i_l}}{\partial y^\xi_{j_1 \ldots j_q}} \frac{\partial}{\partial y^\eta_{i_1 \ldots i_l}} +$$

$$+\sum_{s=1}^{l} a^{\eta}\frac{\partial^{s+1}b^{\xi}}{\partial y^{\eta}\,\partial y^{\gamma_1}\dots\partial y^{\gamma_s}}\,H^{\gamma_1\dots\gamma_s}_{j_1\dots j_l}\frac{\partial}{\partial y^{\xi}_{j_1\dots j_l}}-\sum_{r=1}^{l}\frac{\partial^{r+1}a^{\eta}}{\partial y^{\xi}\,\partial y^{\alpha_1}\dots\partial y^{\alpha_r}}\,b^{\xi}H^{\alpha_1\dots\alpha_r}_{i_1\dots i_l}\frac{\partial}{\partial y^{\eta}_{i_1\dots i_l}}=$$

$$=\sum_{\substack{r=1\\ r+s-1\leq l}}^{l}\sum_{s=1}^{l}\frac{\partial^{r}a^{\eta}}{\partial y^{\alpha_1}\dots\partial y^{\alpha_r}}\frac{\partial^{s}b^{\xi}}{\partial y^{\gamma_1}\dots\partial y^{\gamma_s}}\left(\sum_{k=r}^{l}H^{\alpha_1\dots\alpha_r}_{i_1\dots i_k}\frac{\partial H^{\gamma_1\dots\gamma_s}_{j_1\dots j_l}}{\partial y^{\eta}_{i_1\dots i_k}}\right)\frac{\partial}{\partial y^{\xi}_{j_1\dots j_l}}-$$

$$-\sum_{\substack{r=1\\ r+s-1\leq l}}^{l}\sum_{s=1}^{l}\frac{\partial^{s}b^{\xi}}{\partial y^{\gamma_1}\dots\partial y^{\gamma_s}}\frac{\partial^{r}a^{\eta}}{\partial y^{\alpha_1}\dots\partial y^{\alpha_r}}\left(\sum_{q=s}^{l}H^{\gamma_1\dots\gamma_s}_{j_1\dots j_q}\frac{\partial H^{\alpha_1\dots\alpha_r}_{i_1\dots i_l}}{\partial y^{\xi}_{j_1\dots j_q}}\right)\frac{\partial}{\partial y^{\eta}_{i_1\dots i_l}}+\sum_{s=1}^{l}\square-\sum_{r=1}^{l}\square$$

where $\square$ stands for the terms which remain unchanged. It suffices to take the sum over $r+s-1\leq l$ because for $r+s-1>l$ the derivatives of the polynomials H vanish. Now the right-hand side. We have $p_l([X,Y])=p_l(a^{\eta}(\partial b^{\xi}/\partial y^{\eta})(\partial/\partial y^{\eta}))-$ $-p_l(b^{\xi}(\partial a^{\eta}/\partial y^{\xi})(\partial/\partial y^{\eta}))$ and we shall start calculating $p_l(a^{\eta}(\partial b^{\xi}/\partial y^{\eta})(\partial/\partial y^{\xi}))$. Let us denote by $P(k)$ the group of all permutations of k elements.

$$p_l\left(a^{\eta}\frac{\partial b^{\xi}}{\partial y^{\eta}}\frac{\partial}{\partial y^{\xi}}\right)=\sum_{k=1}^{l}\frac{\partial^{k}\left(a^{\eta}\dfrac{\partial b^{\xi}}{\partial y^{\eta}}\right)}{\partial y^{\alpha_1}\dots\partial y^{\alpha_k}}\,H^{\alpha_1\dots\alpha_k}_{j_1\dots j_l}\frac{\partial}{\partial y^{\xi}_{j_1\dots j_l}}=$$

$$=\sum_{k=1}^{l}a^{\eta}\frac{\partial^{k+1}b^{\xi}}{\partial y^{\eta}\,\partial y^{\alpha_1}\dots\partial y^{\alpha_k}}\,H^{\alpha_1\dots\alpha_k}_{j_1\dots j_l}\frac{\partial}{\partial y^{\xi}_{j_1\dots j_l}}+$$

$$+\sum_{k=1}^{l}\left(\sum_{r=1}^{k}\frac{1}{r!\,(r-k)!}\sum_{\pi\in P(k)}\frac{\partial^{r}a^{\eta}}{\partial y^{\alpha_{\pi(1)}}\dots\partial y^{\alpha_{\pi(r)}}}\frac{\partial^{k-r+1}b^{\xi}}{\partial y^{\eta}\,\partial y^{\alpha_{\pi(r+1)}}\dots\partial y^{\alpha_{\pi(k)}}}\right)H^{\alpha_1\dots\alpha_k}_{j_1\dots j_l}\frac{\partial}{\partial y^{\xi}_{j_1\dots j_l}}=$$

$$=\Delta+\sum_{k=1}^{l}\left(\sum_{r=1}^{k}\frac{1}{r!\,(r-k)!}\sum_{\pi\in P(k)}\frac{\partial^{r}a^{\eta}}{\partial y^{\alpha_{\pi(1)}}\dots\partial y^{\alpha_{\pi(r)}}}\frac{\partial^{k-r+1}b^{\xi}}{\partial y^{\eta}\,\partial y^{\alpha_{\pi(r+1)}}\dots\partial y^{\alpha_{\pi(k)}}}H^{\alpha_{\pi(1)}\dots\alpha_{\pi(k)}}_{j_1\dots j_l}\right)\cdot$$

$$\cdot\frac{\partial}{\partial y^{\xi}_{j_1\dots j_l}}=\Delta+\sum_{k=1}^{l}\left(\sum_{r=1}^{k}\frac{1}{r!\,(r-k)!}\sum_{\pi\in P(k)}\frac{1}{k-r+1}\frac{\partial^{r}a^{\eta}}{\partial y^{\alpha_1}\dots\partial y^{\alpha_r}}\frac{\partial^{k-r+1}b^{\xi}}{\partial y^{\gamma_1}\dots\partial y^{\gamma_{k-r+1}}}\cdot\right.$$

$$\left.\cdot\left(\sum_{t=1}^{k-r+1}\delta^{\gamma_t}_{\eta}H^{\alpha_1\dots\alpha_r\gamma_1\dots\hat{\gamma}_t\dots\gamma_{k-r+1}}_{j_1\dots j_l}\right)\right)\frac{\partial}{\partial y^{\xi}_{j_1\dots j_l}}=$$

$$=\Delta+\sum_{k=1}^{l}\sum_{r=1}^{k}\frac{k!}{r!\,(r-k)!}\frac{1}{k-r+1}\frac{\partial^{r}a^{\eta}}{\partial y^{\alpha_1}\dots\partial y^{\alpha_k}}\frac{\partial^{k-r+1}b^{\xi}}{\partial y^{\gamma_1}\dots\partial y^{\gamma_{k-r+1}}}\cdot$$

$$\cdot\left(\sum_{t=1}^{k-r+1}\delta^{\gamma_t}_{\eta}H^{\alpha_1\dots\alpha_r\gamma_1\dots\hat{\gamma}_t\dots\gamma_{k-r+1}}_{j_1\dots j_l}\right)\frac{\partial}{\partial y^{\xi}_{j_1\dots j_l}}.$$

