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MATRIX REPRESENTATION OF TORSION-FREE RINGS

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Dedicated to Professor VLADIMÍR KOŘÍNEK *on the occasion of his seventieth birthday*

1. INTRODUCTION

The aim of this paper is to give a brief account on a characterization of torsion-free rings in terms of matrix rings over division rings. In particular, the main results allow an immediate deduction of generalized GOLDIE and WEDDERBURN - ARTIN Theorems.

We refer for some basic definitions and results to [3]; however, contrary to [3], we understand throughout this paper by a ring an associative ring not necessarily with unity. In accordance with [3], a ring R is said to be (left) *torsion-free* if no essential left ideal of R is a (left) annihilator ideal of a non-zero element in R, and it is said to be *tidy* if, moreover, it contains a direct sum of its uniform left ideals which is essential in R. In the same sense, these terms are used for R-modules, in particular for (left) ideals of R. Here, an R-module U is said to be *uniform* if each of its non-zero R-submodules is essential in U.

In [3], for every associative ring R^1 with unity, the concept of a generalized prime $\mathscr{P}^1$ was introduced as a minimal "closed" set of (meet-)irreducible left idels. Alternatively, a generalized prime is the set of all orders of elements of $\sim$-equivalent uniform R^1-modules; here, two uniform R^1-modules are understood to be $\sim$-equivalent if they contain R^1-isomorphic non-zero R^1-submodules. Recall that one of the results of [3] asserts that, for every (left unital) R^1-module M and every generalized prime $\mathscr{P}^1$, the cardinality of the set of all the summands whose elements have orders from $\mathscr{P}^1$, in a maximal direct sum of uniform R^1-submodules of M is an invariant, the $\mathscr{P}^1$-*rank* $\mathrm{r}_{\mathscr{P}^1}(M)$ of M.

Here we modify the previous definitions in the following way:

Definition 1.1. Two (left) uniform R-modules U_1 and U_2 are said to be $\sim$*-equivalent* if they contain non-zero R-submodules $V_1 \subseteq U_1$ and $V_2 \subseteq U_2$ which are R-isomorphic. Let U be a uniform R-module; the set $\mathscr{P}_U$ of all orders (i.e. left annihilators) $O_R(x)$ of elements x belonging to uniform R-modules X $\sim$-equivalent to U is said to be a *generalized prime of R associated with U* (or with the corresponding class of $\sim$-equivalent uniform R-modules).

Denote by Π^R, and $\Pi_\circ^R$, the set of all generalized primes of R, and the set of all *relevant generalized primes* of R, i.e. those containing left annihilators of non-zero elements of R, respectively. We shall see that, for a torsion-free R and every $\mathscr{P} \in \Pi_\circ^R$, the $\mathscr{P}$-rank $\mathrm{r}_{\mathscr{P}}(R)$ of R is an invariant of the ring R and that, similar to the unital case, a division ring $D_{\mathscr{P}}$ (unique up to an isomorphism) is associated with every $\mathscr{P} \in \Pi_\circ^R$.

Definition 1.2. The cardinal function f defined on $\Pi_\circ^R$:

$$\mathrm{f}(P) = \mathrm{r}_P(R)$$

will be called *the basic invariants function of R.*

The fundamental concepts of the present paper are those of essential and rigid embeddings; we shall see that in a torsion-free case, they, in fact, coincide.

Definition 1.3. A triple (R', η, R'') of rings R', R'' and a monomorphism $\eta : R' \to R''$ is said to be *an essential embedding* if

(e) $R'\eta$ is an essential R'-submodule of R'', i.e. for every $0 \neq \varrho'' \in R''$, there is $\varrho' \in R'$ such that

$$0 \neq (\varrho'\eta)\,\varrho'' \in R'\eta\,.$$

(R', η, R'') is said to be *a rigid embedding* if, moreover, the following condition holds:

(r) Let L'' be the left ideal of R'' generated by $L'\eta$, where L' is a left ideal of R'. Then, for every $0 \neq \lambda'' \in L''$, there is $\varrho' \in R'$ such that

$$0 \neq (\varrho'\eta)\,\lambda'' \in L'\eta\,.$$

If (R', η, R'') is an essential, or a rigid embedding, $R'\eta$ is often called an essential, or a rigid subring of R'' and R'' an essential, or a rigid extension of $R'\eta$, respectively.

The main theorem then reads

Theorem. *Let R be a tidy torsion-free ring with the basic invariants function* $\mathrm{f} : \mathrm{f}(\mathscr{P}) = \mathrm{r}_{\mathscr{P}}(R)$ *for* $\mathscr{P} \in \Pi_\circ^R$; *let* $D_{\mathscr{P}}$ *be a division ring associated wth* $\mathscr{P}$. *Then R is isomorphic to a rigid subring of the direct product*

$$\prod_{\mathscr{P}\in\Pi_\circ{}^R} \mathbf{M}^\circ(\mathrm{r}_{\mathscr{P}}(R), D_{\mathscr{P}})$$

of the full $\mathrm{r}_{\mathscr{P}}(R) \times \mathrm{r}_{\mathscr{P}}(R)$ *finite-rows matrix rings* $\mathbf{M}^\circ(\mathrm{r}_{\mathscr{P}}(R), D_{\mathscr{P}})$ *over* $D_{\mathscr{P}}$.

On the other hand, every essential (and for that matter rigid) subring R of the direct product

$$\prod_{\bar\omega\in\bar\Omega} \mathbf{M}^\circ(\mathfrak{c}_{\bar\omega}, D_{\bar\omega})$$

of the (non-zero) full $\mathfrak{c}_{\bar\omega} \times \mathfrak{c}_{\bar\omega}$ *finite-rows matrix rings* $\mathbf{M}^\circ(\mathfrak{c}_{\bar\omega}, D_{\bar\omega})$ *over a division ring* $D_{\bar\omega}$ *is a tidy torsion-free ring R such that the set* $\Pi_\circ^R$ *of all relevant generalized* *primes of R can be indexed by $\bar{\Omega}$ and $\mathfrak{c}_{\bar{\omega}} = \mathfrak{r}_{\mathscr{P}_{\bar{\omega}}}(R)$ for $\bar{\omega} \in \bar{\Omega}$; moreover, $D_{\bar{\omega}}$ is a division ring associated with $\mathscr{P}_{\bar{\omega}}$.*

