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ON SOME PROPERTIES OF A SOLUTION OF THE SCHWARZIAN DIFFERENTIAL EQUATION

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1. Let $q(z)$ be analytic in $K : |z| < 1$. Between the solutions of the equation

$$u'' + q(z)\,u = 0 \tag{1}$$

and those of the Schwarzian differential equation

$$\{w, z\} = q(z)\,, \tag{2}$$

where the Schwarzian derivative $\{w, z\} = (w''/2w')' - (w''/2w')^2$, there is a relation such that the ratio $u(z)/v(z)$ of two linearly independent solutions $u(z)$, $v(z)$ of (1) is a solution of (2) and conversely, each solution of (2) can be written as the quotient of two linearly independent solutions of (1). In a series of papers ([1], [2], [3], [4]) Z. NEHARI used this connection to obtain some criteria of univalence for the solutions of (2). In this way R. F. GABRIEL obtained a criterion of convexity [5]. In this paper some other properties of a solution of the equation (2) will be derived. Their proof will be given on the basis of some comparison theorems in the real domain given in the next section. We recall that the solutions of the equation (2) are uniquely determined by the initial conditions.

2. The first comparison theorem which is to be used represents a simple generalization of the well-known comparison theorem.

Lemma 1. *Let the real function $f(t)$ have the following properties:*

1. $f(t) \in C_0(\langle a, b\rangle)$ *and* $f(t) > 0$ *for* $t \in (a, b)$, $f(a) = f(b) = 0$.

2. *On the sum of the intervals* $S = (a, c) \cup (c, b)$, $a < c < b$, $f(t)$ *satisfies the differential equation* $y'' + p_1(t)\,y = 0$, *where* $p_1(t) \in C_0(S)$ *is real.*

3. *There exist* $f'(a + 0)$, $f'(b - 0)$, $f'(c + 0)$, $f'(c - 0)$, *whereby* $f'(c + 0) - f'(c - 0) \geqq 0$.

Let the real function $p_2(t) \in C_0(\langle a, b\rangle)$ satisfy the inequality $p_2(t) \geqq p_1(t)$ for each $t \in S$. Then each (real) solution $g(t)$ of the differential equation

$$(3) \qquad y'' + p_2(t)\, y = 0$$

has at least one zero-point in (a, b) with the exception to the case $p_1(t) = p_2(t)$ on S and $g(t) = k f(t)$, $t \in \langle a, b\rangle$, $k \neq 0$ is a constant.

Proof. Suppose there exists a solution $g(t)$ of (3) positive in (a, b). Let $\varepsilon > 0$ be sufficiently small. By the known device we get

$$\int_{a+\varepsilon}^{c-\varepsilon} [f''(t)\, g(t) - g''(t) f(t)]\, dt + \int_{c+\varepsilon}^{b-\varepsilon} [f''(t)\, g(t) - g''(t) f(t)]\, dt =$$

$$= \int_{a+\varepsilon}^{c-\varepsilon} [p_2(t) - p_1(t)] f(t)\, g(t)\, dt + \int_{c+\varepsilon}^{b-\varepsilon} [p_2(t) - p_1(t)] f(t)\, g(t)\, dt \geqq 0.$$

Since $f''(t)\, g(t) - g''(t) f(t) = [f'(t)\, g(t) - f(t)\, g'(t)]'$, from the last inequality letting $\varepsilon \to 0$ the inequality

$$(4) \qquad f'(b - 0)\, g(b) - f'(a + 0)\, g(a) - g(c)\, [f'(c + 0) - f'(c - 0)] \geqq 0$$

follows. By the assumptions of lemma the last expression is non positive and thus, equal to 0. This implies $p_1(t) = p_2(t)$ on S and $f'(c + 0) = f'(c - 0) = f'(c)$. From the continuity of the functions $f(t)$, $p_2(t)$ at the points a, c, b follows that $f(t)$ satisfies the equation (3) on $\langle a, b\rangle$. With respect to $f(a) = f(b) = 0$ the sharp inequalities $f'(b - 0) < 0$, $f'(a + 0) > 0$ must hold in (4) and thus $g(a) = g(b) = 0$, hence $g(t) = k f(t)$, $t \in \langle a, b\rangle$.

