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ORDERED SET OF CLASSES OF COMPACTIFICATIONS

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INTRODUCTION

In general topology a considerable attention has been paid to compact spaces. There are mostly studied Hausdorff compact spaces which are normal. Hence, it follows that only completely regular spaces from Hausdorff spaces can be densely imbedded into some Hasudorff compact space. On the other hand, A. N. TICHONOV [11] has shown that any completely regular space Q can be densely imbedded into a Hausdorff compact space. Among these compactifications there exists just one, in a certain sense "the largest" one called *Čech-Stone compactification* of the space Q and denoted $\beta(Q)$, that was dealt with by E. ČECH in [2] and M. H. Stone in [10]. In [12] H. WALLMAN studied for any T_1-space Q a certain compact T_1-space $\omega(Q)$, in which Q is dense. The space $\omega(Q)$ is called *Wallman compactification* of the space Q. The concept of a "T_1-space" can be obtained from the concept of a Hausdorff space by a certain weakening of separating axioms. In this paper there are studied the compactifications of separated closure spaces (see [5], 27 A.1, p. 487), whose axioms can be obtained from axioms for Hausdorff spaces by omitting the axiom: $\overline{\overline{M}} = \overline{M}$.

In Section 1 there are introduced basic concepts and theorems referring to the theory of filters and topology used in the following. In Section 2 there is defined, between compactifications (relative compactifications) P_1, P_2 of a topological space Q, a quasi-ordering $\leqq$ and an equivalence $\sim$ so that $P_1 \sim P_2$ if and only if $P_1 \geqq P_2 \geqq P_1$. The relation $\leqq$ on the set of classes $\mathscr{K}(Q)$ ($\mathscr{R}(Q)$) corresponding to this equivalence is an ordering and it is shown that $\mathscr{K}(Q) \cong \mathscr{R}(Q) \cong \mathfrak{N}(I(Q))$ [1]), where $I(Q)$ is the ordered system of filters $\mathfrak{a}$ on Q such that $\mathfrak{a} \neq \mathfrak{o}$, $\mathfrak{a} \wedge \tau q = \mathfrak{o}$ for any point $q \in Q$, while τq denotes the filter of the neighbourhoods of the point q and $\mathfrak{o}$ denotes the filter containing the empty set. For completely regular compactifications P_1, P_2 of a completely regular space Q this definition of the relation $\leqq$ is equivalent to the usual definition of the relation $\leqq$: $P_1 \leqq P_2$ if and only if there exists a continuous mapping f of P_2 into P_1 such that $f(x) = x$ for $x \in Q$ ([5], 41.D.1).

[1]) The definition and properties of ordered set $\mathfrak{N}(G)$ for an ordered set G are described in [9].

Among these compactifications of a topological space Q an important role is given to the $\mathfrak{h}$-compactification denoted by $\mathfrak{h}(Q)$ and described in Section 3. This compactification is characterized by the property that any continuous mapping of the space Q into some compact space has a χ-extension on $\mathfrak{h}(Q)$. Also, a χ-extension of a continuous mapping f of Q into a space R on a space P is a mapping F of P into R with the properties: 1) $F(x) = f(x)$ for $x \in Q$, 2) for $z \in P$ and for a neighbourhood V of the point $F(z)$ there exists a neighbourhood U of the point z such that $F^1(U - Q) \subseteq \subseteq \overline{V \cap f^1(Q)}$ and $F^1(U \cap Q) = f^1(U \cap Q) \subseteq V \cap f^1(Q)$.

In Section 4 there are studied spaces which have a one-point compactification and it is shown that they are exactly the spaces which are non-compact and in which every point has a neighbourhood relatively compact in the given space. Under a one-point compactification we, naturally, mean a topological space mentioned above (i.e. separated closure space in the sense of [5]). It is possible to construct a one-point compactification, the so called Alexandroff's compactification ([7], § 25), for a non-compact topological space in the usual sense. (For locally compact Hausdorff spaces which are not compact, the construction has been presented in [1]). From 4.2 and 4.4 and e.g. from [7] 25.5 it follows that this Alexandroff's compactification coincides with the one-point compactification introduced in this paper, if and only if the original space is regular. In conclusion of this section, there are given necessary and sufficient conditions for a space Q under which ordered sets $\mathscr{K}(Q)$ and $\mathscr{R}(Q)$ are distributive or modular lattices.

1. FUNDAMENTAL CONCEPTS AND ASSERTIONS FROM THE THEORY OF FILTERS AND TOPOLOGY

In this section, there are introduced fundamental concepts and assertions from the theory of filters and topology, which are used in what follows. Naturally, we do not present proofs when using the assertions mentioned in references. The definition of topology is introduced in the same way as in [7], definition 4a; however, we have a more general concept of topology in mind. Assertion 1.9 is almost evident.

Definition 1.1. A non-mepty system $\mathfrak{a}$ of subsets of a set R will be called a *filter* if the following axioms are fulfilled:

(F_1) $A_1, A_2 \in \mathfrak{a} \Rightarrow A_1 \cap A_2 \in \mathfrak{a}$,
(F_2) $A \in \mathfrak{a},\ A \subseteq B \subseteq R \Rightarrow B \in \mathfrak{a}$.
([7], definition 3a).

Filters will be denoted by small German letters.

Definition 1.2. Let $\mathfrak{a}$ be a filter on a set R. A system of subsets $\mathfrak{A}$ of R will be called a *base of the filter* $\mathfrak{a}$, if $\mathfrak{A} \subseteq \mathfrak{a}$ and for any $A \in \mathfrak{a}$ there exists $A' \in \mathfrak{A}$ such that $A' \subseteq A$ ([7], definition 3b).

Definition 1.3. We put $\mathfrak{a} \leqq \mathfrak{b}$ for filters $\mathfrak{a}, \mathfrak{b}$ on a set R, if $B \in \mathfrak{a}$ for any $B \in \mathfrak{b}$ ([7], definition 3c).

The relation $\leqq$ is an ordering and $F(R)$ will stand for the set of all filters on R ordered by means of this relation. (In some sections of this paper F will have another meaning but misunderstanding is excluded in these cases). The system of all subsets of R forms a filter which is the least element in the set $F(R)$ and this filter will be denoted by $\mathfrak{o}$.

1.1. *Let R be an arbitrary set. Then $F(R)$ is a complete, distributive and strongly atomic lattice*[2]) ([7], 3.5, 3.9 and [8], § 7, Theorem 4).

Atoms of the lattice $F(R)$ will be called *ultrafilters.* A filter (ultrafilter) $\mathfrak{a} \in F(R)$ is called a *free filter* (ultrafilter), if $\bigcap A(A \in \mathfrak{a}) = \emptyset$. Infima and suprema in $F(R)$ will, as usually, be denoted by $\wedge, \bigwedge, \vee, \bigwedge$. If there is no danger of misunderstanding the least element in $F(R)$ and in $F(R')$ will be denoted by the same letter $\mathfrak{o}$ for different sets R and R'. $\mathfrak{m}$ being a cardinal number, exp $\mathfrak{m}$ denotes the cardinal number of the system of all subsets of a set the cardinal number of which is equal to $\mathfrak{m}$.

1.2. *Let $\mathfrak{M}$ be the system of all free ultrafilters on an infinite set M. Then* card $\mathfrak{M} =$ $=$ exp exp card M ([7], Exercise 3D).