Let us add a few remarks to terminology and notation of the paper. Unless specified otherwise, "ideal" and "module" means always "left ideal" and "left module". If n is an element of an R-module N, then the *order* $\{\varrho \in R \mid \varrho n = 0\}$ of n in R is denoted by $O_R(n)$; thus, in particular, $O_R(\varrho)$ denotes the (left) annihilator of $\varrho \in R$ in R. By *Dorroh's embedding* of a ring R, we understand the triple $(R, {}^*\eta, {}^*R)$, where ${}^*\eta$ is the monomorphism of R into the ring ${}^*R = Z \times R$ (Z stands for the integers) defined by

$$\varrho\, {}^*\eta = (0, \varrho) \quad \text{for every} \quad \varrho \in R\,;$$

the addition in ${}^*R$ is component-wise and the multiplication is given by

$$(r_1, \varrho_1)(r_2, \varrho_2) = (r_1 r_2, r_1 \times \varrho_2 + r_2 \times \varrho_1 + \varrho_1 \varrho_2)\,.$$

Similarly, (R, η^Q, R^Q) is said to be a *quotient embedding*, if R^Q is a (left classical) quotient ring for R and η^Q is the corresponding canonical monomorphism of R into R^Q (see e.g. JACOBSON [7]).

2. RIGID EMBEDDINGS

The concept of a rigid embedding is, in fact, a concept of the module theory; here, we treat it only briefly in connection with our future needs. We start with a simple lemma which will be used repeatedly.

Lemma 2.1. *Let (R', η, R'') be an essential embedding. Then, for every $\varrho \in R''$, there is a left essential ideal L' of R' such that*

$$(L'\eta)\, \varrho \subseteq R'\eta\,.$$

Proposition 2.2. *Let (R', η, R'') be an essential embedding. Then the following four statements are equivalent:*

(i) *R' is torsion-free.*
(ii) *R'' is torsion-free.*
(iii) *(R', η, R'') is rigid and R' is torsion-free.*
(iv) *(R', η, R'') is rigid and R'' is torsion-free.*

Thus, if (R', η, R'') is essential, then R' is torsion-free if and only if R'' is torsion-free and, moreover, (R', η, R'') is then rigid.

Proof. (i) → (ii). Suppose that R'' is not torsion-free. Then, there is $0 \neq \varrho \in R''$ such that $K'' = O_{R''}(\varrho)$ is essential in R''. Evidently, one can assume that $\varrho \in R'\eta$:

$$\varrho = \varrho'\eta \quad \text{with} \quad \varrho' \in R'\,.$$

In view of our hypothesis, $K' = O_{R'}(\varrho')$ is not essential in R' and hence

$$K' \cap R'\lambda' = 0 \quad \text{with} \quad R'\lambda' \neq 0 \quad \text{for a suitable} \quad \lambda' \in R' .$$

However, since K'' is essential in R'', there exists $\sigma \in R''$ such that

$$0 \neq \sigma(\lambda'\eta) \in R'\eta \cap K'' = K'\eta ;$$

put $\sigma(\lambda'\eta) = \sigma'\eta$ with $\sigma' \in K'$. Now, by Lemma 2.1, there is an essential ideal L' of R' such that

$$(L'\eta)\,\sigma \subseteq R'\eta ,$$

and thus,

$$L'\sigma' \subseteq K' \cap R'\lambda' = 0 ,$$

a contradiction.

(ii) → (iv). Let L'' be the ideal generated by $L'\eta$, where L' is an ideal of R'; let $0 \neq \lambda'' \in L''$. Then,

$$\lambda'' = \sum_{\text{finite}} \varrho_i''(\lambda_i'\eta) \quad \text{with} \quad \varrho_i'' \in R'' \quad \text{and} \quad \lambda_i' \in L' .$$

By Lemma 2.1, there are essential ideals L'_i of R' such that

$$(L'_i\eta)\,\varrho_i'' \subseteq R'\eta , \quad \text{i.e.} \quad (L'_i\eta)\,\varrho_i''(\lambda_i'\eta) \subseteq L'\eta .$$

Hence, $L'_0 = \bigcap_{\text{finite}} L'_i$ is essential in R' and

$$0 \neq (L'_0\eta)\,\lambda'' \subseteq L'\eta ;$$

the validity of (r) follows.

(iv) → (iii). For, if R'' is torsion-free, then, for every $0 \neq \varrho' \in R'$,

$$[O_{R'}(\varrho')]\,\eta = O_{R''}(\varrho'\eta) \cap R'\eta$$

cannot be essential in $R'\eta$ in view of (e).

(iii) → (i) is trivial.

Proposition 2.3. *Let (R', η', R'') and (R'', η'', R''') be essential embeddings. If one of the rings R', R'' or R''' is torsion-free, then $(R', \eta'\eta'', R''')$ is an essential embedding of torsion-free rings and therefore rigid.*

Proof. According to Proposition 2.2, all the rings R', R'' and R''' are torsion-free. It is therefore sufficient to show that $(R', \eta'\eta'', R''')$ satisfies (e).

Let $0 \neq \varrho \in R'''$. Then, there exist subsequently $\varrho'' \in R''$ and $\varrho' \in R'$ such that

$$0 \neq \sigma'(\eta'\eta'') = [\varrho'(\eta'\eta'')]\,(\varrho''\eta'')\,\varrho \in R'(\eta'\eta'') \quad \text{with} \quad \sigma' \in R' .$$

Since R' is torsion-free, there is an element $\lambda' \in R'$ such that

$$O_{R'}(\sigma') \cap R'\lambda' = 0 \quad \text{and} \quad R'\lambda' \neq 0 .$$

There is also $\varkappa' \in R'$ such that

$$0 \neq \tau'\eta' = (\varkappa'\eta')\,[(\lambda'\varrho')\,\eta']\,\varrho'' = [(\varkappa'\lambda'\varrho')\,\eta']\,\varrho'' \in R'\eta' \quad \text{with} \quad \tau' \in R' .$$

Hence,

$$0 \neq [\tau'(\eta'\eta'')]\,\varrho = (\varkappa'\lambda'\sigma')(\eta'\eta'') \in R'(\eta'\eta''),$$

as required.

The following definition will help to simplify formulations of our next results.

