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MUTANTS IN SEMIGROUPS

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1. Introduction. In [5] MULLIN has posed research problems and defined a mutant in a grupoid $(A, *)$ as the following: A subset M of A is called a mutant of $(A, *)$ if $M * M \subseteqq A \setminus M$, where $M * M = (a * b\colon a \in M \text{ and } b \in M)$ and $A \setminus M$ is the set of all elements of A not in M. DOYLE and WARNE have defined an antigroupoid of a groupoid as a mutant of a groupoid in [2]. ISEKI (4) has made a definition of a mutant in a semigroup as the following: A subset A of a semigroup S is an (m, n) mutant of S if and only if $A^m \subset S \setminus A^n$. Iseki has established a theorem which states that if A and B are (m, n) mutants in semigroups S and T, respectively, then $A \times B$ is an (m, n) mutant of $S \times T$. KOCH and WALLACE have proved the existence of a maximal ideal in a compact semigroup in [6]. In this paper, we shall follow a definition of a mutant by Mullin [5] and refering to [6], we shall prove, among others, that in a topological semigroup S, for any non-idempotent a in S, there exists a maximal open mutant containing a.

2. TOPOLOGICAL SEMIGROUPS

A topological semigroup [3, 1.2] is an ordered triple consisting of a non-empty set S, a function $(x, y) \to xy$ from $S \times S$ into S, and a Hausdorff topology on S such that

(a) $x(yz) = (xy) z$ for all x, y, z in S,

(b) $(x, y) \to xy$ is continuous. In addition, if S is a compact space, then S will be called a compact semigroup.

Definition. Let M be a subset of a semigroup S. M is a mutant of S if and only if $MM \subseteqq S \setminus M$.

It is clear that a mutant M of a semigroup S does not contain any idempotent of S.

If $T \subseteqq S$, define $E(T) = (e \in T\colon e^2 = e)$. $A \setminus B = (a \in A\colon a \notin B)$.

Lemma. *Let S be a topological semigroup. If $E(S) \neq S$ and if $a \in S \setminus E(S)$, then there exists an open mutant $M(a)$ of S containing a.*

Proof. Let $a \in S \setminus E(S)$ and let $aa = b \neq a$. Let $V_1(b)$ be an open set containing b. Then there exists an open neighborhood $U_1(a)$ of a such that $U_1(a) U_1(a) \subset V_1(b)$. Since S is a Hausdorff space, for $a \neq b$, there exist two neighbourhoods $U_2(a)$ and $V_2(b)$ of a and b, respectively, such that $U_2(a) \cap V_2(b) = \emptyset$, the empty set. Let $V_1(b) \cap V_2(b) = V_3(b)$ and let $U_1(a) \cap U_2(a) = U_3(a)$. For $V_3(b)$, there exists a neighborhood $U_4(a)$ of a such that $U_4(a) U_4(a) \subset V_3(b)$. Letting $U_4(a) \cap U_3(a) = U_5(a)$, we have that $U_5(a) U_5(a) \subset U_4(a) U_4(a) \subset V_3(b)$. We claim that $U_5(a) \cap V_3(b) = \emptyset$. If $z \in (U_5(a) \cap V_3(b)) \neq \emptyset$, then $z \in V_2(b)$ and $z \in U_3(a) \subset U_2(a)$. Hence we have that $z \in (U_2(a) \cap V_2(b))$, which is a contradiction. Consequently, we have shown that $U_5(a)$ is an open mutant containing a.

Theorem 1. *Let S be a topological semigroup with $S \neq E(S)$. For each $a \in S \setminus E(S)$, there exists a maximal open mutant $M(a)$ of S containing a.*

Proof. Let F be the family of all open mutants of S containing a. By the above lemma, F is non-empty. F is partially ordered by inclusion. Applying Hausdorff maximal principle, there exists a maximal chain F_0. Then $M = \bigcup(M(a)\colon M(a) \in F_0)$ is a maximal open mutant containing a. To show this, consider MM. Assume, by way of contradiction, that $MM \cap M \neq \emptyset$. Let x and y be two elements of M such that $xy \in M$. Then there exist M_1, M_2, and M_3 in F_0 such that $x \in M_1$, $y \in M_2$, and $xy \in$ $\in M_3$. Since F_0 is a chain, either $M_1 \supseteq M_2$ or $M_2 \subseteq M_1$. Without loss of generality, we can assume that $M_1 \subseteq M_2$. From $M_2 M_2 \cap M_2 = \emptyset$, and $x, y \in M_2$, it follows that $xy \notin M_2$ by the definition of a mutant M_2. Again, since F_0 is a chain, either $M_3 \supseteq M_2$ or $M_3 \subseteq M_2$. It follows from $xy \in M_3$ and $xy \notin M_2$ that $M_2 \subseteq M_3$. Then M_3 can not be a mutant of S. This contradiction implies that $xy \notin M$ and M is an open mutant of S. Finally, let N be an open mutant of S containing a such that $N \supset M$. Then $N \in F_0$ and hence $N = M$. This proves the theorem.