1.3. *Let R be an arbitrary set, $\mathfrak{a}_\iota \in F(R)$ for $\iota \in I \neq \emptyset$, $\mathfrak{a}, \mathfrak{b} \in F(R)$. Then the system of all sets of the form*

$$V = \bigcup A_\iota(\iota \in I)\,, \quad A_\iota \in \mathfrak{a}_\iota$$

is a filter $\mathfrak{i} = \bigvee\{\mathfrak{a}_\iota \mid \iota \in I\}$ *and* $\mathfrak{a} \wedge \mathfrak{b} = \mathfrak{o}$ *if and only if* $A \in \mathfrak{a}$, $B \in \mathfrak{b}$ *exist such that* $A \cap B = \emptyset$ (Follows from [7], 3.6).

Definition 1.4. Let φ be a mapping of a set R into a set R'. If $\mathfrak{a} \in F(R)$, then the system of sets $\{\varphi^1(A) \mid A \in \mathfrak{a}\}$ forms a basis of a filter on R' which will be denoted by $\varphi(\mathfrak{a})$. If $\mathfrak{a}' \in F(R')$, then the system of sets $\{\varphi^{-1}(A') \mid A' \in \mathfrak{a}'\}$ forms a basis of a filter on R which will be denoted by $\varphi^{-1}(\mathfrak{a}')$ ([7], § 3, pp. 37).

1.4. *Let φ be a mapping of a set R into R' and let $\mathfrak{a}$ be an ultrafilter on R. Then $\varphi(\mathfrak{a})$ is an ultrafilter on R'* ([7], 3.15).

Definition 1.5. Under the topology on a set R we shall understand a mapping τ of the set R into $F(R)$ with the following properties:

$p \in R,\ U \in \tau p \Rightarrow p \in U,$

$p, q \in R,\ p \neq q \Rightarrow \tau p \wedge \tau q = \mathfrak{o}.$

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[2]) A lattice S will be called *strongly atomic* if for $s_1, s_2 \in S$, $s_1 \neq o \neq s_2 \neq s_1$ there is $\emptyset \neq$ $\neq \{a \mid a \leqq s_1, a \in A\} \neq \{a \mid a \leqq s_2, a \in A\} \neq \emptyset$, where A is the set of all atoms of the lattice S and o is the least element of S.

A pair (R, τ) is called a *topological space* or briefly *space*, elements from R are called *points of the space* (R, τ) and sets from the filter τp for $p \in R$ are called *neighbourhoods of the point p.*

The other fundamental concepts, especially the concept of the closure will be defined in the usual way. In the paper, fundamental theorems introduced in [4] or [5] will be used without quotation. The topological space will sometimes be denoted by the letter R only. The closure of a subset X of the space R will be denoted by $\bar{X}$. E. Čech called the space introduced in definition 1.5 an *AH-space* in [3], an *H-space* in [4] and a *separated closure space* in [5]. If more is valid, i.e. if $p \in R \Rightarrow \bigwedge_{U \in \tau p} \bigvee_{q \in U} \tau q \leqq \tau p$, we get a topological Hausdorff space ([7], definition 8b).

Definition 1.6. Let R be a topological space, $\mathfrak{a} \in F(R)$, $\mathfrak{a} \neq \mathfrak{o}$ and x be an element of R. We shall say that x is a *cluster point* of $\mathfrak{a}$ *in* R, if x belongs to $\bigcap \bar{A} (A \in \mathfrak{a})$. R is said to be *compact* if every filter $\mathfrak{a} \in F(R)$, $\mathfrak{a} \neq \mathfrak{o}$ has a cluster point in R ([5], 41 A.1 and 41 A.3).

1.5. *Every compact set of a topological space is closed in this space* ([5], 41 A.11)

1.6. *Every closed set of a compact space is compact* ([5], 41 A.10).

1.7. *A one-to-one continuous mapping of a compact space on a topological space is a homeomorphism.* ([5], Corrollary of 41 C.5).

Definition 1.7. A system $\mathscr{S}$ of sets of a space (R, τ) is called a *cover* (*interior cover*) of a set $M \subseteqq R$, if for any $p \in M$ there exists $X \in \mathscr{S}$ such that $p \in X (X \in \tau p)$. (Similarly like in [5], 1 E. 12, 12 A.1, 17 A.17 and in [4], 8.1.1).

1.8. *A space is compact iff each interior cover contains a finite cover* ([5], Exercises, Sections 17–18, 5).

Definition 1.8. Let R be a topological space. A set $Q \subseteqq R$ is called *relatively compact in the space* R, if any interior cover of the space R contains a finite cover of the set Q ([4], definition 8.5.2).

1.9. *Let R be a topological space, $Q \subseteqq P \subseteqq R$ and let P be relatively compact in R. Then also Q is relatively compact in the space R. A set-theoretic sum of a finite number of relatively compact sets in R is again a relatively compact set in the space R,* (a similar assertions may be found in [7], the note following definition 11c and assertion 11.8).

Definition 1.9. Let R be a topological space and Q be its subspace. The space R is called a *compactification* (*relative compactification*) of the space Q if R is compact (Q is relatively compact in R) and Q is dense in R.

1.10. *A locally compact Hausdorff space is regular* ([7], 12.8).

Let M be a subset in R and let $\mathfrak{a} \in F(R)$. The system of sets $\{A \cap M \mid A \in \mathfrak{a}\}$ is a filter on M and we are going to denote it by $Sp_M(\mathfrak{a}) = Sp_M\mathfrak{a}$ ([7], § 9).

Let Q be a subset of a space (R, τ). Then, we shall denote $S(R, Q) = S((R, \tau), Q) = \{Sp_Q\tau p \mid p \in R - Q\}$.

Further in this paper I presuppose the knowledges of fundamental concepts and assertions of my paper [9].

2. ORDERED SET OF CLASSES OF COMPACTIFICATIONS

For a topological space (Q, τ) we denote the set $\{\mathfrak{a} \mid \mathfrak{a} \in F(Q),\ \mathfrak{a} \neq \mathfrak{o},\ \mathfrak{a} \wedge \tau q = \mathfrak{o}$ for each point $q \in (Q, \tau)\}$ by $I(Q, \tau) = I(Q) = I$.

2.1. *For any space Q, the set $I(Q) \cup (\mathfrak{o})$ ordered by means of inclusion in such a way that $\mathfrak{o}$ is the least element, is a distributive, strongly atomic lattice*[2]) *complete from below with the least element $\mathfrak{o}$.*

This lattice is a sublattice of the lattice $F(Q)$. Every atom of the lattice $I(Q) \cup (\mathfrak{o})$ is an ultrafilter on Q.

Proof. Let $\mathfrak{a}, \mathfrak{b} \in I$. Then $(\mathfrak{a} \vee \mathfrak{b}) \wedge \tau q = (\mathfrak{a} \wedge \tau q) \vee (\mathfrak{b} \wedge \tau q) = \mathfrak{o}$ for each point $q \in Q$ according to 1.1. Let $\mathscr{K} \neq \emptyset$, $\mathfrak{a}_k \in I$ for $k \in \mathscr{K}$. Then, according to 1.1, $\mathfrak{a} = \bigwedge \mathfrak{a}_k (k \in \mathscr{K})$ exists and, evidently, $\mathfrak{a} \wedge \tau q = \mathfrak{o}$ for each point $q \in Q$. Thus, $I \cup (\mathfrak{o})$ is a sublattice of the lattice $F(Q)$ and it is complete from below. From this it follows, according to 1.1, that it is also distributive. If $\mathfrak{b}$ is an ultrafilter on Q and $\mathfrak{a} \in I$, $\mathfrak{b} \leqq \mathfrak{a}$, then $\mathfrak{b} \in I$. Hence it follows, by 1.1, that $I \cup (\mathfrak{o})$ is a strongly atomic lattice and any atom of this lattice is an ultrafilter on Q.