Definition 2.4. Let (R', η, R'') be an (essential) embedding. Define the relation $\Theta = \Theta_{(R',\eta,R'')}$ between the left ideals of R' and R'', respectively, by

$$(L', L'') \in \Theta \equiv L'' \quad \text{is the left ideal of } R'' \text{ generated by } L'\eta .$$

Proposition 2.5. *Let (R', η, R'') be a rigid embedding.*

(i) *If $(L'_\omega, L''_\omega) \in \Theta$ for every $\omega \in \Omega$, then $\{L'_\omega \mid \omega \in \Omega\}$ is an independent set of left ideals of R' if and only if $\{L''_\omega \mid \omega \in \Omega\}$ is an independent set of left ideals of R''.*

(ii) *If $(L', L'') \in \Theta$, then L' is an essential left ideal of R' if and only if L'' is an essential left ideal of R''.*

(iii) *If $(L'_\omega, L''_\omega) \in \Theta$ for every $\omega \in \Omega$, then $\{L'_\omega \mid \omega \in \Omega\}$ is a maximal independent set of left ideals of R' if and only if $\{L''_\omega \mid \omega \in \Omega\}$ is a maximal independent set of left ideals of R''.*

(iv) *If $(L', L'') \in \Theta$, then L' is a uniform left ideal of R' if and only if L'' is a uniform left ideal of R''.*

(v) *R' is tidy if and only if R'' is tidy.*

(vi) *If $(L', L'') \in \Theta$, then L' is torsion-free (as an R'-module) if and only if L'' is torsion-free (as an R''-module).*

Proof. (i) Let $\{L'_\omega \mid \omega \in \Omega\}$ be independent and let $\sum_{\text{finite}} \lambda''_i = 0$ be a sum of non-zero elements λ''_i from distinct ideals L''_{ω_i}. Then, it is easy to establish by induction the existence of an element $\varrho \in R''$ (in fact, even $\varrho \in R'\eta$) such that

$$\sum_{\text{finite}} \varrho\lambda''_i = 0\,, \quad \varrho\lambda''_i \in L'_{\omega_i}\eta \quad \text{and} \quad \varrho\lambda''_{i_0} \neq 0 \quad \text{for a suitable } i_0\,,$$

a contradiction of our assumption. The converse of the statement is trivial.

Now, (ii) and (iv) is a simple consequence of (i). So is (iii); for, $(\bigoplus_{\omega\in\Omega} L'_\omega, \bigoplus_{\omega\in\Omega} L''_\omega) \in \Theta$. Furthermore, (v) combines (iii) together with (iv). Finally, (vi) follows immediately from (ii). For, if L'' is not torsion-free, then there is $0 \neq \lambda' \in L'$ such that $O_{R''}(\lambda'\eta)$ is, essential in R''; hence, $O_{R'}(\lambda')$ is essential in R'. The converse is trivial.

Proposition 2.6. *Let (R', η, R'') be a rigid embedding. Let $(L'_i, L''_i) \in \Theta$ for $i = 1, 2$, where L'_i and L''_i are uniform torsion-free ideals of the respective rings. Then L'_1 and L'_2 are $\sim$-equivalent (as R'-modules) if and only if L''_1 and L''_2 are $\sim$-equivalent (as R''-modules).*

Proof. In view of the previous Proposition 2.5, the proof of the "if" part is of routine character. In order to prove the "only if" part, take $\lambda'_i \in L'_i$ $(i = 1, 2)$ such that ψ defined by

$$(\varrho'\lambda'_1)\,\psi = \varrho'\lambda'_2 \quad \text{for every} \quad \varrho' \in R'$$

is an R'-isomorphism of $R'\lambda'_1$ and $R'\lambda'_2$. Such λ'_1 and λ'_2 exist in accordance with our assumption. Now, "extend" the mapping ψ to the following mapping φ of $R''(\lambda'_1\eta)$ onto $R''(\lambda'_2\eta)$:

$$[\varrho''(\lambda'_1\eta)]\,\varphi = \varrho''(\lambda'_2\eta) \quad \text{for every} \quad \varrho'' \in R''\,.$$

First, φ is an isomorphism between the underlying abelian groups; this follows form the fact that, for some $\varrho'' \in R''$,

$$\varrho''(\lambda'_1\eta) = 0 \quad \text{if and only if} \quad \varrho''(\lambda'_2\eta) = 0\,.$$

Indeed, if $\varrho''(\lambda'_1\eta) = 0$, then $(L'\eta)\,\varrho''(\lambda'_1\eta) = 0$, where L' is an essential ideal of R' such that $(L'\eta)\,\varrho'' \subseteq R'\eta$; but, then $(L'\eta)\,\varrho''(\lambda'_2\eta) = 0$ because of the isomorphism ψ, and thus $\varrho''(\lambda'_2\eta) = 0$, as required. Now, it is easy to see that φ is, in fact, an R''-isomorphism.

In Proposition 2.6, we have established the fact that the $\sim$-equivalence of uniform torsion-free ideals is stable under Θ; this yields immediately a one-to-one correspondence between the respective $\sim$-equivalent classes of ideals or, alternatively, a one-to-one correspondence between the respective generalized primes (Definition 1.1). Here, we shall formulate this result in the particular case of torsion-free rings.

Proposition 2.7. *Let (R', η, R'') be a rigid embedding of torsion-free ringe R' and R''. Then there is a one-to-one correspondence Φ between the sets $\Pi_o^{R'}$ and $\Pi_o^{R''}$ of all relevant generalized primes of R' and R'', respectively, induced by Θ, i.e.*

$$\mathscr{P}'\Phi = \mathscr{P}'' \quad \textit{if and only if} \quad L' \in \mathscr{P}' \quad \textit{and} \quad L'' \in \mathscr{P}''$$

$$\textit{exist such that} \quad (L', L'') \in \Theta\,.$$

Moreover, for every $\mathscr{P}' \in \Pi_o^{R'}$,

$$\mathrm{r}_{\mathscr{P}'}(R') = \mathrm{r}_{\mathscr{P}'\Phi}(R'')\,.$$

Proof. The first part follows from Proposition 2.6. The statement on the ranks is a consequence of the results of [3]; indeed, the rings can be considered as unital modules over their respective Dorroh's extensions.