Corollary. *Let S be a topological semigroup. If $E(S) \neq \emptyset$, then $E(S)$ is closed.*
The proof of Corollary follows from Theorem 1 and also see [2].

3. ALGEBRAIC SEMIGROUPS

In this section, we shall discuss mutants in a semigroup. If M is a mutant of a semigroup S, then any subset N of M is a mutant of S. In general, a union of two mutants of a semigroup is not a mutant.

Theorem 2. *Let S be a semigroup.*
(i) S has no a decomposition $S = M_1 \cup M_2$ into two disjoint mutants M_1 and M_2 of S.

(ii) S has no a decomposition $S = M_1 \cup M_2 \cup M_3$ into three disjoint mutants M_i $(i = 1, 2, 3)$ of S.

Proof. (i) Let M_1 be a mutant of S and $M_2 = S \setminus M_1$ be a mutant. Let $a \in M_1$. Then $a^2 \in M_2$, and hence $a^4 \in M_1$, $a^5 \in M_2$. We claim that $a^3 \in M_2$. Assume that $a^3 \in M_1$. Then $a^4 \in M_2$, contrary to that $a^4 \in M_1$. Thus $a^3 \in M_2$, and hence $a^5 \in M_1$. This contradiction proves the part (i) of Theorem 2.

(ii) Let $x \in S$. We shall use the following symbol:

```
baba(1, 3, 5,        )
    (2,       8,     )
    (4, 6,           ) (10).
```

This symbol denotes that in case of *baba* (dictionally ordered), a mutant M_1 contains elements x, x^3, and x^5, a mutant M_2 contains x^2, and a mutant M_3 contains x^4 and x^6. Hence M_2 contains x^8. Then there is no any mutant M_i $(i = 1, 2, 3)$ containing x^{10}. If S is finite, then $E(S) \neq \emptyset$. Hence we assume that S is infinite and $E(S = = \emptyset$. We have the following tree of cases (see p. 89).