Thus the assertion is proved.

2.2. *The space (Q, τ) is compact iff $I(Q) = \emptyset$.*

Proof. $I(Q)$ is the set of filters $\mathfrak{a} \in F(Q)$, $\mathfrak{a} \neq \mathfrak{o}$ which fail to have a cluster point in Q. From this the assertion follows.

2.3. *Do not let (Q, τ) be a compact space and let (P, τ') be its relative compactification. Then $S(P, Q) \in \mathfrak{N}(I(Q))$.* (Definition and properties of the operator $\mathfrak{N}$ are described in [9].)

Proof. I. Evidently, $Sp_Q(\tau' p) \in I(Q)$ for all points $p \in P - Q$; consequently $S(P, Q) \subseteqq I(Q)$.

II. Let $p_1, p_2 \in P - Q$, $p_1 \neq p_2$. Then there exist disjoint neighbourhoods V_1, V_2 of points p_1, p_2. From this we get that $Sp_Q(\tau' p_1) \wedge Sp_Q(\tau' p_2) = \mathfrak{o}$. Consequently, $S(P, Q)$ fulfils the axiom $(I\mathfrak{N})$.

III. Let $\mathfrak{a} \in I(Q) - S(P, Q)$, $(\mathfrak{a}) \cup S(P, Q)$ have the property (h). Let us denote $\mathscr{P}$ the system of all sets of the form $P - X$, where $X \in \mathfrak{a}$. $\mathscr{P}$ is evidently the interior cover of the space P; consequently, there exists a finite cover $Y_1, \ldots, Y_n$ of the space Q, where $Y_\nu \in \mathscr{P}$ for $1 \leqq \nu \leqq n$. Then for $\nu = 1, 2, \ldots, n$ we have $P - Y_\nu \in \mathfrak{a}$, thus $A = \bigcap_{\nu=1}^{n} (P - Y_\nu) \in \mathfrak{a}$. Since $Q \subseteqq \bigcup_{\nu=1}^{n} Y_\nu = P - \bigcap_{\nu=1}^{n} (P - Y_\nu) = P - A$, we have $A \cap \cap Q = \emptyset$ which is a contradiction. Consequently, $S(P, Q)$ fulfils the axiom (II $\mathfrak{N}$).

2.4. *Let (Q, τ) be a dense subspace in a space (P, τ'). Let $S(P, Q) \in \mathfrak{N}(I(Q))$. Then Q is relatively compact in (P, τ').*

Proof. Let $\mathscr{P}$ be an interior cover of the space (P, τ') which contains no finite subcover of Q. Then the system of sets $Q - \bigcup X (X \in \mathscr{K})$, where $\mathscr{K} \subseteqq \mathscr{P}$ and card $\mathscr{K} < \aleph_0$, forms a basis of a filter $\mathfrak{m} \in I(Q)$. According to 2.3 [(9]), $\mathfrak{n} \in S(P, Q)$ and $\mathfrak{g} \in I(Q)$ exist such that $\mathfrak{g} \leqq \mathfrak{m}$, $\mathfrak{g} \leqq \mathfrak{n}$. $X_0 \in \mathscr{P}$ exists such that X_0 is a neighbourhood of the point $p \in P - Q$, where $Sp_Q(\tau p) = \mathfrak{n}$. Then $X_0 \cap Q \in \mathfrak{g}$, $Q - X_0 \in \mathfrak{m}$; consequently, $Q - X_0 \in \mathfrak{g}$ which is a contradiction.

2.5. *Let $N \in \mathfrak{N}(I(Q))$, where $Q = (Q, \tau)$ is a topological space. Then there exists at least one compactification (P, τ') of the space (Q, τ) such that $N = S(P, Q)$.*

Proof. Put $P = N \cup Q$. According to 2.2, [9], $N \neq \emptyset$. Let $\mathfrak{n}_0 \in N$. The system of sets $\{X \mid X = Y \cup Z \cup (\mathfrak{n}_0),\ Y = N - K,\ \text{card}\, K < \aleph_0,\ K \subseteqq N,\ Z \in \mathfrak{n}_0\}$ forms a basis of a filter $\tau'\mathfrak{n}_0 \in F(P)$. The system of sets $\{X \mid X = (\mathfrak{n}) \cup Z,\ Z \in \mathfrak{n}\}$ forms for $\mathfrak{n} \in N - (\mathfrak{n}_0)$ a basis of a filter $\tau'\mathfrak{n} \in F(P)$. For $q \in Q$ the filter $\tau q \in F(Q)$ forms a basis of a filter $\tau' q \in F(P)$. Evidently (P, τ') is a topological space, (Q, τ) is a dense subspace of the space (P, τ') and $S(P, Q) = N$.

Let $\mathfrak{a} \in I(P)$. If $A \cap Q \neq \emptyset$ for each $A \in \mathfrak{a}$, then $Sp_Q \mathfrak{a} \in I(Q)$; consequently, according to 2.3, [9], $\mathfrak{n} \in N$ and $\mathfrak{g} \in I(Q)$ exist such that $\mathfrak{g} \leqq \mathfrak{n}$, $\mathfrak{g} \leqq Sp_Q \mathfrak{a}$. But in this case $\tau'\mathfrak{n} \wedge \mathfrak{a} > \mathfrak{o}$, which is a contradiction to the assumption $\mathfrak{a} \in I(P)$. If $A_0 \in \mathfrak{a}$ exists such that $A_0 \cap Q = \emptyset$, then $Sp_N \mathfrak{a} \in I(N)$. The subspace N is compact, by 1.8; consequently, according to 2.2, we have $I(N) = \emptyset$ which is a contradiction.

Thus $I(P) = \emptyset$ and by 2.2, (P, τ') is a compact space.

Definition 2.1. If P_1 and P_2 are compactifications (relative compactifications) of a topological space Q, let us put $P_1 \lesssim P_2$, if there exists a mapping f of P_2 into P_1 such that $f(x) = x$ for $x \in Q$ with the following properties: if $z \in P_2$ and V is a neighbourhood of $f(z)$, then a neighbourhood U of the point z exists such that $U \cap Q \subseteqq V$.

Remark. For complete regular compactifications P_1, P_2 of a completely regular space Q we usually define $P_1 \lesssim P_2$ iff there exists a continuous mapping f of P_2 into P_1 such that $f(x) = x$ for $x \in Q$ ([5], 41 D.1). This definition of the relation $\lesssim$, for completely regular compactifications of a completely regular space, is equivalent to definition 2.1: Let P_1, P_2 be completely regular compactifications of the completely regular space Q, $P_1 \leqq P_2$ be in the sense of definition 2.1, let f denote a mapping mentioned in definition 2.1, let $z \in P_2$ and V be a closed neighbourhood of the point $f(z)$. Then, there exists an open neighbourhood U of the point z such that $U \cap Q \subseteqq V$. If $z' \in U$ exists such that $f(z') \notin V$, then a neighbourhood W of the point $f(z')$ exists such that $W \cap V = \emptyset$. Then a neighbourhood U' of the point z' exists such that $U' \cap Q \subseteqq W$. Hence, it follows that $U \cap U' \cap Q = \emptyset$ which is a contradiction, because $U \cap U'$ is a neighbourhood of the point z'. Thus, $f^1(U) \subseteqq V$ from which it follows that f is a continuous mapping.