We conclude this section with some results on quotient embeddings; first, notice (see e.g. [7]) that

Proposition 2.8. *A quotient embedding (R, η^Q, R^Q) is always rigid.*

For the sake of future application we include also a statement slightly extending the fact that a direct product ΠR_ω has a quotient ring if and only if all the rings R_ω have quotient rings.

Proposition 2.9. *Let $(R_{\bar\omega}, \eta^Q_{\bar\omega}, \mathbf{M}_{\bar\omega})$ be quotient embeddings for $\bar\omega \in \bar\Omega$ denote by η^Q the corresponding induced monomorphism of $\prod_{\bar\omega\in\bar\Omega} R_{\bar\omega}$ into $\prod_{\bar\omega\in\bar\Omega} \mathbf{M}_{\bar\omega}$. Then*

$$(R, \eta, \prod_{\bar\omega\in\bar\Omega} \mathbf{M}_{\bar\omega})$$

is a quotient embedding for every ring R and every monomorphism such that $\eta'\eta = \eta^Q$ for a suitable monomorphism $\eta' : \prod_{\bar\omega\in\bar\Omega} R_{\bar\omega} \to R$.

On the other hand, if $(R, \eta, \prod_{\bar\omega\in\bar\Omega} \mathbf{M}_{\bar\omega})$ is a quotient embedding, then, for each $\bar\omega \in \bar\Omega$, $(R_{\bar\omega}, \eta_{\bar\omega}, \mathbf{M}_{\bar\omega})$, where $\eta_{\bar\omega}$ is a monomorphism and $R_{\bar\omega}\eta_{\bar\omega} = R\eta \cap \mathbf{M}_{\bar\omega}$, is a quotient embedding.

Proof. The first part is obvious. In order to prove the second part, notice first that either $R_{\bar\omega}\eta_{\bar\omega} = \mathbf{M}_{\bar\omega}$ or there is a regular element $\varrho_{\bar\omega}$ in $R_{\bar\omega}$ (in these sense that $\varrho_{\bar\omega}$ is neither a left nor a right zero divisor in $R_{\bar\omega}$). For, if $\varrho \in \mathbf{M}_{\bar\omega} \setminus R_{\bar\omega}\eta_{\bar\omega}$, then there is a regular element $\varkappa$ of R and $\lambda \in R$ such that

$$\varrho = (\varkappa\eta)^{-1}(\lambda\eta), \quad \text{i.e.} \quad (\varkappa\eta)\varrho = \lambda\eta .$$

Hence, if ϱ is regular in $\mathbf{M}_{\bar\omega}$, then $\lambda\eta = \lambda'_{\bar\omega}\eta_{\bar\omega}$ with a regular element $\lambda'_{\bar\omega} \in R_{\bar\omega}$. Furthermore, for a regular element $\varrho_{\bar\omega}$ in $R_{\bar\omega}$ and an arbitrary $\varrho \in \mathbf{M}_{\bar\omega}$,

$$\varrho = (\varkappa\eta)^{-1}(\varrho_{\bar\omega}\eta_{\bar\omega})^{-1}(\varrho_{\bar\omega}\eta_{\bar\omega})(\lambda\eta) = (\varkappa_{\bar\omega}\eta_{\bar\omega})^{-1}(\lambda_{\bar\omega}\eta_{\bar\omega})$$

with a regular element $\varkappa_{\bar\omega} \in R_{\bar\omega}$ and $\lambda_{\bar\omega} \in R_{\bar\omega}$, as required.

Finally, let us formulate the following characterization of quotient embeddings in matrix rings.

Proposition 2.10. *Let $\mathbf{M} = \mathbf{M}(k, D)$ be a full $k \times k$ matrix ring over a division ring D. Then an essential embedding $(R, \eta, \mathbf{M})$ is a quotient embedding if and only if R is prime (or semiprime).*

Proof. For the proof of the "only if" part we refer to [7]. In order to establish the converse assume that R is prime and notice that, in view of Lemma 2.1, we only need to show that every essential ideal L of R contains a regular element μ, i.e. – since $\mu\eta \in \mathbf{M}(k, D)$ – an element μ such that $O_R(\mu) = 0$.

If this is not the case, pick up in the set $\{O_{\mathbf{M}}(\lambda) \mid \lambda \in L\}$ of non-zero ideals a minimal one

$$O_{\mathbf{M}}(\lambda_0) \quad \text{for a certain} \quad \lambda_0 \in L.$$

Evidently $\mathbf{M}(\lambda_0\eta) \neq \mathbf{M}$ and hence, $R\lambda_0$ is not essential in R; therefore a non-zero ideal U of R exists such that

$$R\lambda_0 \cap U = 0 \quad \text{and} \quad U \subseteq L.$$

As a consequence, $O_{\mathbf{M}}(\lambda_0) \supseteq O_{\mathbf{M}}(\lambda)$ for every $\lambda \in U$ and thus,

$$O_R(\lambda_0)\, U = 0 \quad \text{with} \quad [O_R(\lambda_0)]\,\eta = O_{\mathbf{M}}(\lambda_0\eta) \cap R\eta \neq 0\,,$$

contradicting our hypothesis R to be prime.

The fact that R to be semiprime is sufficient for the same conclusion follows from the following

Lemma 2.11. *Let (R', η, R'') be an essential embedding and R'' be prime. Then R' is prime if and only if it is semiprime.*

Proof. Only the "if" part requires a proof. Let

$$\varkappa' R' \lambda' = 0 \quad \text{for suitable} \quad \varkappa' \neq 0 \quad \text{and} \quad \lambda' \neq 0 \quad \text{of} \quad R'.$$

Since R' is semiprime, $R'\varkappa' \neq 0$ and thus – because R is prime –

$$L = (R'\lambda')\eta \quad R''(\varkappa'\eta) \neq 0\,.$$

Hence, the intersection $L_0 = L \cap R'\eta$ of the R'-module L with $R'\eta$ is a non-zero ideal of $R'\eta$; therefore, $L_0^2 \neq 0$. But, on the other hand, every element of L_0 is a sum of products of the form

$$[(\varrho_1'\lambda')\eta]\, \varrho_1''\, [(\varkappa'\varrho_2'\lambda')\eta]\, \varrho_2''(\varkappa'\eta)$$

with suitable $\varrho_i' \in R'$ and $\varrho_i'' \in R''$ $(i = 1, 2)$, and equals therefore zero.