Again Δ stands for the term which remains unchanged. Setting $s=k-r+1$ we get easily

$$p_l\left(a^{\eta}\frac{\partial b^{\xi}}{\partial y^{\eta}}\frac{\partial}{\partial y^{\xi}}\right)=\Delta+\sum_{\substack{r=1\\ r+s-1\leq l}}^{l}\sum_{s=1}^{l}\frac{1}{s}\binom{r+s-1}{r}\frac{\partial^{r}a^{\eta}}{\partial y^{\alpha_1}\dots\partial y^{\alpha_r}}\left(\sum_{t=1}^{s}\delta^{\gamma_t}_{\eta}H^{\alpha_1\dots\alpha_r\gamma_1\dots\hat{\gamma}_t\dots\gamma_s}_{j_1\dots j_l}\right)\frac{\partial}{\partial y^{\xi}_{j_1\dots j_l}}.$$

In the same way we can calculate $p_l(b^\xi(\partial a^\eta/\partial y^\xi)(\partial/\partial y^\eta))$. Now the assertion follows immediately from lemma 4.

Corollary. *Let $\mathscr{F} \subset \mathscr{S}(M)$ be a subsheaf of Lie algebras. It follows immediately from proposition 6 that $\mathscr{F}^l$ is also a sheaf of Lie algebras.*

4. SHEAVES OF INVARIANTS

From now on we shall consider a locally finitely generated subsheaf $\mathscr{F} \subset \mathscr{S}(M)$ of Lie algebras such that dim $\mathscr{F}_\xi$ is constant on M. This constant will be denoted by k.

Definition 7. Let N be a differentiable manifold. A *pseudodistribution* D on N is a mapping assigning to every point $p \in N$ a subspace $D_p \subset T_p(N)$ (not necessarily of the same dimension at every point). A vector field X defined on a subset $U \subset N$ is said to lie in D if for every $p \in U$ there is $X_p \in D_p$. A pseudodistribution D is called *differentiable* if for any $p \in N$ there exists its open neighborhood U and a finite number of differentiable vector fields $X_1, \dots, X_r$ defined on U such that each of them lies in D and for every $q \in U$ the vectors $(X_1)_q, \dots, (X_r)_q$ span the subspace D_q. The r-tuple $(X_1, \dots, X_r)$ is called *set of local generators* of the pseudodistribution D on the neighborhood of p.

The sheaf $\mathscr{F}$ gives us in the natural way a pseudodistribution $\tilde{D}^l$ on $\tilde{J}^l$: let $x \in \tilde{J}^l$; a vector $Y_x \in T_x(\tilde{J}^l)$ belongs to $\tilde{D}^l_x$ if and only if there exists a differentiable vector field X defined on an open neighborhood of x such that $g_x(X) \in \mathscr{F}^l$ and $Y_x = X_x$. As $\mathscr{F}^l$ is locally finitely generated, the pseudodistribution $\tilde{D}^l$ is obviously differentiable. It is also clear that $\tilde{D}^l$ is vertical, i.e. for any $x \in \tilde{J}^l$ any vector from $\tilde{D}^l_x$ is a vertical vector on the fibered manifold $(\tilde{J}^l, \pi^l_{-1}, B)$.

Let us denote by $\tilde{\mathscr{D}}^l$ the sheaf of germs of all differentiable functions on $\tilde{J}^l$. $\tilde{\mathscr{D}}^l$ is obviously a sheaf of rings. Let us define a subsheaf $\tilde{\mathscr{A}}^l \subset \tilde{\mathscr{D}}^l$ in the following way: let $g_x(f) \in \tilde{\mathscr{D}}^l$, where $x \in \tilde{J}^l$ and f is a differentiable function defined on an open neighborhood U_1 of x; $g_x(f) \in \tilde{\mathscr{A}}^l$ if and only if for any differentiable vector field X defined on an open neighborhood U_2 of x and lying in $\tilde{D}^l$ there exists a neighborhood $U \subset \subset U_1 \cap U_2$ of x on which $Xf = 0$. Obviously $g_x(f) \in \tilde{\mathscr{A}}^l$ if and only if $Xf = 0$ on a neighborhood of x for all elements X of a set of local generators of the pseudodistribution $\tilde{D}^l$. It is also clear that $\tilde{\mathscr{A}}^l \subset \tilde{\mathscr{D}}^l$ is a subsheaf of rings.

Definition 8. The sheaf $\tilde{\mathscr{A}}^l$ will be called the *sheaf of invariants of order l*.