Another comparison theorem will be useful, too.

Lemma 2. *Let the real functions $p_i(t) \in C_0(\langle 0, 1))$, $i = 1, 2$, and satisfy $p_1(t) \leqq p_2(t)$ for $0 \leqq t < 1$. Let $y_i(t)$, $i = 1, 2$, be the solution of the equation*

$$(5) \qquad y'' + p_i(t)\, y = 0$$

satisfying the initial conditions $y_i(0) = 1$, $y_i'(0) = 0$. Let

$$(6) \qquad y_2(t) > 0 \quad for \quad 0 \leqq t < 1\,.$$

Then $y_1(t) \geqq y_2(t)$ there and

$$y_1^2(\tau) \int_\tau^t \frac{1}{y_1^2(p)}\, dp \leqq y_2^2(\tau) \int_\tau^t \frac{1}{y_2^2(p)}\, dp$$

for $0 \leqq \tau \leqq t < 1$.

Proof. $y_1(t)$ satisfies the equation

$$y_1(t) = y_2(t) + \int_0^t G(t, \tau)\,(p_2(\tau) - p_1(\tau))\, y_1(\tau)\, d\tau\,, \tag{7}$$

where $G(t, \tau)$ — as a function of the variable t — is the solution of the equation (5) for $i = 2$ and $G(\tau, \tau) = 0$, $\partial G(\tau, \tau)/\partial t = 1$. From (6) it follows that $G(t, \tau) > 0$ for $0 \leqq \tau < t < 1$ and hence, on basis of the equation (7), the inequality $y_1(t) \geqq y_2(t)$ can be obtained. Further, applying the Comparison Theorem ([6], p. 23) to the solutions $Y_i(t) = y_i'(t)/y_i(t)$, $i = 1, 2$, of the equation $Y' = -p_i(t) - Y^2$, we get that $Y_2(t) \leqq Y_1(t)$, $0 \leqq t < 1$. Finally the function $G_i(t) = y_i^2(\tau) . \int_\tau^t (1/y_i^2(p))\, dp$ is the solution of the equation $G'' + 2Y_i(t)\, G' = 0$ determined by $G_i(\tau) = 0$, $G_i'(\tau) = 1$, $i = 1, 2$. Therefore, $G_1(t) = G_2(t) + 2 \int_\tau^t K(t, u)\,(Y_2(u) - Y_1(u))\, G_1'(u)\, du$, where $K(t, u) = y_2^2(u) \int_u^t (1/y_2^2(p))\, dp > 0$ and $G_1'(u) = y_1^2(\tau)/y_1^2(\tau) > 0$ for $0 \leqq \tau \leqq u < < t < 1$. This finishes the proof of Lemma 2.

3. We return to the differential equation (1). On each segment starting at the origin its solutions define two real functions which satisfy certain differential equations.

Lemma 3. *Let $u(z)$ be a solution of the differential equation (1), $\alpha \in \langle 0, 2\pi)$ be a number and $I_\alpha = \{t : t \in \langle 0, 1),\ u_\alpha(t) = u(e^{i\alpha}t) \neq 0\}$. Then if we denote $u_\alpha(t) = \varrho(t)\, e^{i\sigma(t)}$ ($\varrho(t) = \varrho(t, \alpha) > 0$, $\sigma(t) = \sigma(t, \alpha)$ are real and continuous on I_α) for each $t \in I_\alpha$, the equalities*

$$\varrho'(t) = \frac{1}{\varrho(t)} \operatorname{Re}\{\overline{u(e^{i\alpha}t)}\, u'(e^{i\alpha}t)\, e^{i\alpha}\}\,, \tag{8}$$

$$\sigma'(t) = \frac{1}{\varrho^2(t)} \operatorname{Im}\{\overline{u(e^{i\alpha}t)}\, u'(e^{i\alpha}t)\, e^{i\alpha}\}$$

hold on I_α, $\varrho(t)$ satisfies the differential equation

$$\varrho'' + (-\sigma'^2(t) + \operatorname{Re}\{q(e^{i\alpha}t)\, e^{2i\alpha}\})\, \varrho = 0 \tag{9}$$

and $\sigma(t)$ the differential equation

$$\sigma'' + \frac{2\varrho'(t)}{\varrho(t)}\, \sigma' = -\operatorname{Im}\{q(e^{i\alpha}t)\, e^{2i\alpha}\}\,, \tag{10}$$

both functions on I_α. Here $\bar{z}$ denotes the complex conjugate number to z.