3. THE UNITAL EMBEDDING

In this section, we deal with a particular type of rigid embeddings. As before, R denotes a ring; notice that $R_* = \{\varrho \in R \mid O_R(\varrho) = R\}$ is a two-sided ideal of R – the *defect ideal* of R. A ring R is said to be (left) *defect-free* if $R_* = 0$. Thus, for example a ring with unity, a torsion-free ring or a semiprime ring are defect-free rings. Recall also the definition of a (left) *faithful ring* as a ring R for which $\varrho R \neq 0$ whenever $\varrho \neq 0$.

Although we shall be concerned mainly with defect-free rings in this section, let us introduce the concept of a unital embedding in general.

Definition 3.1. An essential embedding (R, η^1, R^1) is said to be a (left) unital embedding (of R) if

(u) R^1 has a unity ε^1 and is generated by $R\eta^1$ and ε^1.

From the definition, one can deduce immediately that R is commutative if and only if R^1 is commutative and also that L is a left, right or two-sided ideal of R if and only if $L\eta^1$ is a left, right, or two-sided ideal of R^1, respectively. Thus, in particular, $R\eta^1$ is a two-sided ideal of R^1.

Proposition 3.2. *Let R be a defect-free ring. Then there exists a unital embedding (R, η^1, R^1) of R; moreover, it is unique in the following sense: If $(R, \bar{\eta}, \bar{R})$ is another unital embedding of R, then there is an isomorphism $\varphi : R^1 \to \bar{R}$ such that $\eta^1\varphi = \bar{\eta}$.*

Proof. Take Dorroh's embedding $(R, {}^*\eta, {}^*R)$ and consider

$$ {}^*I = \{(r, \varrho) \in {}^*R \mid r \times \chi + \chi\varrho = 0 \quad \text{for all} \quad \chi \in R\} \subseteq {}^*R . $$

Clearly, ${}^*I$ is a two-sided ideal of ${}^*R$ and

$$ {}^*I \cap R{}^*\eta = 0 ; $$

in fact, ${}^*I$ is the greatest ideal possessing the latter property. Now denote by π the canonical epimorphism ${}^*R \to R^1 = {}^*R/{}^*I$ and define $\eta^1 : R \to R^1$ by $\eta^1 = {}^*\eta\pi$. It is easy to verify that (R, η^1, R^1) is a unital embedding; (u) of Definition 3.1 is obvious and if $0 \neq (r, \varrho)\,\pi \in R'$, then there obviously exists $\chi_0 \in R$ such that

$$ 0 \neq (0, \chi_0)\,(r, \varrho)\,\pi = (\chi_0\eta^1)\,(r, \varrho)\,\pi \in R\eta^1 , $$

as required in (e) of Definition 1.3.

Now, let $(R, \bar{\eta}, \bar{R})$ be another unital embedding of R; let $\psi : {}^*R \to \bar{R}$ be the epimorphism defined by

$$ (r, \varrho)\,\psi = r \times \bar{\varepsilon} + \varrho\bar{\eta} , \quad \text{where } \bar{\varepsilon} \text{ is the unity of } \bar{R} . $$

It is easy to see that $(r, \varrho) \in \operatorname{Ker}\psi$ if and only if $r \times \chi + \chi\varrho = 0$ for all $\chi \in R$, i.e. if and only if $(r, \varrho) \in {}^*I$. Hence, ψ can be factored through $R^1 : \psi = \pi\varphi$ with an isomorphism φ between R^1 and R. Finally,

$$ \eta^1\varphi = {}^*\eta\pi\varphi = {}^*\eta\psi = \bar{\eta} , $$

completing the proof.

Notice that a consequence of the preceding Proposition 3.2 states that if R is a ring with unity, then (R, η, R) where η is an automorphism of R, is the up-to-an-isomorphism unique unital embedding of R. Also, if (R, η^1, R^1) is a unital embedding and R is faithful, then for every $0 \neq \varrho \in R^1$ there is $\chi_0 \in R$ such that $0 \neq \varrho(\chi_0\eta^1) \in R\eta^1$; thus, in this case, the unital embedding is "two-sided".

Proposition 3.3. *Let* (R, η^1, R^1) *be a unital embedding. Then*

(i) *R has no divisors of zero if and only if R^1 has no divisors of zero.*
(ii) *R is prime if and only if R^1 is prime.*
(iii) *R is semiprime if and only if R^1 is semiprime.*

Proof. The "only if" part of (i) follows from the preceding Proposition 3.2. The rest of Proposition 3.3. is obvious.

4. ENDOMORPHISM RING OF A TORSION-FREE MODULE

Throughout this section, let R^1 be a ring with unity and all R^1-modules unital. We start with a general result which will be later applied to torsion-free situation; since the proof is of a routine nature, it is omitted. Recall only the definition of a *pseudo-intersection* $\overline{\bigcap} N_\omega$ of submodules N_ω of an R^1-module N (cf. [1]):

$$\overline{\bigcap_{\omega\in\Omega}} N_\omega = \{n \in N \mid n \in N_\omega \text{ for all but a finite number of } \omega\text{'s}\} .$$

Also, let us remark that, for a (possibly infinite) index set Ω, by a $\Omega \times \Omega$ matrix are understood here, a (not necessarily ordered) graph of a function whose domain is $\Omega \times \Omega$; the addition and multiplication (if possible) are defined in the usual way.

Proposition 4.1. (cf. [1]). *Let* $N = \bigoplus_{\omega\in\Omega} N_\omega$ *be a direct decomposition of an R^1-module N. Then the endomorphism ring* $E_N = \mathrm{Hom}_{R^1}(N, N)$ *is isomorphic to the ring of all $\Omega \times \Omega$ matrices* $(\varphi_{\omega'\omega''})$ *such that*

(i) $\varphi_{\omega'\omega''} \in \mathrm{Hom}_{R^1}(N_{\omega'}, N_{\omega''})$
and
(ii) $\overline{\bigcap_{\omega\in\Omega}} \mathrm{Ker}\, \varphi_{\omega'\omega} = N_{\omega'}$.

The following simple lemma will be found useful on several occasions.