More generally let Q be a differentiable manifold, dim $Q = q$, let $\mathscr{D}(Q)$ be the sheaf of germs of all differentiable functions on Q and let $\mathscr{A} \subset \mathscr{D}(Q)$ be a subsheaf of rings. For $x \in Q$ we denote by $\mathscr{A}_x$ the fiber of $\mathscr{A}$ over x. Now we shall introduce several useful concepts. A subsheaf $\mathscr{A}$ is called *φ-closed* if for any $x \in Q$ the following assertion holds: if $g_x(f_1), \dots, g_x(f_r) \in \mathscr{A}_x$ where $f_1, \dots, f_r$ are differentiable functions defined on an open neighborhood of x and if F is a differentiable function defined on an open neighborhood of the point $(f_1(x), \ldots, f_r(x)) \in \mathbf{R}^r$ then $g_x(F(f_1, \ldots, f_r)) \in \in \mathscr{A}_x$. Let $(x^1, \ldots, x^q)$ be a coordinate system defined on an open neighborhood of x and let $g_x(f_1), \ldots, g_x(f_s) \in \mathscr{A}_x$. The germs $g_x(f_1), \ldots, g_x(f_s)$ are called *φ-independent* if the matrix $\|(\partial f_j/\partial x^i)_x\|_{i=1,\ldots,q}^{j=1,\ldots,s}$ has maximal rank. It can be immediately seen that φ-independence of germs does not depend on the choice of a coordinate system around x. s-tuple of germs $(g_x(f_1), \ldots, g_x(f_s))$ is said to be a *set of φ-generators* of a fiber $\mathscr{A}_x$ if for any $g_x(f) \in \mathscr{A}_x$ there exists a differentiable function F defined on an open neighborhood of the point $(f_1(x), \ldots, f_s(x)) \in \mathbf{R}^s$ such that $g_x(f) = g_x(F(f_1, \ldots \ldots, f_s))$. s-tuple of germs $(g_x(f_1), \ldots, g_x(f_s))$ is called *φ-basis* of $\mathscr{A}_x$ if it is a set of φ-generators and if the germs $g_x(f_1), \ldots, g_x(f_s)$ are φ-independent. We can prove easily that any two φ-basis of the fiber $\mathscr{A}_x$ consist of the same number of elements, and so we are entitled to define the *φ-dimension* of $\mathscr{A}_x$. We can also easily check on examples that the fiber $\mathscr{A}_x$ need not have any φ-basis. If $\mathscr{A}_x$ has a φ-basis, then obviously $\dim \mathscr{A}_x \leqq q$. We say that a sheaf has a *local φ-basis* around $x \in Q$ if there exists an open neighborhood U of x and differentiable functions $f_1, \ldots, f_s$ defined on U such that for every $y \in U$ the s-tuple $(g_y(f_1), \ldots, g_y(f_s))$ is a φ-basis of the fiber $\mathscr{A}_y$. A sheaf $\mathscr{A}$ is called *differentiable* if to any $x \in Q$ there exists a local φ-basis around x and φ-dim $\mathscr{A}_x$ is constant on Q.

Definition 9. A point $x \in \tilde{J}^l$ is called regular if $\dim D_x^l = k$. The set of all regular points in $\tilde{J}^l$ will be denoted by $\bar{J}^l$.

Proposition 7. *$\bar{J}^l$ is an open subset of $\tilde{J}^l$.* The proof is obvious.

Proposition 8. *Let $l_1 \geqq l_2 \geqq 0$ be integers. Let $x \in \tilde{J}^{l_1}$, $y \in \bar{J}^{l_2}$, $y = \pi_{l_2}^{l_1}(x)$. Then $x \in \bar{J}^{l_1}$.*

Proof. Let V and W be coordinate neighborhoods of points $\xi = q\pi_0^{l_1}(x)$ and $\pi_{-1}^{l_1}(x)$ with coordinates $(y^1, \ldots, y^m)$ and $(x^1, \ldots, x^n)$ respectively. On $(\pi_0^{l(j)})^{-1}(V \times W)$ there is the associated coordinate system $(x^i, y^\alpha, y_{i_1}^\alpha, \ldots, y_{i_1 \ldots i_{l(j)}}^\alpha)$ $j = 1, 2$. As $y \in \bar{J}^{l_2}$ we can find an open neighborhood $V_1 \subset V$ of ξ and differentiable vector fields $X_1, \ldots, X_k$ defined on V_1 such that $g_\xi(X_1), \ldots, g_\xi(X_k) \in \mathscr{F}$ and vectors $(X_1^{l_2})_y, \ldots \ldots, (X_k^{l_2})_y$ are linearly independent. In terms of the associated coordinate system we can write (see § 3) $X_r^{l_1} = \sum_{i=0}^{l_1} p_i(X_r)$, $X_r^{l_2} = \sum_{i=0}^{l_2} p_i(X_r)$, $1 \leqq r \leqq k$. Hence it is clear that the vectors $(X_1^{l_2})_x, \ldots, (X_k^{l_2})_x$ are linearly independent and therefore $x \in \bar{J}^{l_1}$.

Now let $f : M' \to M$ be a differentiable mapping, and let us denote by $f^* T(M)$ the induced bundle of $T(M)$ under f (see [4], p. 18, Def. 5.3). We denote by $\mathscr{T}(M', f, M)$ the sheaf of germs of all local cross sections of the bundle $f^* T(M)$. Let us define a subsheaf $\mathscr{F}(M', f, M) \subset \mathscr{T}(M', f, M)$ in this way: let $\xi \in M'$, $g_\xi(\tau) \in \mathscr{T}(M', f, M)$ where τ is a local cross section defined on an open neighborhood U_1 of ξ; $g_\xi(\tau) \in$

$\in \mathscr{F}(M', f, M)$ if and only if there exists a local cross section σ of $T(M)$ defined on an open neighborhood U_2 of $f(\xi)$ such that on some neighborhood $U \subset U_1 \cap f^{-1}(U_2)$ of ξ there is $\tau \mid U = (f^*\sigma) \mid U$ (for the definition of $f^*\sigma$ see [4], p. 19, Prop. 5.10). $\mathscr{F}(M', f, M) \subset \mathscr{T}(M', f, M)$ is obviously a subsheaf of vector spaces.

Proposition 9. *Let σ be a local cross section of the fibered manifold $(M \times B, p, B)$ defined on an open neighborhood W of $a \in B$. If $\dim \mathscr{F}_a(W, q \circ \sigma, M) < k$ then for every $l \geqq 0$ there is $j_a^l(\sigma) \in \bar{J}^l - J^l$. If all structures are analytic and if $\dim \mathscr{F}_a(W, q \circ \circ \sigma, M) = k$ then there exists $l \geqq 0$ such that $j_a^l(\sigma) \in \bar{J}^l$.*

Proof. For any $l \geqq 0$ let $(x^i, y^\alpha, y^\alpha_{i_1}, \ldots, y^\alpha_{i_1 \ldots i_l})$ be the same associated coordinate system as in the proof of the preceding proposition. Let $X_1, \ldots, X_k$ be differentiable vector fields defined on an open neighborhood of $\xi = (q \circ \sigma)(a)$ such that $(g_\xi(X_1), \ldots \ldots, g_\xi(X_k))$ is a basis of $\mathscr{F}_\xi$. In terms of our coordinate system we can write $X_i = a_i^\eta (\partial/\partial y^\eta)$. If $\dim \mathscr{F}_a(W, q \circ \sigma, M) < k$ then there exists a non-zero vector $(\lambda_1, \ldots, \lambda_k)$ such that $\sum_{i=1}^{k} \lambda_i(a_i^\eta \circ q \circ \sigma) = 0$, $\eta = 1, \ldots, m$, on a neighborhood of a. By mere taking the derivative we get from the last equality

$$0 = \frac{\partial^r}{\partial x^{i_1} \ldots \partial x^{i_r}} \Big(\sum_{i=1}^{k} \lambda_i(a_i^\eta \circ q \circ \sigma)\Big)_a = \sum_{i=1}^{k} \lambda_i \sum_{s=1}^{r} \frac{\partial^r a_i^\eta}{\partial y^{\alpha_1} \ldots \partial y^{\alpha_s}} (\xi)\, H^{\alpha_1 \ldots \alpha_s}_{i_1 \ldots i_r}(j_a^s(\sigma))$$

for all integers $r \geqq 1$. It is immediately clear that the vectors $X_1^l, \ldots, X_k^l$ are linearly dependent for any $l \geqq 0$.