Proof. Obviously the functions $\varrho(t)$, $\sigma(t)$ have the derivatives of all orders on I_α. Further $u_\alpha'(t) = e^{i\sigma(t)}[\varrho'(t) + i\, \varrho(t)\, \sigma'(t)]$ is valid there, from which (8) follows. Differentiating we obtain $u_\alpha''(t) = e^{i\sigma(t)}[\varrho''(t) - \varrho(t)\, \sigma'^2(t) + i.\ (2\varrho'(t)\, \sigma'(t) + \varrho(t)\, .\ \sigma''(t))]$. Because $u(z)$ is a solution of (1), this expression is equal to $-\ q(e^{i\alpha}t)\, e^{2i\alpha}\, \varrho(t)$.

Comparing of the real and imaginary parts yields that $\varrho(t)$ satisfies the equation (9) and $\sigma(t)$ the equation (10).

Corollary. *If* $u_\alpha(t) \neq 0$, $0 \leqq t < 1$, *then*

$$\sigma(t) = \sigma(0) + \sigma'(0) \int_0^t \frac{\varrho^2(0)}{\varrho^2(\tau)} \, d\tau - \int_0^t \left[\int_\tau^t \frac{1}{\varrho^2(p)} \, dp \right] \varrho^2(\tau) \, \mathrm{Im} \, \{q(e^{i\alpha}\tau) \, e^{2i\alpha}\} \, d\tau \,,$$
$$0 \leqq t < 1 \,.$$

Proof. In fact, the function $H(t, \tau) = \varrho^2(\tau) \int_\tau^t (1/\varrho^2(p)) \, dp$ satisfies (as a function of the variable t) the corresponding homogeneous equation to the equation (10) and $H(\tau, \tau) = 0$, $\partial H(\tau, \tau)/\partial t = 1$.

Together with the differential equation (1) its majorant equation

$$(11) \qquad y'' + M(t) \, y = 0 \,, \quad -1 < t < 1 \,,$$

will be considered where $M(t) = \max_{|z| \leqq t} |q(z)|$ for $0 \leqq t < 1$ and $M(t) = M(-t)$ for $-1 < t < 0$.

The main theorem of the paper is the following

Theorem 1. *If each non-trivial solution of the equation* (11) *has at most one zero-point in* $(-1, 1)$, *then the solution* $w(z)$ *of the equation* (2) *satisfying the conditions*

$$(12) \qquad w(0) = 0 \,, \quad w'(0) = 1 \,, \quad w''(0) = 0$$

is holomorphic in K, *satisfies the relation*

$$(13) \qquad \mathrm{Re} \left\{ \frac{w(z)}{z} \right\} > 0$$

there and is one-to one on each segment starting at the origin.

Proof. At first we shall prove that from the assumption of the theorem two assertions follow:

1. On each segment (without endpoint) $z = e^{i\alpha}t$, $0 \leqq t < 1$, $\alpha \in \langle 0, 2\pi)$ is an arbitrary number, no non-trivial solution $u(z)$ of the equation (1) has two zero-points. Especially the solution $u_1(z)$ of this equation given by the initial conditions $u_1(0) = 0$, $u_1'(0) = 1$ has no other zero in K than at the origin.

2. For each non-trivial solution $u(z)$ of the equation (1) with at least one zero-point $ae^{i\alpha}$, $a > 0$, $0 \leqq \alpha < 2\pi$ which is normed by $u(0) = 1$ the inequality

$$(14) \qquad \mathrm{Re} \, \{u'(0) \, e^{i\alpha}\} < 0$$

holds. Especially the solution $u_2(z)$ of the equation (1) which satisfies the conditions $u_2(0) = 1$, $u_1'(0) = 0$ has no zero-point in K.