Lemma 4.2. *Let* $\varphi \in \mathrm{Hom}_{R^1}(U, V)$, *where U is a uniform and V a torsion-free R^1-module. Then either $\varphi = 0$ or φ is a monomorphism.*

Proof. For, if $0 \neq u_0 \in \mathrm{Ker}\, \varphi$, then for every $u \in U$ there is an essential ideal L in R^1 such that

$$\lambda u \in \langle u_0 \rangle \quad \text{for every} \quad \lambda \in L;$$

hence, $L(u\varphi) = 0$ which implies $u\varphi = 0$, i.e. $u \in \mathrm{Ker}\, \varphi$, as required.

Now, the previous Lemma 4.2 together with Proposition 4.1 provides immediately a proof for

Proposition 4.3. *Let Π^{R^1} be the set of all generalized primes and*

$$N = \bigoplus_{\omega \in \Omega} N_\omega$$

a direct sum of torsion-free uniform injective R^1-modules N_ω. Let

$$\mathfrak{r}_{\mathscr{P}^1}(N) = \mathfrak{c}_{\mathscr{P}^1} \quad \text{for} \quad \mathscr{P}^1 \in \Pi^{R^1}$$

thus, $\sum_{\mathscr{P}^1 \in \Pi^{R^1}} \mathfrak{c}_{\mathscr{P}^1} = \operatorname{card} \Omega$. *Then the endomorphism ring* $\mathrm{E}_N = \mathrm{Hom}_{R^1}(N, N)$ *is isomorphic to the direct product*

$$\prod_{\mathscr{P}^1 \in \Pi^{R^1}} \mathbf{M}^\circ(\mathfrak{c}_{\mathscr{P}^1}, D_{\mathscr{P}^1}),$$

where $\mathbf{M}^\circ(\mathfrak{c}_{\mathscr{P}^1}, D_{\mathscr{P}^1})$ *is the ring of all* $\mathfrak{c}_{\mathscr{P}^1} \times \mathfrak{c}_{\mathscr{P}^1}$ *finite-rows matrices over a division ring* $D_{\mathscr{P}^1}$ *associated with* $\mathscr{P}^1$ *(i.e. over the endomorphism ring of a uniform injective* R^1*-module of* $\mathscr{P}^1$*-rank* 1).

We conclude this section with a result which will enable us to apply Proposition 4.3 in the proof of our Theorem. To this end, let R^1 be now a torsion-free ring with unity, $H(R^1)$ its (in what follows fixed) injective hull and $\mathrm{E}_{H(R^1)} = \mathrm{Hom}_{R^1}(H(R^1), H(R^1))$. It is well known that

$$(R^1, \eta^\circ, \mathrm{E}_{H(R^1)})$$

is an essential and thus, in view of Proposition 2.2, a rigid embedding; here, for each $\varrho \in R^1$, $\varrho\eta^\circ \in \mathrm{E}_{H(R^1)}$ is a (unique) extension of $\psi_\varrho \in \mathrm{Hom}_{R^1}(R^1, R^1)$ defined by

$$\chi\psi_\varrho = \chi\varrho \quad \text{for every} \quad \chi \in R^1 .$$

We are going to present $\mathrm{E}_{H(R^1)}$ as the endomorphism ring of a direct sum.

Proposition 4.4. *Let R^1 be a tidy torsion-free ring and let $\{U_\omega \mid \omega \in \Omega\}$ be a maximal independent set of uniform left ideals of R^1. Let, for each $\omega \in \Omega$, $H(U_\omega) \subseteq H(R^1)$ be an injective hull of U_ω. Then*

$$G = \bigoplus_{\omega \in \Omega} H(U_\omega)$$

is an (essential) fully invariant R^1-submodule of $H(R^1)$ (i.e. invariant under every R^1-endomorphism of $H(R^1)$); thus, the rings $\mathrm{E}_G = \mathrm{Hom}_{R^1}(G, G)$ *and* $\mathrm{E}_{H(R^1)}$ *are (canonically) isomorphic. As a consequence,*

$$(R^1, \eta^0, \mathrm{E}_G)$$

is a rigid embedding.

Proof. Since every R^1-endomorphism of G can be extended to an R^1-endomorphism of $H(R^1)$, we get immediately an isomorphism (cf. [1]).

$$\mathrm{E}_G \cong \mathrm{E}_{[H(R^1),G]}/\mathrm{I}_{[H(R^1),G]},$$

where $E_{[H(R^1),G]} = \{\varphi \in E_{H(R^1)} \mid H(R^1)\varphi \subseteq G\}$ and the two-sided ideal $I_{[H(R^1),G]} = \{\varphi \in E_{H(R^1)} \mid G\varphi = 0\}$. There are no non-trivial R^1-endomorphisms of $H(R^1)$ with essential kernels (Lemma 4.2) and thus $I_{[H(R^1),G]} = 0$. The fact that $E_{[H(R^1),G]} = E_{H(R^1)}$, i.e. that G is fully invariant in $H(R^1)$, then follows easily from the following

Lemma 4.5. *Retain the notation of Proposition* 4.4. *Let* $\varphi \in E_{H(R^1)}$ *and* U *a uniform* R^1*-submodule of the (torsion-free) hull* $H(R^1)$. *Then*

(i) $U\varphi$ *is a uniform* R^1*-submodule*

and

(ii) $U \subseteq G$.

In fact, Lemma 4.5 is a consequence of Lemma 4.2. The statement (i) follows readily. Furthermore, for every $0 \neq u \in U$, there is a finite direct sum $G^\circ = \bigoplus_{\text{finite}} H(U_{\omega_i}) \subseteq G$ such that

$$0 \neq \varrho u \in G^\circ \quad \text{for a suitable} \quad \varrho \in R^1 .$$

Let $\varphi \in \mathrm{Hom}_{R^1}(\langle u \rangle, G^\circ)$ be the extension of the (identical) embedding of $\langle \varrho u \rangle$ into the (injective) G°. Then, applying Lemma 4.2 again, we deduce easily that $u = u\varphi \in G$, as required.

This completes also the proof of Proposition 4.4.

5. THE PROOF OF THEOREM

Here, we are going to combine the previous results to prove our Theorem.