In the second part of the proof all structures are supposed to be analytic. Let $\dim \mathscr{F}_a(W, q \circ \sigma, M) = k$ and let us suppose that for every $l \geqq 0$ the vectors $(X_1^l)_{x^l}, \ldots \ldots, (X_k^l)_{x^l}$, where $x^l = j_a^l(\sigma)$, are linearly dependent. Let us denote by $B^l \subset \mathbf{R}^k$ the set of all vectors $(\lambda_1, \ldots, \lambda_k)$ such that $\sum_{i=1}^{k} \lambda_i(X_i^l)_{x^l} = 0$. $B^l \subset \mathbf{R}^k$ is obviously a non-zero subspace. Let $l_1 \geqq l_2$. In terms of our coordinate system we can write $X_i^{l_1} = X_i^{l_2} + \sum_{j=l_2+1}^{l_1} p_j(X_i)$, $1 \leqq i \leqq k$, and hence it is clear that $B^{l_1} \subseteq B^{l_2}$. According to our assumption $\dim B^l \geqq 1$ and therefore $\bigcap_{l=0}^{\infty} B^l$ is a non-zero-subspace, i.e. there exists a non-zero-vector $(\lambda_1, \ldots, \lambda_k)$ such that $\sum_{i=1}^{k} \lambda_i(X_i^l)_{x^l} = 0$ for all $l \geqq 0$. Let us write again $X_i = a_i^\eta (\partial/\partial y^\eta)$, $1 \leqq i \leqq k$. From the last equality it follows immediately

$$\frac{\partial^r}{\partial x^{i_1} \ldots \partial x^{i_r}} \Big(\sum_{i=1}^{k} \lambda_i(a_i^\eta \circ q \circ \sigma)\Big)_a = 0$$

for all $r \geqq 0$, all admissible r-tuples $(i_1, \ldots, i_r)$ and $\eta = 1, \ldots, m$, and hence we have $\sum_{i=1}^{k} \lambda_i(a_i^\eta \circ q \circ \sigma) = 0$ on a neighborhood of a. But this is the contradiction.

Let us denote by $\bar{D}^l$ resp. $\bar{\mathscr{D}}^l$ resp. $\bar{\mathscr{A}}^l$ the restriction of $\tilde{D}^l$ resp. $\tilde{\mathscr{D}}^l$ resp. $\tilde{\mathscr{A}}^l$ to $\bar{J}^l$. Clearly $\bar{D}^l$ is a differentiable involutive distribution on $\bar{J}^l$ (see [1], pp. 86, 87, Def. 2, 3, 5).

Proposition 10. *$\bar{\mathscr{A}}^l$ is a φ-closed differentiable sheaf.*

Proof. It is quite obvious that $\bar{\mathscr{A}}^l$ is φ-closed (even $\tilde{\mathscr{A}}^l$ is φ-closed). According to ([1], p. 89, Theorem 1) to any $x \in \bar{J}^l$ there exists its coordinate neighborhood $U \subset \bar{J}^l$ with coordinates $(u^1, \ldots, u^{n_l})$ where $n_l = \dim \bar{J}^l$ (as a differentiable manifold) such that the k-tuple $(\partial/\partial u^1, \ldots, \partial/\partial u^k)$ is a basis of $\bar{D}^l$ on U. It is immediately clear that $(u^{k+1}, \ldots, u^{n_l})$ is a local φ-basis of $\bar{\mathscr{A}}^l$ on the neighborhood U.

Proposition 11. *Let $f_1, \ldots, f_{n_l-k}$ be differentiable functions defined on an open set $U \subset \bar{J}^l$ such that for any $x \in U$ there is $g_x(f_1), \ldots, g_x(f_{n_l-k}) \in \bar{\mathscr{A}}^l$. For any $x \in U$ let the germs $g_x(f_1), \ldots, g_x(f_{n_l-k})$ be φ-independent. Then the $(n_l - k)$-tuple $(f_1, \ldots, f_{n_l-k})$ is a φ-basis of $\bar{\mathscr{A}}^l$ on U.*

The proof is obvious.

Let $U \subset \tilde{J}^l$ be an open subset and let $(x^i, y^\alpha, y^\alpha_{i_1}, \ldots, y^\alpha_{i_1 \ldots i_l})$ be an associated coordinate system on U. Let f be a differentiable function defined on U. The formal derivative $\partial^{x^i}_{\#} f$ is a function defined on $(\pi^{l+1}_l)^{-1} U$ in this way: if $x = j^{l+1}_{a_0}(\sigma) \in (\pi^{l+1}_l)^{-1} U$, where σ is a local cross section defined on an open neighborhood of $a_0 \in B$, we set

$$(\partial^{x^i}_{\#} f)(x) = \left(\frac{\partial}{\partial x^i}(f(j^l_a(\sigma)))\right)_{a=a_0}.$$

For the further properties of formal derivatives see [2], p. 15. Let us keep the just used notation for the next proposition.

Proposition 12. *Let $y = \pi^{l+1}_l(x)$ and let $g_y(f) \in \tilde{\mathscr{A}}^l$. Then $g_x(\partial^{x^i}_{\#} f) \in \tilde{\mathscr{A}}^{l+1}$.*

Proof. Let $\xi = q\pi^{l+1}_0(x)$ and let $X_1, \ldots, X_k$ be differentiable vector fields defined on an open neighborhood of ξ such that $g_\xi(X_1), \ldots, g_\xi(X_k)$ is a basis of $\mathscr{F}_\xi$. As $g_y(f) \in \tilde{\mathscr{A}}^l$ there exists a neighborhood $U' \subset U$ of y on which $X^l_1 f = \ldots = X^l_k f = 0$. Let ${}^1h_t, \ldots, {}^kh_t$ be the local 1-parameter groups generating the fields $X_1, \ldots, X_k$ respectively. For $x' = j^{l+1}_{b_0}(\sigma) \in (\pi^{l+1}_l)^{-1} U'$ and for all $j = 1, \ldots, k$ there is

$$X^{l+1}_j(\partial^{x^i}_{\#} f)(x') = \frac{\mathrm{d}}{\mathrm{d}t}[(\partial^{x^i}_{\#} f \circ {}^jh^{l+1}_t)(x')]_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}[(\partial^{x^i}_{\#} f)(j^{l+1}_{b_0}({}^jh^0_t\sigma))]_{t=0} =$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial x^i} f(j^l_b(({}^jh^0_t)\,\sigma))_{b=b_0}\right]_{t=0} = \frac{\partial}{\partial x^i}\left[\frac{\mathrm{d}}{\mathrm{d}t} f(j^l_b({}^jh^0_t\sigma))_{t=0}\right]_{b=b_0} = \frac{\partial}{\partial x^i}(X^l_j f)_{b=b_0} = 0$$

and the proposition immediately follows.