2.6. *The relation* $\lesssim$ *is a quasiordering and for the mapping* f *occurring in definition* 2.1 *we have* $f^1(P_2 - Q) = P_1 - Q$, *where* P_1, P_2 *and* Q *have the same meaning as in definition* 2.1.

Proof. The assertion that the relation $\lesssim$ is reflexive and transitive is evident. Let (P_1, τ_1'), (P_2, τ_2') be compactifications (relative compactifications) of a non-compact space Q and let $P_1 \lesssim P_2$. Evidently, $f^1(P_2 - Q) \subseteqq P_1 - Q$. For $p \in P_2 - Q$ we have $Sp_Q(\tau_2' p) \leqq Sp_Q(\tau_1' f(p))$. According to 2.3 we have $S(P_1, Q)$, $S(P_2, Q) \in \in \mathfrak{N}(I(Q))$. Consequently, from [9], 2.6 if follows that $\bigcup\{Sp_Q(\tau_1' f(p))\}$ $(p \in P_2 - Q) = S(P_1, Q)$. Thus $f^1(P_2 - Q) = P_1 - Q$.

If Q is a compact space, then the assertion follows from 1.5.

Definition 2.2. Let us put $P_1 \sim P_2$ for compactifications (relative compactifications) P_1, P_2 of a topological space Q, if $P_1 \lesssim P_2$ and $P_2 \lesssim P_1$. The relation $\sim$ is an equivalence and the relation $\lesssim$ for classes of this equivalence is an ordering. Let us denote $\mathscr{K}(Q)$ $(\mathscr{R}(Q))$ the set of all classes of this equivalence ordered by means of the relation $\lesssim$.

2.7. *Let* Q *be a non-compact space. Then* $\mathscr{K}(Q) \cong \mathscr{R}(Q) \cong \mathfrak{N} I(Q)$.

Proof. For compactifications (relative compactifications) P_1, P_2 of a space Q we have, according to 2.3, $S(P_1, Q)$, $S(P_2, Q) \in \mathfrak{N}(I(Q))$ and, according to 2.6, $P_1 \lesssim P_2$ exactly if $S(P_1, Q) \lesssim S(P_2, Q)$. From this and from 2.5 the introduced assertion follows.

Remark. If Q is a compact space and P its compactification (relative compactification), then, according to 1.5, $P = Q$; consequently, $\mathscr{K}(Q) = \mathscr{R}(Q) = \{Q\}$.

3. $\mathfrak{h}$-COMPACTIFICATION

Definition 3.1. Let Q be a topological space embedded into a topological space P. Let f be a continuous mapping of Q into a space R. The mapping F of the space P into R is a *χ-extension of the mapping f on P* if F has the following properties:

1) $F(x) = f(x)$ for $x \in Q$,

2) for $z \in P$ and for a neighbourhood V of the point $F(z)$ there exists a neighbourhood U of the point z such that $F^1(U - Q) \subseteqq \overline{V \cap f^1(Q)}$ and $F^1(U \cap Q) = f^1(U \cap \cap Q) \subseteqq V \cap f^1(Q)$.[3])

3.1. *Let Q be a dense subspace in a space P and let f be a continuous mapping of Q into a space R. Then there exists at most one χ-extension of the mapping f on P.*

Proof. Let F, G be two different χ-extensions of f on P. Then $z \in P$ exists such that $F(z) \neq G(z)$. Then there exist disjoint neighbourhoods V_F, V_G of points $F(z)$ and $G(z)$. Neighbourhoods U_F and U_G of the point z exist such that $F^1(U_F \cap Q) \subseteqq V_F$, $G^1(U_G \cap \cap Q) \subseteqq V_G$. The set $U = U_F \cap U_G$ is a neighbourhood of the point z and $\emptyset \neq \neq f^1(U \cap Q) = F^1(U \cap Q) \subseteqq V_F$, $\emptyset \neq f^1(U \cap Q) = G^1(U \cap Q) \subseteqq V_G$ which is a contradiction.

Definition 3.2. Let (Q, τ) be a topological space and $N \in \mathfrak{N}(I(Q))$. Put $P(N) = = N \cup Q$ and $\varphi_N(X) = \{\mathfrak{a} \mid \mathfrak{a} \in N,\ X \in \mathfrak{a}\} \cup X$ for $X \subseteqq Q$. Denote $\sigma\mathfrak{a} = \mathfrak{a}$ for $\mathfrak{a} \in I(Q)$ and $\sigma q = \tau q$ for $q \in Q$. Since $\varphi_N(X \cap Y) = \varphi_N(X) \cap \varphi_N(Y)$ for $X \subseteqq Q$, $Y \subseteqq Q$, the system of sets $\{\varphi_N(U) \mid U \in \sigma p\}$ forms a basis of a filter $\tau_N p \in F(P(N))$ for any element $p \in P(N)$. The pair $(P(N), \tau_N)$ will, sometimes, be denoted concisely by $P(N)$.

3.2. *Let (Q, τ) be a topological space and let $N \in \mathfrak{N}(I(Q))$. Then $P(N) = (P(N), \tau_N)$ is a relative compactification of the space (Q, τ).*

Proof. The pair $(P(N), \tau_N)$ is obviously a topological space in which (Q, τ) is dense and $S(P(N), Q) = N \in \mathfrak{N}(I(Q))$. According to 2.4, $P(N)$ is a relative compactification of the space (Q, τ).

3.3. *Let f be a continous mapping of a space (Q, τ) into a space (R, τ^+). Then $M = \{f^{-1}(Sp_{f^1(Q)}\tau^+ r) \mid r \in \overline{f^1(Q)} - f^1(Q)\} \subseteqq I(Q)$ and M has the property (h) in $I(Q)$.*

Let $N \in \mathfrak{N}(I(Q))$, and let for any $\mathfrak{m} \in M$ at least one filter $\mathfrak{n} \in N$ exist such that $\mathfrak{n} \leqq \mathfrak{m}$. Let $M' = \{\mathfrak{m}' \mid \mathfrak{m}' \in N$, there exists $\mathfrak{m} \in M$ such that $\mathfrak{m} \geqq \mathfrak{m}'\}$. Then the following assertions are equivalent:

(A) *There exists the χ-extension F of the mapping f on the space $P(N)$ such that $F^1(M') = \overline{f^1(Q)} - f^1(Q)$, $F^1(N - M') \subseteqq f^1(Q)$.*

(B) *For any filter $\mathfrak{n} \in N - M'$ there exists $q' \in f^1(Q)$ such that $f(\mathfrak{n}) \leqq \tau^+ q'$.*

[3]) The mapping f of a space X into a space Y is called *θ-continuous*, if for any point $x \in X$ and any neighbourhood V of the point $f(x)$ there exists a neighbourhood U of the point x such that $f^1(U) \subseteq \overline{V}$. This concept was introduced by *S.* Fomin in [6]; he considered, however, topological spaces in the common sense (i.e. *AU*-spaces, according to [3], or general *F*-spaces according to [4] or topological spaces in the sense of [5]). Evidently a χ-extension of a continuous mapping is a θ-continuous mapping.