Let R be a tidy torsion-free ring. Consider the unital embedding (R, η^1, R^1) of section 3 which, by Proposition 3.2, exists and is rigid (Proposition 2.2). Furthermore, in view of Proposition 2.5, R^1 is tidy and torsion-free. Also, by Proposition 2.7, there is a one-to-one correspondence Φ between the sets $\Pi_\circ^R$ and $\Pi_\circ^{R^1}$ of all relevant generalized primes of R and R^1, respectively, and

$$r_{\mathscr{P}}(R) = r_{\mathscr{P}\Phi}(R^1) \quad \text{for all} \quad \mathscr{P} \in \Pi_\circ^R .$$

Now, by Propositions 4.4 and 4.3,

$$\left(R^1, \eta^\circ, \prod_{\mathscr{P}^1 \in \Pi_\circ^{R^1}} \mathbf{M}^\circ(r_{\mathscr{P}^1}(R^1), D_{\mathscr{P}^1})\right)$$

is a rigid embedding; remark that $r_{\mathscr{P}^1}(R^1) = r_{\mathscr{P}^1}(G)$ for every $\mathscr{P}^1 \in \Pi_\circ^{R^1}$. Finally, applying Proposition 2.3, we get that

$$\left(R, \eta^1\eta^0, \prod_{\mathscr{P} \in \Pi_\circ^{R1}} \mathbf{M}^0(r_{\mathscr{P}}(R), D_{\mathscr{P}})\right)$$

is a rigid embedding.

Consider on the other hand an essential embedding

$$(R, \eta, \mathbf{M}) \quad \text{with} \quad \mathbf{M} = \prod_{\bar{\omega} \in \bar{\Omega}} \mathbf{M}^\circ(\mathfrak{c}_{\bar{\omega}}, D_{\bar{\omega}}) .$$

Put $\Omega = \{\omega = (\bar{\omega}, \bar{\mathfrak{c}}) \mid \bar{\omega} \in \bar{\Omega} \text{ and } 1 \leqq \bar{\mathfrak{c}} \leqq \mathfrak{c}_{\bar{\omega}}\}$; thus, the elements of $\mathbf{M}$ are $\Omega \times \Omega$ matrices. Furthermore, denote by $C_\omega = C_{(\bar{\omega}, \bar{\mathfrak{c}})}$, $\omega \in \Omega$, the (left minimal) uniform ideal of all $\Omega \times \Omega$ matrices $(\varphi_{\omega'\omega''})$ such that

$$\varphi_{\omega'\omega''} = 0 \quad \text{for all} \quad \omega'' \neq \omega .$$

It is easy to see that two ideals $C_{(\bar{\omega}_1, \bar{\mathfrak{c}}_1)}$ and $C_{(\bar{\omega}_2, \bar{\mathfrak{c}}_2)}$ are $\sim$-equivalent if and only if $\bar{\omega}_1 = \bar{\omega}_2$ and also that

$$\Pi_\circ^{\mathbf{M}} = \{\mathscr{P}_{\bar{\omega}} \mid \bar{\omega} \in \bar{\Omega}\} \quad \text{with} \quad C_{(\bar{\omega}, \bar{\mathfrak{c}})} \in \mathscr{P}_{\bar{\omega}} .$$

The rest then follows from Propositions 2.2, 2.5, 2.6 and 2.7. The proof of Theorem is completed.

For the sake of possible applications, let us formulated the following additional statement which can be easily verified in the course of the previous proof.

Addendum. (i) *The direct product* $\prod_{\mathscr{P} \in \Pi_\circ^R} \mathbf{M}^\circ(\mathrm{r}_{\mathscr{P}}(R), D_{\mathscr{P}})$ *of Theorem is an injective hull of the ring R.*

(ii) *If* $R = \bigoplus_{\omega \in \Omega} L_\omega$ *is a direct sum of uniform (torsion-free) left ideals, then the rigid embedding* $(R, \eta, \mathbf{M})$ *of Theorem can be chosen so that*

$$L_\omega \eta \subseteq C_\omega \quad \textit{for all} \quad \omega \in \Omega .$$

6. SOME APPLICATIONS

In this final section, we intend to make a few remarks with regard to consequences of our Theorem.

First, referring to the notation used in the proof of Theorem, we can easily verify that, for each $\bar{\omega} \in \bar{\Omega}$,

$$\mathbf{M}_{\bar{\omega}}^{\circ\circ} = \bigoplus_{1 \leqq \bar{\mathfrak{c}} \leqq \mathfrak{c}_{\bar{\omega}}} L_{(\bar{\omega}, \bar{\mathfrak{c}})} \subseteq \mathbf{M}_{\bar{\omega}}^\circ = \mathbf{M}^\circ(\mathfrak{c}_{\bar{\omega}}, D_{\bar{\omega}})$$

is a two-sided ideal containing any other two-sided ideal of $\mathbf{M}_{\bar{\omega}}^\circ$. In fact, $\mathbf{M}_{\bar{\omega}}^{\circ\circ}$ is also the least left essential ideal of $\mathbf{M}_{\bar{\omega}}^\circ$, i.e. $\mathbf{M}_{\bar{\omega}}^{\circ\circ}$ is the (left) socle of $\mathbf{M}_{\bar{\omega}}^\circ$. Notice that $\mathbf{M}_{\bar{\omega}}^{\circ\circ}\mu = 0$ implies $\mu = 0$ for every $\mu \in \mathbf{M}_{\bar{\omega}}^\circ$. Hence, in particular,

Proposition 6.1. *For each* $\bar{\omega} \in \bar{\Omega}$, *the ring* $\mathbf{M}^\circ(\mathfrak{c}_{\bar{\omega}}, D_{\bar{\omega}})$ *is prime. Thus,* $\mathbf{M} = \prod_{\bar{\omega} \in \bar{\Omega}} \mathbf{M}^\circ(\mathfrak{c}_{\bar{\omega}}, D_{\bar{\omega}})$ *is semiprime.*

In order to be able to describe a tidy torsion-free ring R by means of the matrix ring M more closely it seems, therefore, to be quite natural to impose on R the additional condition to be semiprime.

Proposition 6.2. *Let R be a semiprime tidy torsion-free ring and let*

$$\left(R, \eta, \prod_{\mathscr{P} \in \Pi^{\circ R}} \mathbf{M}^{\circ}(\mathrm{r}_{\mathscr{P}}(R), D_{\mathscr{P}})\right)$$

be the rigid embedding of Theorem.