Each case in the above tree has the following symbol.

```
a(1, 4, )   aa(1, 4,      )   aaa(1, 4, 6, )   aaaa(1, 4, 6,            )
 (2,    )     (2, 3,      )      (2, 3,    )        (2, 3,    10,       )
 (      )     (      5,   )      (5,       )        (5,            7,   ) (12).

aab(1, 4, )   aaba(1, 4,    10,          )       aabb(1, 4,     )
   (2, 3, )       (2, 3, 8,              )           (2, 3,     )
   (5, 6, )       (5, 6,                 ) (11).     (5, 6, 8,  )

aabba(1, 4, 10,     13,        )        aabbb(1, 4,      12,       )
     (2, 3,     11,      12,   )             (2, 3, 10,       11,  )
     (5, 6,  8,                ) (14).       (5, 6,  8,            ) (13).

ab(1, 4, )   aba(1, 4, )   abaa(1, 4, 6,      )
  (2,    )      (2, 5, )       (2, 5,         )
  (3,    )      (3,    )       (3,       7,   ) (10).

abab(1, 4,       11,          )        abb(1, 4, )   abba(1, 4, 6,       )
    (2, 5, 6,                 )           (2,    )       (2,       8,    )
    (3,      8,     12, 10,   ) (7).      (3, 5, )       (3, 5,          ) (10).

abbb(1, 4, )        b(1, )   ba(1, 3, )   baa(1, 3, 8, )
    (2, 6, )         (2, )     (2,    )      (2, 6,    )
    (3, 5, ) (8).    (4, )     (4,    )      (4,       )

baaa(1, 3, 8,      )        baab(1, 3, 8,         )        bab(1, 3, )
    (2, 6, 5,      )            (2, 6,    9,      )           (2,    )
    (4,       7,   ) (11).      (4, 5,       7,   ) (11).     (4, 6, )
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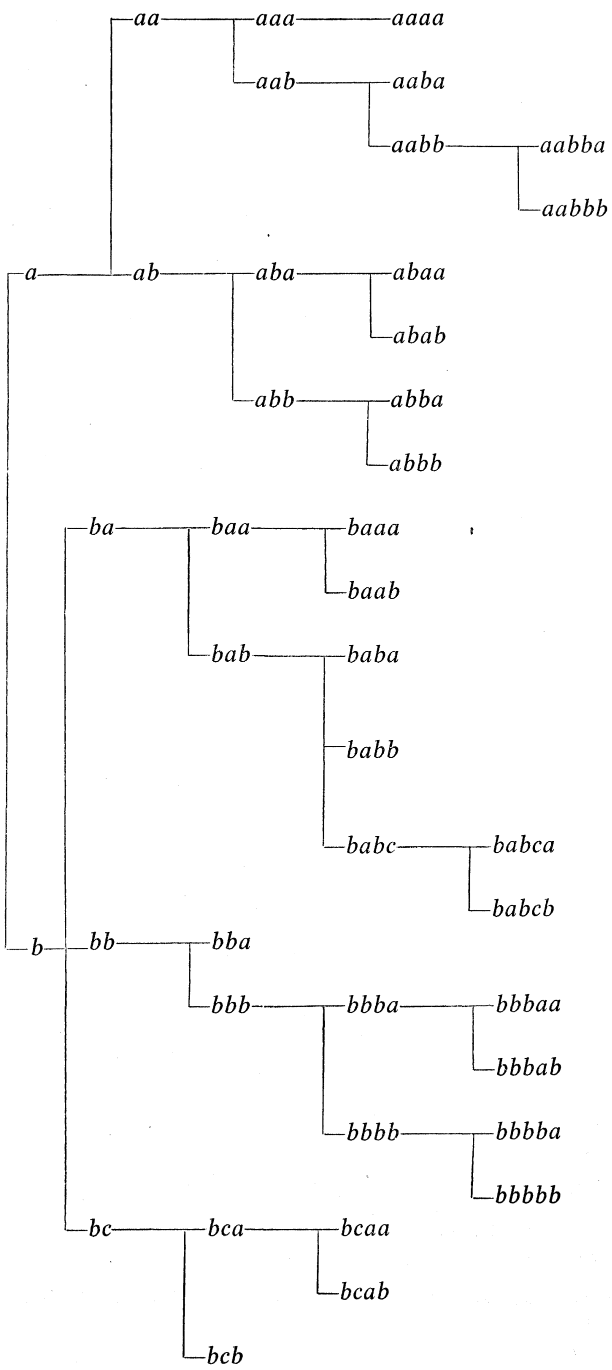
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bbbab
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bbbba
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bc
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bcaa
bcab
bcb

baba(1, 3, 5,) *babb*(1, 3, 10,) *babc*(1, 3,)
(2, 8,) (2, 5, 11,) (2,)
(4, 6,) (10). (4, 6, 7, 9,) (13). (4, 5, 6,)

babca(1, 3, 9,) *babab*(1, 3, 11,) *bb*(1,)
(2, 10,) (2, 9, 10,) (2, 3,)
(4, 5, 6,) (12). (4, 5, 6,) (8). (4,)

bba(1, 5, 8, 12,) *bbb*(1,) *bbba*(1, 6,)
(2, 3, 10,) (2, 3,) (2, 3,)
(4, 6, 7,) (13). (4, 5,) (4, 5,)

bbbaa(1, 6, 10,) *bbbab*(1, 6, 10,) *bbbb*(1,)
(2, 3, 7,) (2, 3, 12,) (2, 3,)
(4, 5,) (9). (4, 5, 7,) (9). (4, 5, 6,)

bbbba(1, 8, 11,) *bbbbb*(1, 10,) *bc*(1,)
(2, 3, 9, 10,) (2, 3, 8,) (2,)
(4, 5, 6, 7,) (12). (4, 5, 6,) (11). (4, 3,)

bca(1, 6,) *bcaa*(1, 6,) *bcab*(1, 6, 9,)
(2,) (2, 5,) (2, 7, 8,)
(3, 4,) (3, 4,) (7). (3, 4, 5,) (10).

bcb(1, 8, 5,)
(2, 6, 7,)
(3, 4, 9,) (13).

This proves the theorem.

Remark. In the above theorem, we can replace a semigroup S by a power associative groupoid.

Conjecture. Any semigroup S has no decomposition $S = \bigcup M_i$ into a finite number of disjoint mutants M_i $(i = 1, 2, \ldots, n)$ of S.

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