Let us define a subset $K \subset \bar{\mathscr{D}}^{l+1}$ in this way: $g_x(f) \in \bar{\mathscr{D}}^{l+1}$, where f is a differentiable function defined on an open neighborhood of $x \in \bar{J}^{l+1}$, belongs to K if and only if there exists either a differentiable function f' defined on an open neighborhood of $y = \pi_l^{l+1}(x)$ such that $g_y(f') \in \bar{\mathscr{A}}^l$ and $g_x(f) = g_x(f' \circ \pi_l^{l+1})$ or a differentiable function f'' defined on an open neighborhood U of y and an associated coordinate system $(x^i, y^\alpha, y^\alpha_{i_1}, \ldots, y^\alpha_{i_1 \ldots i_l})$ on U such that $g_y(f'') \in \bar{\mathscr{A}}^l$ and for some $1 \leqq i \leqq n$ there is $g_x(f) = g_x(\partial^{x^i}_{\#} f'')$. $K \subset \bar{\mathscr{D}}^{l+1}$ is clearly a subsheaf of sets. Let us denote by $p\bar{\mathscr{A}}^l$ the smallest φ-closed subsheaf of $\bar{\mathscr{D}}^{l+1}$ containing K. The subsheaf of $p\bar{\mathscr{A}}^l$, which is obviously a sheaf of rings, will be called the *formal prolongation* of $\bar{\mathscr{A}}^l$. Proposition 12 gives us immediately the inclusion $p\bar{\mathscr{A}}^l \subset \bar{\mathscr{A}}^{l+1}$. For $x \in \bar{J}^l$ let us denote by $Q_x\bar{J}^l$ the subspace of $T_x(\bar{J}^l)$ which is the kernel of the mapping $(\pi^l_{l-1})_* : T_x(\bar{J}^l) \to T_y(\bar{J}^{l-1})$ where $y = \pi^l_{l-1}(x)$. Let $\mathscr{B} \subset \bar{\mathscr{D}}^l$ be a subsheaf. Similarly as in [2] (p. 13, Def. 3.4) we introduce the subspace $C^l_x(\mathscr{B}) \subseteq Q_x\bar{J}^l$. Let $X \in Q_x\bar{J}^l$; $X \in C^l_x(\mathscr{B})$ if and only if for any differentiable function f defined on an open neighborhood of x and such that $g_x(f) \in \mathscr{B}$ there is $Xf = 0$. If $x \in \bar{J}^l$ and if $(f_1, \ldots, f_{n_l-k})$ is a local φ-basis of $\bar{\mathscr{A}}^l$ on a neighborhood of x, then obviously $X \in Q_x\bar{J}^l$ belongs to $C^l_x(\bar{\mathscr{A}}^l)$ if and only if $Xf_1 = \ldots = Xf_{n_l-k} = 0$. In terms of an associated coordinate system we can write $X = \sum a^\xi_{i_1 \ldots i_l}(\partial/\partial y^\xi_{i_1 \ldots i_l})_x$ where the sum is taken over all admissible l-tuples $(i_1, \ldots, i_l)$. Clearly $X \in C^l_x(\bar{\mathscr{A}}^l)$ if and only if $\sum a^\xi_{i_1 \ldots i_l}(\partial f_j/\partial y^\xi_{i_1 \ldots i_l})(x) = 0$ for all $j = 1, \ldots, n_l - k$. The reader can easily verify that for $x \in \bar{J}^l$ there is $C^l_x(\bar{\mathscr{A}}^l) = Q_x\bar{J}^l \cap \bar{D}^l_x$.

Definition 10. A point $x \in \bar{J}^l$ is called *proper* if $C^l_x(\bar{\mathscr{A}}^l) = 0$. The set of all proper points in $\bar{J}^l$ will be denoted by J^l.

Proposition 13. *J^l is an open subset of $\bar{J}^l$.* The proof is obvious.

Proposition 14. *Let $l_1 > l_2 \geqq 0$ be integers. Let $x \in \bar{J}^{l_1}$, $y \in J^{l_2}$, $\pi^{l_1}_{l_2}(x) = y$. Then $x \in J^{l_1}$.*

Proof. Let $\xi = (q\pi^{l_1}_0)(x)$ and let $X_1, \ldots, X_k$ be differentiable vector fields defined on an open neighborhood of ξ such that $g_\xi(X_1), \ldots, g_\xi(X_k)$ is a basis of $\mathscr{F}_\xi$. Let $z \in \bar{J}^{l_2+1}$ be such that $\pi^{l_2+1}_{l_2}(z) = y$ and let $(x^i, y^\alpha, y^\alpha_{i_1}, \ldots, y^\alpha_{i_1 \ldots i_{l(2)}})$ and $(x^i, y^\alpha, y^\alpha_{i_1}, \ldots, y^\alpha_{i_1 \ldots i_{l(2)+1}})$ be associated coordinate systems defined on an open neighborhood of y and z respectively. In terms of our coordinate systems we can write $X^{l_2+1}_j = X^{l_2}_j + p_{l_2+1}(X_j)$, $j = 1, \ldots, k$. Taking $X \in C^{l_2+1}_z(\bar{\mathscr{A}}^{l_2+1}) = Q_z\bar{J}^{l_2+1} \cap \bar{D}^{l_2+1}_z$ we have obviously $X = \sum_{j=1}^k \lambda_j(X^{l_2+1}_j)_z = \sum_{j=1}^k \lambda_j(X^{l_2}_j)_y + \sum_{j=1}^k \lambda_j p_{l_2+1}(X_j)$, where $(\lambda_1, \ldots, \lambda_k) \in \mathbf{R}^k$. But according to the fact that $X \in Q_z\bar{J}^{l_2+1}$ there must be $\sum_{j=1}^k \lambda_j(X^{l_2}_j)_y = 0$ which implies $\lambda_1 = \ldots = \lambda_k = 0$, for $y \in J^{l_2}$. Thus we have $X = 0$ and now the assertion follows easily.