Proof. I. Let $r_1, r_2 \in \overline{f^1(Q)}$, $r_1 \neq r_2$. Then $Sp_{f^1(Q)}\tau^+ r_1 \wedge Sp_{f^1(Q)}\tau^+ r_2 = \mathfrak{o}$, thus $f^{-1}(Sp_{f^1(Q)}\tau^+ r_1) \wedge f^{-1}(Sp_{f^1(Q)}\tau^+ r_2) = \mathfrak{o}$. For $r_1 \in f^1(Q)$ we have $f^{-1}(Sp_{f^1(Q)}\tau^+ r_1) \geqq \tau q_1$, where $q_1 \in Q$, $f(q_1) = r_1$. For $r_2 \in \overline{f^1(Q)} - f^1(Q)$ we have then $f^{-1}(Sp_{f^1(Q)}\tau^+ r_2) \wedge \tau q = \mathfrak{o}$ for any $q \in Q$. Since $f^{-1}(Sp_{f^1(Q)}\tau^+ r_2) \neq \mathfrak{o}$, we have also $M \subseteqq I(Q)$ and M has the property (h) in $I(Q)$.

II. Let $N \in \mathfrak{N}(I(Q))$ and let N have the introduced property.

a) Let the assertion (A) hold and let $\mathfrak{n} \in N - M'$. Let us put $q' = F(\mathfrak{n})$. Then $q' \in f^1(Q)$. Let $V \in \tau^+ q'$. Then, there exists a neighbourhood U of the point $\mathfrak{n} \in P(N)$ such that $f^1(U \cap Q) \subseteqq V$. Since $U \cap Q \in \mathfrak{n}$, we have $f^1(U \cap Q) \in f(\mathfrak{n})$; thus $V \in f(\mathfrak{n})$ which means that $f(\mathfrak{n}) \leqq \tau^+ q'$.

b) Let the statement (B) hold. For $\mathfrak{m}' \in M'$ exactly one filter $\mathfrak{m} \in M$ exists such that $\mathfrak{m} \geqq \mathfrak{m}'$. Furthermore, exactly one element $r \in \overline{f^1(Q)} - f^1(Q)$ exists such that $f^{-1}(Sp_{f^1(Q)}\tau^+ r) = \mathfrak{m}$. Let us put $F(\mathfrak{m}') = r$. Put $F(q) = f(q)$ for $q \in Q$. For $\mathfrak{n} \in N - M'$, $q' \in f^1(Q)$ exists such that $f(\mathfrak{n}) \leqq \tau^+ q'$. Let us put $F(\mathfrak{n}) = q'$. (Since for $q'' \in f^1(Q) - (q')$ we have $\tau^+ q' \wedge \tau^+ q'' = \mathfrak{o}$, there exists exactly one point q').

Evidently, $F^1(M') \subseteqq \overline{f^1(Q)} - f^1(Q)$ and $F^1(N - M') \subseteqq f^1(Q)$. (From the assumption on the set N it follows that $F^1(M') = \overline{f^1(Q)} - f^1(Q)$). Let $z \in P(N)$ and V be a neighbourhood of the point $F(z)$. Denote $X = f^{-1}(V \cap f^1(Q))$. If $z \in Q$, then $X \in \tau z$. If $z \in M'$, then $X \in \mathfrak{m}$, where $\mathfrak{m} \in M$ and $\mathfrak{m} \geqq z$. Thus $X \in z$. If $z \in N - M'$, then $V \cap f^1(Q) \in f(z)$ (f is now considered as a mapping of filters), thus $f^{-1}(V \cap f^1(Q)) = X \in z$. Consequently $X \in \sigma z$. (σ has the same meaning as in definition 3.2). The set $U = \varphi_N(X)$ is a neighbourhood of the point z and $F^1(U \cap Q) = f^1(U \cap Q) = f^1(X) = V \cap f^1(Q)$. Let $z' \in U - Q$, and let V' be a neighbourhood of the point $F(z')$. Then $X \in z'$ and $V' \cap f^1(Q) \in Sp_{f^1(Q)}\tau^+ F(z')$. Thus $f^{-1}(V' \cap f^1(Q)) \in z'$; hence, it follows that $f^{-1}(V' \cap f^1(Q)) \cap X \neq \emptyset$, consequently $V' \cap f^1(Q) \cap V = V' \cap f^1(Q) \cap f^1(X) \neq \emptyset$. Thus $F(z') \in \overline{V \cap f^1(Q)}$. The mapping F is therefore the χ-extension of the mapping f on the space $P(N)$.

Thus, the assertion is proved.

3.4. *Let (P, τ') be a relative compactification of a non-compact space (Q, τ) and let $N \in \mathfrak{N}(I(Q))$, $N \geqq S(P, Q)$ ($S(P, Q) \in \mathfrak{N}(I(Q))$ according to 2.3). Then there exists a mapping f of the space $P(N)$ on P with the following properties:*

a) *$f(x) = x$ for $x \in Q$,*

b) *$f^1(P(N) - Q) = P - Q$,*

c) *for $x \in P(N)$ and a neighbourhood V of the point $f(x)$ there exists a neighbourhood U of the point x such that $f^1(U - Q) \subseteqq \overline{V \cap Q}$ and $f^1(U) \cap Q \subseteqq V \cap Q$.*

Proof. The assertion follows from 3.3 if we put $(P, \tau') = (R, \tau^+)$ and $f(x) = x$. Then we have $M = S(P, Q)$ and by 2.5, [9] the mentioned condition for the set N is valid. The set M' is equal to the set N.

Definition 3.3. A space (P, τ') is called an $\mathfrak{h}$-*compactification of a space* (Q, τ), if (P, τ') is a compactification of the space (Q, τ) and if to any continuous mapping of the space Q into some compact space there exists a χ-extension on P.

According to 3.5 any topological space possesses an $\mathfrak{h}$-*compactification* and, according to 3.6, it is "in the essential" one-to-one defined; we are going to denote it by $\mathfrak{h}(Q) = \mathfrak{h}(Q, \tau)$.

3.5. *For any space Q there exists an $\mathfrak{h}$-compactification. If Q is a non-compact space and N_0 the set of all minimal elements of the set $I(Q)$, then $N_0 \in \mathfrak{N}(I(Q))$, N_0 is the largest element of the system $\mathfrak{N}(I(Q))$ and the space $P(N_0)$ is an $\mathfrak{h}$-compactification of the space Q.*

Proof. I. If Q is a compact space, then the assertion follows from the remark following 2.7.

Suppose that Q is a non-compact space.

II. From 2.1 and [9], 2.8 it follows that $N_0 \in \mathfrak{N}(I(Q))$ and that N_0 is the largest element of the system $\mathfrak{N}(I(Q))$.