(i) *For each $\mathscr{P} \in \Pi_{\circ}^{R}$, the subring (which is a two-sided ideal) $R_{\mathscr{P}} \subseteq R$ such that $R_{\mathscr{P}}\eta = R\eta \cap \mathbf{M}^{\circ}(\mathrm{r}_{\mathscr{P}}(R), D_{\mathscr{P}})$ is prime.*

(ii) *If, for a certain $\mathscr{P} \in \Pi_{\circ}^{R}$, $\mathrm{r}_{\mathscr{P}}(R)$ is finite, then $(R_{\mathscr{P}}, \eta_{\mathscr{P}}, \mathbf{M}^{\circ}(\mathrm{r}_{\mathscr{P}}(R), D_{\mathscr{P}}))$ with $\eta_{\mathscr{P}}$ induced on $R_{\mathscr{P}}$ by η is a quotient embedding.*

(iii) *If, for a certain $\mathscr{P} \in \Pi_{\circ}^{R}$, $\mathrm{r}_{\mathscr{P}}(R)$ is finite and $R_{\mathscr{P}}$ has no proper (i.e. $\neq R_{\mathscr{P}}$) essential ideals, then $\eta_{\mathscr{P}}$ is an isomorphism of $R_{\mathscr{P}}$ and $\mathbf{M}^{\circ}(\mathrm{r}_{\mathscr{P}}(R), D_{\mathscr{P}})$.*

Proof. (i) Obviously, $(R_{\mathscr{P}}, \eta_{\mathscr{P}}, \mathbf{M}^{\circ}(\mathrm{r}_{\mathscr{P}}(R), D_{\mathscr{P}}))$ is an essential embedding. Moreover, $R_{\mathscr{P}}$ is semiprime; for, if $0 \neq \varrho_1 \in R_{\mathscr{P}}$, then there exists evidently $\varrho_0 \in R_{\mathscr{P}}$ such that $\varrho_0\varrho_1 \neq 0$ and thus, $\varrho_1 R_{\mathscr{P}} \varrho_1 = 0$ would imply $\varrho_2 R \varrho_2 = 0$ with $\varrho_2 = (\varrho_0\varrho_1)\eta$. Hence, by Proposition 6.1 and Lemma 2.11, $R_{\mathscr{P}}$ is prime.

(ii) This is a consequence of (i) and Proposition 2.10.

(iii) In view of (ii), necessarily $R_{\mathscr{P}}\eta_{\mathscr{P}} = \mathbf{M}^{\circ}(\mathrm{r}_{\mathscr{P}}(R), D_{\mathscr{P}})$.

Remark 6.3. It may be appropriate to remark here that the two-sided ideals $R_{\mathscr{P}}$, $\mathscr{P} \in \Pi_{\circ}^{R}$, can be defined without any reference to the embedding described in Theorem. Using some results of [2], we can easily deduce that there is a (unique) R-family (idempotant topological sets of ideals) $\mathscr{P}^*$ corresponding to $\mathscr{P}$ such that $R_{\mathscr{P}}$ is just the set of all elements of R whose orders belong to $\mathscr{P}^*$; we can thus refer to $R_{\mathscr{P}}$ as to the *$\mathscr{P}$-primary part* of R. Notice that $\mathrm{r}_{\mathscr{P}}(R) = \mathrm{r}(R_{\mathscr{P}})$; in particular, if R is Artinian, then all $\mathscr{P}$-primary parts have finite ranks.

Proposition 6.4. (cf. GOLDIE [4], [5]). *Let R be a tidy semiprime (or, prime) torsion-free ring such that the ranks $\mathrm{r}_{\mathscr{P}}(R)$ are finite for all $\mathscr{P} \in \Pi_{\circ}^{R}$; then $\prod_{\mathscr{P} \in \Pi_{\circ}^{R}} \mathbf{M}(\mathrm{r}_{\mathscr{P}}(R), D_{\mathscr{P}})$ is the quotient ring for R (or, $\Pi_{\circ}^{R} = \{\mathscr{P}\}$, $\mathrm{r}_{\circ}^{\mathscr{P}}(R) = \mathrm{r}(R)$, $D_{\mathscr{P}} = D_R$ and $\mathbf{M}(\mathrm{r}(R), D_R)$ is the quotient ring for R, respectively.).*

On the other hand, if $\prod_{\bar{\omega} \in \bar{\Omega}} \mathbf{M}(k_{\bar{\omega}}, D_{\bar{\omega}})$, with integers $k_{\bar{\omega}} \geqq 1$, is the quotient ring for a ring R, then R is a tidy semiprime, torsion-free ring such that the elements of $\Pi_{\circ}^{R}$ can be indexed by $\bar{\Omega}$ and

$$\mathrm{r}_{\mathscr{P}_{\bar{\omega}}}(R) = k_{\bar{\omega}} \quad \textit{for every} \quad \bar{\omega} \in \bar{\Omega}\,;$$

if, in particular, $\bar{\Omega}$ consists of a single element, then R is prime.

Proposition 6.4 is an immediate consequence of our Theorem and Proposition 6.2, 2.9 and 2.10. Also, it is not difficult to see that if a ring R is semiprime, then the original Goldie's maximal condition for left annihilators (even when restricted to essential annihilators) of R implies that R is torsion-free (cf. MICHLER [8]).

Proposition 6.5. (cf. WEDDERBURN-ARTIN Theorem). *If R is a tidy semiprime ring which has no proper (i.e. $\neq R$) essential ideals and whose $\mathscr{P}$-primary parts $R_{\mathscr{P}}$ are of finite ranks $\mathrm{r}(R_{\mathscr{P}}) = k_{\mathscr{P}}$, then*

$$R = \prod_{\mathscr{P}\in\Pi_0^R} \mathbf{M}(k_{\mathscr{P}}, D_{\mathscr{P}}),$$

and vice versa. Thus, R is a direct product of full finite-dimensional matrix rings over division rings if and only if R is semiprime and every its product-indecomposable direct factor is Artinian.

Again, Proposition 6.5 follows from our Theorem immediately (use Proposition 6.2 and notice that an Artinian semiprime ring has no essential ideals and is of finite rank). We like to conclude the paper with a brief note that Theorem (together with Addendum) can also be used to derive simply Goldie's characterization of Artinian torsion-free generalized uniserial rings in [6].

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