Let us denote by D^l resp. $\mathscr{D}^l$ resp. $\mathscr{A}^l$ the restriction of $\bar{D}^l$ resp. $\bar{\mathscr{D}}^l$ resp. $\bar{\mathscr{A}}^l$ to J^l.

Proposition 15. *Let $x \in J^l$ and let $f_1, \ldots, f_{n_l-k}$ be differentiable functions defined on an open neighborhood U of x such that $(f_1, \ldots, f_{n_l-k})$ is a local φ-basis of $\mathscr{A}^l$ on U. Let $(x^i, y^\alpha, y^\alpha_{i_1}, \ldots, y^\alpha_{i_1 \ldots i_l})$ be an associated coordinate system defined on U. Let $y \in J^{l+1}$, $\pi_l^{l+1}(y) = x$. Then we can choose a subsystem S' of the system $S = \{f_j \circ \pi_l^{l+1}, \partial^{x^i}_{\#} f_j;\ i = 1, \ldots, n;\ j = 1, \ldots, n_l - k\}$ such that S' is a local φ-basis of $\mathscr{A}^{l+1}$ on an open neighborhood of y.*

Proof. For $v \in U$, $w \in (\pi_l^{l+1})^{-1}\, U$ we introduce matrices

$$B^l(v) = \left(\frac{\partial(f_1, \ldots, f_{n_l-k})}{\partial(x^i, y^\alpha, \ldots, y^\alpha_{i_1 \ldots i_l})}\right), \qquad p\, B^l(w) = \left(\frac{\partial(f_j \circ \pi_l^{l+1}, \partial^{x^i}_{\#} f_j)}{\partial(x^i, y^\alpha, \ldots, y^\alpha_{i_1 \ldots i_{l+1}})}\right)_w .$$

As the functions $f_1, \ldots, f_{n_l-k}$ are φ-independent on U the matrix $B^l(v)$ has for every $v \in U$ maximal rank. The matrix $p\, B^l(y)$ can be obviously written in the following way

$$p\, B^l(y) = \left(\begin{array}{c|c} B^l(x) & 0 \\ \hline \bar{B}^{l+1}(y) & \bar{\bar{B}}^{l+1}(y) \end{array}\right).$$

Let us set $a = \pi^l_{-1}(x)$. We can verify in the same way as in [2] (p. 17, Prop. 4.3) that under the fundamental identification (see [2], p. 6) there is $C_y^{l+1}(p\mathscr{A}^l) = (C_x^l(\mathscr{A}^l) \otimes T_a^*) \cap (Q_x J^l \otimes S^2(T_a^*))$. Because $x \in J^l$ there is $C_x^l(\mathscr{A}^l) = 0$ and from the last formula we have $C_y^{l+1}(p\mathscr{A}^l) = 0$. But this means nothing else than that the matrix $\bar{\bar{B}}^{l+1}(y)$ has linearly independent columns. Thus by removing a certain number of rows from the $n(n_l - k)$ last ones of $p\, B^l(y)$ we get a matrix $B^{l+1}(z)$ of type $(n_{l+1} - k, n_{l+1})$, which has maximal rank. And now according to Proposition 14 our assertion is obvious.

5. PSEUDOGROUP ASSOCIATED TO THE SHEAF

Let Γ be a pseudogroup on M (see [3], p. 8, Def. 1.1). For every integer $l \geqq 0$ we associate with Γ a pseudogroup Γ^l on $\tilde{J}^l$ defined in the following way. Let Δ^l be the set of local diffeomorphisms of $\tilde{J}^l$ such that $\psi \in \Delta^l$ if and only if there exists $\varphi \in \Gamma$ such that $\psi = \varphi^l$. Now we define Γ^l to be the smallest subpseudogroup of the pseudogroup of all local diffeomorphisms of $\tilde{J}^l$ containing Δ^l.

Definition 11. Γ^l is called the *l-th prolongation of Γ*.

A sheaf $\mathscr{F}$ induces a pseudogroup $\Gamma(\mathscr{F})$ on M in the following way (see also [3], pp. 9, 10). Let Θ be the set of local diffeomorphisms of M such that $\varphi : U_1 \to U_2$ belongs to Θ if and only if there exists a local 1-parameter group of transformations $h_t : U_1 \times (-\varepsilon, \varepsilon) \to M$ and $t_0 \in (-\varepsilon, \varepsilon)$ such that a) if X is a differentiable vector field generated by h_t on $h_t(U_1 \times (-\varepsilon, \varepsilon))$ then for any $\xi \in h_t(U_1 \times (-\varepsilon, \varepsilon))$ there is

$g_\xi(X) \in \mathscr{F}$, b) $h_{t_0}(U_1) = U_2$ and $h_{t_0} = \varphi$. We define $\Gamma(\mathscr{F})$ to be the smallest pseudogroup on M containing Θ.

Definition 12. $\Gamma(\mathscr{F})$ is called the *pseudogroup associated to the sheaf* $\mathscr{F}$.

Definition 13. Let Q be a differentiable manifold and let Γ be a pseudogroup on Q. A differentiable function f defined on an open set $U \subset Q$ is called *Γ-invariant* if and only if for any $\varphi \in \Gamma : U_1 \to U_2$ such that $U_1, U_2 \subset U$ there is $f \mid U_1 = (f \mid U_2) \circ \varphi$.

Proposition 16. *A differentiable function f defined on an apen set $U \subset \bar{J}^l$ is $\Gamma(\mathscr{F})^l$-invariant if and only if for any $x \in U$ there is $g_x(f) \in \bar{\mathscr{A}}^l$.*

Proof. Let f be $\Gamma(\mathscr{F})^l$-invariant. Let $x \in U$ and let $U' \subset U$ be an open neighborhood of x. Let X be a differentiable vector field defined on U' and lying in $\bar{\mathscr{F}}^l$ generated by a local 1-parameter group of transformations $h_t : U' \times (-\varepsilon, \varepsilon) \to \bar{J}^l$ such that $h_t(U' \times (-\varepsilon, \varepsilon)) \subset U$. As $h_t \in \Gamma(\mathscr{F})^l$ there is $f = f \circ h_t$ and so $Xf = 0$ on U'. Thus f lies in $\bar{\mathscr{A}}^l$.