III. From II and 3.2 it follows that the space $P(N_0)$ is a relative compactification of the space Q. Let $\mathscr{P}$ be an interior cover of the space $P(N_0)$ and let us suppose that for any point $p \in P(N_0)$ there exists $U \in \sigma p$ such that $\varphi_{N_0}(U) \in \mathscr{P}$ (σ and φ_{N_0} have the same meaning as in definition 3.2). Thus there exist points $p_1, \ldots, p_n \in P(N_0)$ and sets $U_1 \in \sigma p_1, \ldots, U_n \in \sigma p_n$ such that $\varphi_{N_0}(U_i) \in \mathscr{P}$ for $1 \leqq i \leqq n$ and $\bigcup_{i=1}^{n} U_i = Q$. For $\mathfrak{n} \in N_0$, $i_0 (1 \leqq i_0 \leqq n)$ exists such that $U \cap U_{i_0} \neq \emptyset$ for any $U \in \mathfrak{n}$. The system of sets $\{U \cap U_{i_0} \mid U \in \mathfrak{n}\}$ forms the basis of the filter $\mathfrak{n}'$ on the set Q. Obviously, $\mathfrak{o} < \mathfrak{n}' \leqq \mathfrak{n}$. Since $\mathfrak{n}$ is, according to 2.1, an ultrafilter on Q, we have $\mathfrak{n}' = \mathfrak{n}$; thus $U_{i_0} \in \mathfrak{n}$, from whence it follows that $\mathfrak{n} \in \varphi_{N_0}(U_{i_0})$. Consequently $N_0 \subseteqq \bigcup_{i=1}^{n} \varphi_{N_0}(U_i)$. Therefore $P(N_0) = \bigcup_{i=1}^{n} \varphi_{N_0}(U_i)$; thus $P(N_0)$ is, according to 1.8, a compactification of the space Q.

IV. Let f be a continuous mapping of the space Q into a compact space (R, τ^+). Let us put $M = \{f^{-1}(Sp_{f^1(Q)}\tau^+ r) \mid r \in \overline{f^1(Q)} - f^1(Q)\}$, $M' = \{\mathfrak{m}' \mid \mathfrak{m}' \in N_0, \mathfrak{m} \in M$ exists such that $\mathfrak{m} \geqq \mathfrak{m}'\}$. Let $\mathfrak{n} \in N_0 - M'$. Let us admit that $f(\mathfrak{n}) \wedge \tau^+ q' = \mathfrak{o}$ for any point $q' \in f^1(Q)$. Put $\mathscr{P} = \{X \mid X = R - X^*, X^* \in f(\mathfrak{n})\}$. For $q' \in f^1(Q)$ there exists $V \in \tau^+ q'$, $X^* \in f(\mathfrak{n})$ such that $V \cap X^* = \emptyset$. Therefore $X = R - X^* \in \mathscr{P} \cap \cap \tau^+ q'$. For $r \in \overline{f^1(Q)} - f^1(Q)$ we have $f^{-1}(Sp_{f^1(Q)}\tau^+ r) \wedge \mathfrak{n} = \mathfrak{o}$; so there exist disjoint sets $X' \in f^{-1}(Sp_{f^1(Q)}\tau^+ r)$, $X'' \in \mathfrak{n}$. Next, $f^1(X') \in Sp_{f^1(Q)}\tau^+ r$, $f^1(X'') \in f(\mathfrak{n})$ and $f^1(X') \cap f^1(X'') = \emptyset$. Consequently $X = R - f^1(X'') \in \mathscr{P} \cap \tau^+ r$. If $r \in R - - \overline{f^1(Q)}$, then $X = R - f^1(Q) \in \mathscr{P} \cap \tau^+ r$. Thus, $\mathscr{P}$ is an interior cover of the space R. There exists, according to 1.8, a finite cover $\mathscr{K} = \{X_1, \ldots, X_n\} \subseteqq \mathscr{P}$ of the space $f^1(Q)$ (see proof 3.7). For $1 \leqq i \leqq n$ we have $R - X_i \in f(\mathfrak{n})$ and since $\bigcup_{i=1}^{n} X \supseteq$ $\supseteq f^1(Q)$, we have $\bigcup_{i=1}^{n} (R - X_i) \subseteq R - f^1(Q)$ which is a contradiction.

For this reason $q' \in f^1(Q)$ exists such that $f(\mathfrak{n}) \wedge \tau^+ q' > \mathfrak{o}$. If there exists $V \in$ $\in \tau^+ q' - f(\mathfrak{n})$, then the system of sets $\{X \cap V \mid X \in f(\mathfrak{n})\}$ forms the basis of the filter $\mathfrak{n}'$ on the set R. We have $\mathfrak{o} < \mathfrak{n}' < f(\mathfrak{n})$ which is a contradiction because, according to 1.4, $f(\mathfrak{n})$ is an ultrafilter on R.

Thus $f(\mathfrak{n}) \leqq \tau^+ q'$ and according to 3.3, there exists a χ-extension of the mapping f on the space $P(N_0)$. Thus, the assertion is proved.

3.6. *Let P_1, P_2 be $\mathfrak{h}$-compactifications of a space Q. Then there exists a homeomorphism h of the space P_1 on the space P_2 such that for $x \in Q$ we have $h(x) = x$.*

Proof. If Q is a compact space, then the assertion follows from the remark following 2.7.

Let us suppose, in the following part of the proof, that Q is a non-compact space and that (P, τ') is an $\mathfrak{h}$-compactification of Q. Let N_0 have the same meaning as in 3.5. According to 3.5, the space $P(N_0)$ is also an $\mathfrak{h}$-compactification of Q. Let us put $f(x) =$ $= x$ for $x \in Q$. Then f is a continuous mapping of the space Q into the space $P(N_0)$. Thus, there exists a χ-extension h of the mapping f on P.

h is a one-to-one mapping. Actually, if there exist different points $p_1, p_2 \in P$ such that $h(p_1) = h(p_2) = p \in P(N_0)$, then $Sp_Q\tau p_1 \leqq \sigma p$ (σ and in what follows φ_{N_0} as well, have the same meaning as in definition 3.2), $Sp_Q\tau' p_2 \leqq \sigma p$ and $Sp_Q\tau' p_1 \wedge$ $\wedge\, Sp_Q\tau' p_2 = \mathfrak{o}$. If $p \in N_0$, then p is an ultrafilter and consequently $p = \sigma p =$ $= Sp_Q\tau' p_1 = Sp_Q\tau' p_2$ which is a contradiction. If $p \in Q$, then for example $p_2 \in P -$ $- Q$, thus $Sp_Q\tau' p_2 \wedge \sigma p = \mathfrak{o}$, which is a contradiction, too.

Let $z \in P$, $y = h(z)$ and let W be a neighbourhood of y in $P(N_0)$. Then there exists $U \in \sigma y$ such that $V = \varphi_{N_0}(U) \subseteq W$. Since V is a neighbourhood of the point y, a neighbourhood U' of the point z exists such that $h^1(U' \cap Q) \subseteq V \cap h^1(Q)$ and $h^1(U' - Q) \subseteq \overline{V \cap h^1(Q)} = \overline{U}$. If $\mathfrak{n} \in N_0 \cap \overline{U}$, then for $X \in \mathfrak{n}$ we have $X \cap U \neq \emptyset$ and thus, since $\mathfrak{n}$ is an ultrafilter on Q, $U \in \mathfrak{n}$, and consequently $\mathfrak{n} \in V$. Thus $N_0 \cap$ $\cap\, \overline{U} \subseteq V$ from whence we get (since h is one-to-one), $h^1(U') \subseteq V \subseteq W$. Thus, h is a one-to-one continuous mapping of the space P into the space $P(N_0)$. From 1.7 and 1.5 it follows that h is a homeomorphism P on $P(N_0)$. Hence, the statement is proved.

3.7. *To any continuous mapping f of a space Q into a space R in which the set $f(Q)$ is relatively compact there exists a χ-extension on the space $\mathfrak{h}(Q)$.*

Proof. The statement has been already proved in Section IV in the proof 3.5, because we made use only of the assumption that $f^1(Q)$ is relatively compact in the space R.

Definition 3.4. A topological space P is called a *D-space* if it is Hausdorff and if for any dense set H in P the set $P - H$ is relatively compact in P.