Proof of the point 1. Let us assume that $u(z)$ has (at least) two zero-points $ae^{i\alpha}$, $be^{i\alpha}$, $0 \leqq a < b < 1$, whereby $u(e^{i\alpha}t) \neq 0$ for $a < t < b$. Setting $u(e^{i\alpha}t) = \varrho(t)\, e^{i\sigma(t)}$, where $\varrho(t) > 0$, $\sigma(t)$ are real and continuous in (a, b) and defining $\varrho(a) = \varrho(b) = 0$ we see that $\varrho(t)$ possesses the first property of the function $f(t)$ from Lemma 1 and, on basis of Lemma 3, also the second property. The coefficient of the equation (9) which is satisfied by $\varrho(t)$ is majorized by the function $M(t)$. We have still to prove the existence of $\varrho'(a + 0)$. The existence $\varrho'(b - 0)$ can be analogically proved. On basis of (8) is $\varrho'(a + 0) = \operatorname{Re}\{\lim_{t \to a+} e^{-i\sigma(t)}\, u'(e^{i\alpha}t)\, e^{i\alpha}\}$ and therefore it suffices to prove the existence of $\lim_{t \to a+} e^{-i\sigma(t)}$. This follows from the fact that at the point $t = a$ there exists the right-hand side tangent to the curve $u = u(e^{i\alpha}t)$. Its direction there is given by the expression

$$\frac{|u'(ae^{i\alpha})\, e^{i\alpha}|}{u'(ae^{i\alpha})\, e^{i\alpha}} = \lim_{t \to a+} \frac{u(e^{i\alpha}t)}{|u(e^{i\alpha}t)|} = \lim_{t \to a+} e^{i\sigma(t)}\,.$$

By Lemma 1 there exists then a non-trivial solution of the equation (11) having at least two zero-points in $\langle a, b\rangle$ which is in contradiction with the assumption of the theorem.

Proof of the point 2. Let $u(z)$ be the solution of the equation (1) with a zero at $ae^{i\alpha}$, $a > 0$, $0 \leqq \alpha < 2\pi$ and with $u(0) = 1$, $\operatorname{Re}\{u'(0)\, e^{i\alpha}\} \geqq 0$. By the point 1 $u(z)$ has on the segment $e^{i\alpha}t$, $0 \leqq t < 1$, no other zero-point than $ae^{i\alpha}$. Again we put $u(e^{i\alpha}t) = \varrho(t)\, e^{i\sigma(t)}$, $0 \leqq t < a$, $\varrho(a) = 0$. The function $\varrho(t)$ can be extended on $\langle -a, a\rangle$ as an even function. If the coefficient of the equation is also extended as an even function, then $\varrho(t)$ will satisfy the equation (9) on the intervals $(-a, 0)$, $(0, a)$. By Lemma 3 $\varrho'(0 + 0) = \operatorname{Re}\{u'(0)\, e^{i\alpha}\}$. From it $\varrho'(0 + 0) - \varrho'(0 - 0) = 2 \operatorname{Re}\{u'(0)\, e^{i\alpha}\} \geqq 0$. The existence of $\varrho'(-a + 0)$, $\varrho'(a - 0)$ has been already proved. Thus all assumptions of Lemma 1 being satisfied, there exists a non-trivial solution of the majorant equation (11) which possesses two zeros in $\langle -a, a\rangle$ and this fact is again in contradiction to the assumption of Theorem 1.

Consider now the ratio

$$w(z) = \frac{u_1(z)}{u_2(z)}\,. \tag{15}$$

$w(z)$ is the solution of the equation (2) which satisfies the conditions (12) and, by the point 2, is holomorphic in K. Further from the point 1 it follows that it is one-to-one on each segment starting at the origin. Let us consider the function $W(z) = w(z)/z$. $W(0) = \lim_{z \to 0} W(z) = w'(0) = 1$. Let $z_0 \neq 0$, $z_0 \in K$ be an arbitrary but fixed point. We shall prove that $\operatorname{Re}\{W(z_0)\} > 0$. The solution $u(z)$ of the equation (1) with $u(z_0) = 0$, $u(0) = 1$ can be written in the form $u(z) = u'(0)\, u_1(z) + u_2(z)$ and hence

$u(z) = u'(0)\, u_1(z_0) + u_2(z_0) = 0$. From that $w(z_0) = -1/u'(0)$ and

$$\operatorname{Re}\{W(z_0)\} = \operatorname{Re}\left\{-\frac{1}{u'(0)\, z_0}\right\} = -\frac{1}{|u'(0)|^2\, |z_0|} \operatorname{Re}\{u'(0)\, e^{i\operatorname{Arg} z_0}\} > 0$$

on basis of the inequality (14).