On the contrary let f lie in $\bar{\mathscr{A}}^l$. Obviously it is sufficient to prove that f is Θ-invariant. Let $h_t : U' \times (-\varepsilon, \varepsilon) \to \bar{J}^l$, where $U' \subset U$, a local 1-parameter group of transformations such that $h_t(U' \times (-\varepsilon, \varepsilon)) \subset U$ and let us suppose that the differentiable vector field X defined by h_t on $h_t(U' \times (-\varepsilon, \varepsilon))$ lies in $\bar{\mathscr{A}}^l$. Let us consider the function $h(x, t) = f - f \circ h_t$ defined on $U' \times (-\varepsilon, \varepsilon)$. As f lies in $\bar{\mathscr{A}}^l$ there is $\partial h/\partial t = -Xf = 0$ on $U' \times (-\varepsilon, \varepsilon)$. Obviously $h(x, 0) = 0$ for all $x \in U$ and therefore $h(x, t) = 0$ on $U' \times (-\varepsilon, \varepsilon)$, i.e. f is $\Gamma(\mathscr{F})^l$-invariant.

Definition 14. Let Q be a differentiable manifold and let $\mathscr{B} \subset \mathscr{D}(Q)$ be a subsheaf of rings. Let $\varphi : U_1 \to U_2$, where $U_1, U_2 \subset Q$, be a local diffeomorphism. φ is called a local automorphism of $\mathscr{B}$ if for every $x \in U_1$ the mapping $\varphi_x^* : \mathscr{B}_{\varphi(x)} \to \mathscr{D}_x$, assigning to $g_{\varphi(x)}(f) \in \mathscr{B}_{\varphi(x)}$ an element $g_x(f \circ \varphi) \in \mathscr{D}_x$ is an isomorphism of $\mathscr{B}_{\varphi(x)}$ onto $\mathscr{B}_x$.

Proposition 17. *Let $\varphi : V_1 \to V_2$, where $V_1, V_2 \subset M$, be a local diffeomorphism and let $\psi^l : U_1 \to U_2$, where $U_i = (q\pi_0^l)^{-1} V_i \cap \bar{J}^l$ $(i = 1, 2)$ be the restriction of φ^l to U_1. Let us suppose that ψ^l is a local automorphism of $\mathscr{A}^l$. Then for every $l' \geqq l$ the mapping $\chi^{l'} : (\pi_l^{l'})^{-1} U_1 \to (\pi_l^{l'})^{-1} U_2$ which is the restriction of $\varphi^{l'}$ to $(\pi_l^{l'})^{-1} U_1$ is a local automorphism of $\mathscr{A}^{l'}$.*

Proof. Clearly it is sufficient to prove our assertion for $l' = l + 1$. With regard to the local character of the problem we may suppose that on U_1 and U_2 there are associated coordinate systems $(x^i, y^\alpha, y^\alpha_{i_1}, \ldots, y^\alpha_{i_1 \ldots i_l})$ and $(x^i, \bar{y}^\alpha, \bar{y}^\alpha_{i_1}, \ldots, \bar{y}^\alpha_{i_1 \ldots i_l})$ respectively such that $\bar{y}^\alpha \circ \varphi = y^\alpha$ for all $\alpha = 1, \ldots, m$. Then there is obviously also $\bar{y}^\alpha_{i_1 \ldots i_k} \circ \psi^l = y^\alpha_{i_1 \ldots i_k}$ for all admissible k-tuples $(i_1, \ldots, i_k)$, where $k = 1, \ldots, l$. Further we may suppose that there is a local φ-basis $f_1, \ldots, f_{n_l - k}$ and $\bar{f}_1, \ldots, \bar{f}_{n_l - k}$ of $\mathscr{A}^l$ on U_1

and U_2 respectively and even such that $\bar{f}_j \circ \psi^l = f_j$, for all $j = 1, \ldots, n_l - k$, for ψ^l is a local automorphism of $\mathscr{A}^l$. The reader can easily verify that there is $(\partial_{\#}^{x^i}\bar{f}_j) \circ \chi^{l+1} = \partial_{\#}^{x^i} f_j$ and from this our assertion immediately follows.

Let $U \subset J^l$ be an open set and let $(x^i, y^\alpha, y^\alpha_{i_1}, \ldots, y^\alpha_{i_1 \ldots i_l})$ be an associated coordinate system on U. Let X be a differentiable vector field defined on U such that $(\pi^l_{-1})_* X = 0$. We are going to define a differentiable vector field pX on $(\pi_l^{l+1})^{-1} U$, which we shall call the *formal prolongation of X*. In terms of the associated coordinate system we can write

$$X = a^\eta \frac{\partial}{\partial y^\eta} + \sum_{k=1}^{l} a^\eta_{i_1 \ldots i_k} \frac{\partial}{\partial y^\eta_{i_1 \ldots i_k}},$$

where a^η, $a^\eta_{i_1 \ldots i_k}$ are differentiable functions on U. We define

$$pX = a^\eta \frac{\partial}{\partial y^\eta} + \sum_{k=1}^{l+1} \partial_{\#}^{x^{i(k)}} (a^\eta_{i_1 \ldots i_{k-1}}) \frac{\partial}{\partial y^\eta_{i_1 \ldots i_k}}.$$

If $X = Y^l$, where Y is a differentiable vector field on $q\pi_0^l(U)$, then we can easily verify that there is $pX = Y^{l+1}$.

Proposition 18. *Let $h_t : V \times (-\varepsilon, \varepsilon) \to M$ be a local 1-parameter group of transformations and let $\xi \in V$. Let us suppose that for some $l \geqq 0$ there exist $x \in J^l$ such that $q\pi_0^l(x) = \xi$ and an open neighborhood U of x such that $U \subset (q\pi_0^l)^{-1} V$ and $h_t^l(U \times (-\varepsilon, \varepsilon)) \subset J^l$. For every $t \in (-\varepsilon, \varepsilon)$ let $h_t^l \mid U$ be a local automorphism of $\mathscr{A}^l$. Then there exist an open neighborhood $V' \subset V$ of ξ and $0 < \varepsilon' \leqq \varepsilon$ such that for every $t \in (-\varepsilon', \varepsilon')$ there is $h_t \mid V' \in \Gamma(\mathscr{F})$.*