3.8. *Let Q be a topological space. Then the following statements are equivalent:*

(A) $\mathfrak{h}(Q)$ *is a Hausdorff space,*

(B) Q *is a completely regular space and* $\mathfrak{h}(Q) = \beta(Q)$ [4]*,*

(C) Q *is a D-space.*

Proof. I. Let (A) hold. Then to any bounded continuous function[5] on Q there exists a χ-extension on $\mathfrak{h}(Q)$. Since the set of real numbers in the natural topology is a regular space, then this χ-extension is a continuous extension. Thus $\mathfrak{h}(Q) = \beta(Q)$. Consequently (B) holds.

II. Let (B) hold and let a dense set H exist in Q such that $Q - H$ is not relatively compact in Q. Then an interior cover $\mathscr{P}$ of the space Q exists such that for any finite system $\mathscr{K} \subseteqq \mathscr{P}$ we have $\bigcup X(X \in \mathscr{K})$ non $\supseteq Q - H$. The system of sets $\{Y \mid Y = Q - H - \bigcup X(X \in \mathscr{K}), \mathscr{K} \subseteqq \mathscr{P}, \text{card } \mathscr{K} < \aleph_0\}$ forms a basis of a filter $\mathfrak{a} \in F(Q)$. Evidently $\mathfrak{a} \in I(Q)$. Thus an ultrafilter $\mathfrak{b} \in I(Q)$, $\mathfrak{b} \leqq \mathfrak{a}$ exists. Then $Q - H \in \mathfrak{b}$. If $\emptyset \neq M \subseteqq Q - H$, then $Q - M \supseteq H$; consequently $\overline{Q - M} = Q \neq Q - M$. Therefore M is not an open set so that $\mathfrak{b}$ fails to be an open filter[6]; thus, according to 3.5, $\mathfrak{h}(Q)$ is not a Hausdorff space.

Therefore (B) $\rightarrow$ (C).

III. Let (C) hold and let $\mathfrak{h}(Q)$ fail to be a Hausdorff space. Then, according to 3.5, an ultrafilter $\mathfrak{b} \in I(Q)$ exists such that $\mathfrak{b}$ is not open[6]. $B \in \mathfrak{b}$ exists such that for $Y \in \mathfrak{b}$, $Y \subseteqq B$, Y is not open. Then $X = \overline{Q - B} - (Q - B) \neq \emptyset$. Since any point $x \in X$ is a cluster point of the set $Q - B \subseteqq Q - X$, the set $Q - X$ is dense in Q.

If $Y \in \mathfrak{b}$ exists such that $Y \cap X = \emptyset$, then $Z = Y \cap B \in \mathfrak{b}$ and $Z \cap X = \emptyset$. Consequently $Z \subseteqq Q - \overline{Q - B}$, therefore $Q - \overline{Q - B} \in \mathfrak{b}$ and $Q - \overline{Q - B}$ is an open set (Q is a Hausdorff space). Since $Q - \overline{Q - B} \subseteqq B$, we get a contradiction.

Thus $Y \cap X \neq \emptyset$ for all $Y \in \mathfrak{b}$. Let us denote $\mathscr{P} = \{Q - Y \mid Y \in \mathfrak{b}\}$. Then $\mathscr{P}$ is an interior cover of the space Q. Then $Y_i \in \mathfrak{b}$ $(1 \leqq i \leqq n)$ exist such that $\bigcup_{i=1}^{n} (Q - Y_i) \supseteq X$. Let us put $Y = \bigcap_{i=1}^{n} Y_i$. Then $Y \in \mathfrak{b}$ and $Q - Y \supseteq X$; consequently $X \cap Y = \emptyset$, which is a contradiction.

Thus (C) $\rightarrow$ (A).

The assertion is proved.

3.9. *A D-space is normal.*

Proof. Let Q be a non-compact D-space, $X, Y \subseteqq Q$ disjoint closed sets in Q. If $\mathfrak{a} \in (P(N_0) - Q) \cap \overline{X} \cap \overline{Y}$ exists, where N_0 has the same meaning as in 3.5 and closures refer to the space $P(N_0)$, then systems of sets $\{A \cap X \mid A \in \mathfrak{a}\}$, $\{A \cap Y \mid A \in \mathfrak{a}\}$

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[4]) $\beta(Q)$ denotes Čech-Stone compactification.

[5]) Under a function we understand a mapping into the set of real numbers.

[6]) An open filter is such a filter which has the basis consisting of open sets.

form a basis of different filters $\mathfrak{b}, \mathfrak{c} \in I(Q)$. Since $\mathfrak{b} \leqq \mathfrak{a}$, $\mathfrak{c} \leqq \mathfrak{a}$, we have $\mathfrak{a} = \mathfrak{b} = \mathfrak{c}$ which is a contradiction. Thus $\bar{X} \cap \bar{Y} = \emptyset$. According to 3.5 and 3.8, $P(N_0) = \beta(Q)$; consequently there exist disjoint open sets U, V in the space $P(N_0)$ such that $U \supseteq X$, $V \supseteq Y$. From this it follows that Q is a normal space.

3.10. *Let δ be the discrete topology on Q. Let τ be some topology on Q. Then the space $\mathfrak{h}(Q, \tau) - Q$ is homeomorphic to some subspace of the space $\beta(Q, \delta) - Q$* [7]).

Proof. If (Q, τ) is a compact space, then the assertion follows from the remark following 2.7. Let us suppose, in what follows, that (Q, τ) fails to be a compact space. Let us denote $N_0(N'_0)$ the set of all minimal elements in $I(Q, \tau)$ $(I(Q, \delta))$. From 2.1 it follows that $N_0 \subseteqq N'_0$, and according to 3.5 and 3.8, we have $\mathfrak{h}(Q, \tau) = P(N_0)$, $\beta(Q, \delta) = P(N'_0)$. For $\mathfrak{n} \in \mathfrak{h}(Q, \tau) - Q = N_0$ let us put $h(\mathfrak{n}) = \mathfrak{n} \in N'_0 = \beta(Q, \delta) - Q$. Since for $\mathfrak{n} \in N_0$, $U \in \mathfrak{n}$ we have $\varphi_{N_0}(U) \cap N_0 = \{\mathfrak{a} \mid \mathfrak{a} \in N_0, U \in \mathfrak{a}\} = \varphi_{N'_0}(U) \cap N_0$, h is a homeomorphism $\mathfrak{h}(Q, \tau) - Q$ into $\beta(Q, \delta) - Q$.

4. *l*-COMPACT SPACES

Definition 4.1. The topological space P is called an *l-compact space* if any point in P has a neighbourhood which is relatively compact in P.

4.1. *A topological space P is l-compact iff the system of all neighbourhoods of an arbitrary point $p \in P$, which are relatively compact in P, forms a complete system of neighbourhoods of the point p.*

The proof follows from 1.9.