Remark. Theorem 1 can not be conversed as it is shown by the function

$$w(z) = \frac{2z^2 + 4z}{z^2 + 2z + 4} = \frac{2w_1(z)}{w_1(z) + 2}$$

where $w_1(z) = \frac{1}{2}z^2 + z$. Togetl er with $w_1(z)$ the function $w(z)$ is holomorphic and schlicht in K, satisfies there the d fferential equation

$$\{w, z\} = -\frac{3}{4}\left(\frac{1}{z+1}\right)^2.$$

Because $w_1(0) = 0$, $w_1'(0) = w_1''(0) = 1$, $w(z)$ satisfies the conditions (12). Fruther

$$\frac{w(z)}{z} = \frac{A}{z - z_1} + \frac{\bar{A}}{z - \bar{z}_1},$$

where $z_1 = -1 + i\sqrt{3}$, $A = 1 - (i/\sqrt{3})$ and $\operatorname{Re}\{A/(z - z_1)\} > 0$, $\operatorname{Re}\{\bar{A}/(z - \bar{z}_1)\} > > 0$ for $z \in K$. Thus $w(z)$ satisfies the inequality (13), too. On the other hand the majorant equation

$$y'' + \frac{3}{4}\left(\frac{1}{1 = |t|}\right)^2 y = 0$$

to the equation

$$u'' - \frac{3}{4}\left(\frac{1}{z+1}\right)^2 u = 0$$

is characterized by the fact that its solutions have infinitely many zeros as it can be seen from their form $y(t) = \sqrt{(1-t)}\,\{c_1 \cos \frac{1}{2} \log (1-t) + c_2 \sin \frac{1}{2} \log (1-t)\}$ for $0 < t < 1$.

If the assumption of Theorem 1 is strenghtened, then the solution $w(z)$, mentioned in this theorem will satisfy the inequality $\operatorname{Re}\{w'(z)\} > 0$, $z \in K$.

Theorem 2. *If each non-trivial solution of the equation* (11) *has at most one zero-point in* $(-1, 1)$ *and if for the solution* $y_2(t)$ *of this equation, which is given by the initial conditions* $y_2(0) = 1$, $y_2'(0) = 0$, *the inequality*

$$(16) \qquad \int_0^t \left[\int_\tau^t \frac{1}{y_2^2(p)}\, dp\right] y_2^2(\tau)\, M(\tau)\, d\tau < \frac{\pi}{4}, \quad 0 \leqq t < 1$$

holds, then the solution $w(z)$ *of the equation* (2) *satisfying the conditions* (12) *enjoys the property*

$$\operatorname{Re}\{w'(z)\} > 0\,, \quad z \in K\,. \tag{17}$$

Proof. As it was shown, $u_2(z) \neq 0$ for $z \in K$. For an arbitrary but fixed $\alpha \in \in \langle 0, 2\pi)$ set $u_2(e^{i\alpha}t) = \varrho_\alpha(t)\, e^{i\sigma_\alpha(t)}$, $\varrho_\alpha(t)$, $\sigma_\alpha(t)$ are real and continuous in $\langle 0, 1)$. From $u_2(0) = 1$, $u_2'(0) = 0$ on basis of (8) we can write $\varrho_\alpha(0) = 1$, $\varrho_\alpha'(0) = 0$, $\sigma_\alpha(0) = = 0$, $\sigma_\alpha'(0) = 0$. By Corollary to Lemma 3 then $\sigma_\alpha(t) = -\int_0^t \left[\int_\tau^t (1/\varrho_\alpha^2(p))\, dp\right] . \, . \, \varrho_\alpha^2(\tau) \operatorname{Im}\{q(e^{i\alpha}\tau)\, e^{2i\alpha}\}\, d\tau$, $0 \leqq t < 1$. On the other hand, from (15) it follows that $w'(z) = 1/u_2^2(z)$ and hence, the inequality (17) will be satisfied if and only if $\left|\int_0^t \left[\int_\tau^t (1/\varrho_\alpha^2(p))\, dp\right] \varrho_\alpha^2(\tau) \operatorname{Im}\{q(e^{i\alpha}\tau)\, e^{2i\alpha}\}\, d\tau\right| < \pi/4$ for each $\alpha \in \langle 0, 2\pi)$. By Lemma 2, the left-hand side of this inequality is majorized by the left-hand side of the inequality (16).