Proof. It is obvious that we may suppose that there exist differentiable vector fields $X_1, \ldots, X_k$ on $h_t(V \times (-\varepsilon, \varepsilon))$ such that $(X_1, \ldots, X_k)$ is a local basis of $\mathscr{F}$ on $h_t(V \times (-\varepsilon, \varepsilon))$. We may also suppose that there is an associated coordinate system $(x^i, y^\alpha, y^\alpha_{i_1}, \ldots, y^\alpha_{i_1 \ldots i_l})$ on $\bar{U} = h_t^l(U \times (-\varepsilon, \varepsilon))$. Let X be a differentiable vector field on $h_t(V \times (-\varepsilon, \varepsilon))$ generated by the 1-parameter group h_t. As h_t^l are local automorphisms of $\mathscr{A}^l$ we can see (according to proposition 17) that for all $l' \geqq l$ the vector field $X^{l'}$ defined on $(\pi_l^{l'})^{-1} \bar{U}$ lies in $D^{l'}$. Thus for any $l' \geqq l$ we can write $X^{l'} = \sum_{j=1}^{k} f_j^{l'} X_j^{l'}$, where $f_j^{l'}$ are uniquely determined differentiable functions on $(\pi_l^{l'})^{-1} \bar{U}$. It can be easily seen that for any $l' \geqq l$ there is $f_j^{l'} = f_j^l \circ \pi_l^{l'}, j = 1, \ldots, k$. Therefore we have $pX^l = \sum_{j=1}^{k} (f_j^l \circ \pi_l^{l+1}) X_j^{l+1}$. On $h_t(V \times (-\varepsilon, \varepsilon))$ let us write $X = a^\eta(\partial/\partial y^\eta)$, $X_j = a_j^\eta(\partial/\partial y^\eta)$; $j = 1, \ldots, k$. We have

$$X^l = \sum_{j=1}^{k} f_j^l \left(a_j^\eta \frac{\partial}{\partial y^\eta} + \sum_{r=1}^{l} \sum_{s=1}^{r} \frac{\partial^s a_j^\eta}{\partial y^{\alpha_1} \ldots \partial y^{\alpha_s}} H^{\alpha_1 \ldots \alpha_s}_{i_1 \ldots i_r} \frac{\partial}{\partial y^\eta_{i_1 \ldots i_r}} \right)$$

$$pX^l = \sum_{j=1}^{k} \left\{ f_j^l a_j^\eta \frac{\partial}{\partial y^\eta} + \partial_\#^{x^{i(1)}}(f_j^l a_j^\eta) \frac{\partial}{\partial y_{i_1}^\eta} + \sum_{r=2}^{l+1} \sum_{s=1}^{r-1} \left(f_j^l \frac{\partial^s a_j^\eta}{\partial y^{\alpha_1} \dots \partial y^{\alpha_s}} H_{i_1 \dots i_{r-1}}^{\alpha_1 \dots \alpha_s} \right) \frac{\partial}{\partial y_{i_1 \dots i_r}^\eta} \right\} =$$

$$= \sum_{j=1}^{k} f_j^l \left(a_j^\eta \frac{\partial}{\partial y^\eta} + \sum_{r=1}^{l+1} \sum_{s=1}^{r} \frac{\partial^s a_j^\eta}{\partial y^{\alpha_1} \dots \partial y^{\alpha_s}} H_{i_1 \dots i_r}^{\alpha_1 \dots \alpha_s} \frac{\partial}{\partial y_{i_1 \dots i_r}^\eta} \right) +$$

$$+ \sum_{j=1}^{k} \left\{ (\partial_\#^{x^{i(1)}} f_j^l)\, a_j^\eta \frac{\partial}{\partial y_{i_1}^\eta} + \sum_{r=2}^{l+1} \sum_{s=1}^{r-1} (\partial_\#^{x^{i(r)}} f_j^l) \frac{\partial^s a_j^\eta}{\partial y^{\alpha_1} \dots \partial y^{\alpha_s}} H_{i_1 \dots i_{r-1}}^{\alpha_1 \dots \alpha_s} \frac{\partial}{\partial y_{i_1 \dots i_r}^\eta} \right\} =$$

$$= \sum_{j=1}^{k} (f_j^l \circ \pi_l^{l+1}) X_j^l + \square\,,$$

where $\square$ stands for the term which remains unchanged. But with respect to the equality $pX^l = \sum_{j=1}^{k} (f_j^l \circ \pi_l^{l+1})\, X_j^{l+1}$ there is $\square = 0$ and from this last equality we have for example

$$\sum_{j=1}^{k} (\partial_\#^{x^n} f_j^l) \left(a_j^\eta \frac{\partial}{\partial y_n^\eta} + \sum_{r=1}^{l} \sum_{s=1}^{r} \frac{\partial^s a_j^\eta}{\partial y^{\alpha_1} \dots \partial y^{\alpha_s}} H_{i_1 \dots i_r}^{\alpha_1 \dots \alpha_s} \frac{\partial}{\partial y_{i_1 \dots i_r n}^\eta} \right) = 0\,,$$

where the sum is taken over all admissible r-tuples $(i_1, \dots, i_r)$ where $r = 1, \dots, l$ Now on the basis of this equality we have

$$\sum_{j=1}^{k} (\partial_\#^{x^n} f_j^l) X_j^l = \sum_{j=1}^{k} (\partial_\#^{x^n} f_j^l) \left(a_j^\eta \frac{\partial}{\partial y^\eta} + \sum_{r=1}^{l} \sum_{s=1}^{r} \frac{\partial^s a_j^\eta}{\partial y^{\alpha_1} \dots \partial y^{\alpha_s}} H_{i_1 \dots i_r}^{\alpha_1 \dots \alpha_s} \frac{\partial}{\partial y_{i_1 \dots i_r}^\eta} \right) = 0\,,$$

which with respect to the linear independence of $X_1^l, \dots, X_k^l$ on $\overline{U}$ implies $\partial_\#^{x^n} f_j^l = 0$ on $(\pi_l^{l+1})^{-1}\, \overline{U}$ for $j = 1, \dots, k$. In the same way we can prove that $\partial_\#^{x^i} f_j^l = 0$ on $(\pi_l^{l+1})^{-1}\, \overline{U}$ for all $i = 1, \dots, n; j = 1, \dots, k$. Thus the functions $f_1^l, \dots, f_k^l$ are constant and the field X^l lies in $\mathscr{F}^l$. Likewise X on $q\pi_0^l(\overline{U})$ lies in $\mathscr{F}$ and from this our assertion immediately follows.

References

[1] *C. Chevalley:* Theory of Lie Groups I, Princeton 1946.

[2] *M. Kuranishi:* Lectures on Involutive Systems of Partial Differential Equations, Publicações da Sociedade de Matemática de São Paulo, 1967.

[3] *I. M. Singer, S. Sternberg:* The Infinite Groups of Lie and Cartan, Part I (The Transitive Groups), J. d'Anal. Math., vol. *XV* (1965), pp. 1—114.

[4] *Dale Husemoller:* Fibre Bundles, Mc Graw-Hill Series in Higher Mathematics, 1966.

[5] *Л. В. Овсянников:* Групповые свойства дифференциальных уравнений, Новосибирск 1962.

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