Corollary. *If the coefficient* $M(t)$ *of the equation* (11) *satisfies the inequality* $(0 \leqq)\ M(t) \leqq 1$, $0 \leqq t < 1$, *then the solution* $w(z)$ *of the equation* (2) *satisfying the conditions* (12) *shows the property* (17).

Proof. By Lemma 2, the left-hand side of the inequality (16) is majorized by $\int_0^t \left[\int_\tau^t (1/\cos^2 p)\, \mathrm{dp}\right] \cos^2 \tau\, d\tau = \frac{1}{2} t \operatorname{tg} t$. This implies Corollary because $\frac{1}{2} \operatorname{tg} 1 < < 0{,}780 < 0{,}785 < \pi/4$.

The relation of the equation (11) to the equation

$$\{X, t\} = M(t)\,, \quad -1 < t < 1\,, \tag{18}$$

is completely similar to that of the equation (1) to the equation (2). Between the solutions of the equations (2) and (18) satisfying the conditions (12) the following relation is valid which is asserted by

Theorem 3. *If each non-trivial solution of the equation* (11) *has at most one zero in* $(-1, 1)$ *and* $w(z)$ *is the solution of the equation* (2), $X(t)$ *is the solution of the equation* (18) *which both satisfy the conditions* (12) *then*

$$\max_{|z| \leqq t} |w(z)| \leqq X(t) \quad \text{for each} \quad t \in \langle 0, 1)\,.$$

Proof. Let $u_1(z)$, $u_2(z)$ possess the same meaning as in Theorem 1 and let $y_1(t)$, $y_2(t)$ be the solutions of the equation (11) determined at the point 0 by the same initial conditions as $u_1(z)$, $u_2(z)$, respectively. Similarly like in Theorem 2 denote $u_2(e^{i\alpha}t) = \varrho_\alpha(t)\, e^{i\sigma_\alpha(t)}$. Then $X(t) = y_1(t)/y_2(t)$ and $\varrho_\alpha(t)$ satisfies the equation (9), where $\sigma(t) = \sigma_\alpha(t)$, $0 \leqq t < 1$, with $\varrho_\alpha(0) = 1$, $\varrho_\alpha'(0) = 0$. Because $y_2(t)$ satisfies the condition (6), on basis of Lemma 2 $\varrho_\alpha(t) \geqq y_2(t)$, for all $0 \leqq t < 1$ and $\alpha \in \langle 0, 2\pi)$. From this $|w'(e^{i\alpha}t)| = 1/\varrho_\alpha^2(t) \leqq 1/y_2^2(t) = X'(t)$ and hence $|w(z)| = \left|\int_0^z w'(u)\, du\right| \leqq \leqq \int_0^{|z|} X'(t)\, dt = X(|z|)$. This completes the proof of Theorem 3.

Corollary. *If* $X(t)$ *is bounded in* $(-1, 1)$ (*which is equivalent to the fact that the equation* (11) *has more than one solution without zero-points*) *then* $w(z)$ *is bounded, too* (*and the equation* (1) *has infinitely many solutions without zero-points*).

Proof. $X(t)$ being an odd function the interval $I = X((-1, 1))$ is symmetric with respect to the origin. The roots of the equation $X(t) = c$ are identical to the zero-points of the solution $y_1(t) - c\, y_2(t)$ of the equation (11). Thus the equation (11) has more than one solution without zero-points if and only if there exists c such that $c \notin I$.

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