4.2. *Let Q be a non-compact space. Then there exists a one-point compactification $P = Q \cup (w)$ $(w \notin Q)$ of the space Q iff Q is an l-compact space. In this case the system of sets $(w) \cup (Q - X)$, where X is a relatively compact set in the space Q, forms a complete system of neighbourhoods of the point w.*

Proof. I. Let $P = Q \cup (w)$ $(w \notin Q)$ be a one-point compactification of the space Q.

a) Let $X \subseteqq Q$ be a relatively compact set in the space Q. For any $x \in Q$ there exist disjoint neighbourhoods U_x and V_x of the points x and w in the space P. The system $\{U_x \mid x \in Q\}$ is an interior cover of the space Q, so that there exists a finite number $x_i \in Q$ $(1 \leqq i \leqq n)$ such that $\bigcup_{i=1}^{n} U_{x_i} \supseteqq X$. Since $V = \bigcap_{i=1}^{n} V_{x_i}$ is a neighbourhood of the point w and $V \cap X = \emptyset$, $P - X = (w) \cup (Q - X)$ is a neighbourhood of the point w.

b) Let V be a neighbourhood of the point w and let $\mathscr{P}$ be an interior cover of the

[7]) $\beta(Q, \delta)$ denotes Čech-Stone compactification of the space (Q, δ).

space Q. Put $\mathscr{P}^* = \mathscr{P} \cup \{V\}$. Then $\mathscr{P}^*$ is an interior cover of the space P. So there exists a finite cover $\mathscr{K}^* \subseteqq \mathscr{P}^*$ of the space P. Then $\mathscr{K} = \mathscr{K}^* - \{V\} \subseteqq \mathscr{P}$ and $\bigcup Y(Y \in \mathscr{K}) \supseteqq Q - V$. Thus $Q - V$ is a relatively compact set in Q.

II. Let (Q, τ) be a non-compact, l-compact space. Let us put $P = Q \cup (w)$, where w is a symbol which fails to be an element of the set Q. According to 1.9, the system of sets $\{(w) \cup (Q - X) \mid X$ is a relatively compact set in $Q\}$ forms a basis of a filter $\tau' w \in F(P)$. For $q \in Q$, the system τq forms a basis of a filter $\tau' q \in F(P)$. Evidently, (P, τ') is a topological space and (Q, τ) is a densely embedded subspace of the space (P, τ'). If $\mathscr{P}$ is an interior cover of the space (P, τ'), then there exists $V \in \mathscr{P} \cap \tau' w$. Actually there exists a relatively compact set X in Q such that $X \supseteqq P - V$. A finite system $\mathscr{K} \subseteqq \mathscr{P}$ exists such that $\bigcup Y(Y \in \mathscr{K}) \supseteqq X$. Let us put $\mathscr{K}^* = \mathscr{K} \cup (V)$. Then $\bigcup Y(Y \in \mathscr{K}^*) = P$, $\mathscr{K}^* \subseteqq \mathscr{P}$ and card $\mathscr{K}^* < \aleph_0$. Thus (P, τ') is a compact space.

4.3. *Let Q be a non-compact space. Then Q is an l-compact space iff the set $I(Q)$ has the largest element $\mathfrak{m}$. Then $\{\mathfrak{m}\} \in \mathfrak{N}(I(Q))$ and the space $P(\{\mathfrak{m}\})$ is a one-point compactification of the space Q.*

Proof. I. If $I(Q)$ has the largest element $\mathfrak{m}$, then the set $\{\mathfrak{m}\} \in \mathfrak{N}(I(Q))$ and, according to 3.2, the space $P(\{\mathfrak{m}\})$ is a one-point compactification of the space Q. From 4.2 it then follows that Q is an l-compact space.

II Let Q be an l-compact space. According to 4.2, there exists its one-point compactification $P = Q \cup (w)\,(w \notin Q)$, $P = (P, \tau')$. Let us denote $Sp_Q(\tau' w) = \mathfrak{m}$. We have $\mathfrak{m} \in I(Q)$ and, according to 2.3, $\{\mathfrak{m}\} \in \mathfrak{N}(I(Q))$. According to 3.5, $\mathfrak{m} \geqq \mathfrak{n}$ for any minimal element $\mathfrak{n}$ of the set $I(Q)$. From 2.1, consequently, it follows that $\mathfrak{m}$ is the largest element of the set $I(Q)$. Evidently $(P, \tau') = P(\{\mathfrak{m}\})$.

4.4. *A Hausdorff l-compact space is locally compact iff it is regular.*

Proof. Let Q be a Hausdorff l-compact space. If Q is a locally compact space, then according to 1.10, Q is regular.

Let Q be a regular space and let $q \in Q$. Then there exists a neighbourhood U of the point q that is relatively compact in Q. Let us put $V = \overline{U}$. Then V a neighbourhood of the point q. Let $\mathscr{P}$ be an interior cover of the subspace V. Then to any $x \in V$ there exists a closed neighbourhood W_x such that we have $Y_x \in \mathscr{P}$, $W_x \cap V \subseteqq Y_x$. Let us put $W_x = Q - V$ for $x \in Q - V$. Then the system of sets $\{W_x \mid x \in Q\}$ forms an interior cover of the space Q. Thus, a finite number of points $x_i \in V\ (1 \leqq i \leqq n)$ exists such that $\bigcup_{i=1}^{n} W_{x_i} \supseteqq U$. Since sets $W_{x_i} (1 \leqq i \leqq n)$ are closed, we have $\bigcup_{i=1}^{n} W_{x_i} \supseteqq$ $\supseteqq V$; consequently $\bigcup_{i=1}^{n} Y_{x_i} = V$ and V is, therefore, compact, Q.E.D.

Example 4.1. Let $Q = [0, 1]$ and let R denote the set of rational numbers. For $q \in Q$ and $\varepsilon > 0$ let us denote $U(q, \varepsilon) = Q \cap (q - \varepsilon, q + \varepsilon) \cap (R \cup \{q\})$. The system of sets $\{U(q, \varepsilon) \mid \varepsilon > 0\}$ forms the basis of a filter $\tau q \in F(Q)$. Obviously, (Q, τ) is a Hausdorff space.

Let $\mathscr{P}$ be an interior cover of the space (Q, τ). Then $\varepsilon_q > 0$ and $Y_q \in \mathscr{P}$ exist for any $q \in Q$ such that $U(q, \varepsilon_q) \subseteqq Y_q$. Put $V_q = Q \cap (q - \varepsilon_q, q + \varepsilon_q)$ for any $q \in Q$. Then the system of sets $\{V_q \mid q \in Q\}$ forms an interior cover of the interval $[0, 1]$ in the natural topology. Thus a finite number $q_i \in [0, 1]$ $(1 \leqq i \leqq n)$ exists such that $\bigcup_{i=1}^{n} V_{q_i} = Q$. Then $\bigcup_{i=1}^{n} U(q_i, \varepsilon_{q_i}) \supseteqq R \cap Q$ and consequently $\bigcup_{i=1}^{n} Y_{q_i} \cup Y_x \supseteqq U(x, \varepsilon)$ for an arbitrary point $x \in Q$ and an arbitrary number $\varepsilon > 0$. Therefore, $U(x, \varepsilon)$ is a relatively compact set in the space (Q, τ) for any point $x \in Q$ and any $\varepsilon > 0$. So (Q, τ) is an l-compact space.

Evidently, (Q, τ) is not regular; thus, according to 4.4, (Q, τ) fails to be locally compact.

Definition 4.2. Let P be a topological space. If P is compact, then we put $\mathrm{l}(P) = 0$. Let $n > 0$ be an integer. If there exist, for $1 \leqq \nu \leqq n$, mutually disjoint sets A_ν that are not relatively compact in P, for $M \subseteqq A_\nu$, then either M or $A_\nu - M$ is relatively compact in P and $\bigcup_{\nu=1}^{n} A_\nu = P$; then we put $\mathrm{l}(P) = n$. In the other cases we put $\mathrm{l}(P) = \infty$.[8])

4.5. *Let P be a topological space, $\mathrm{l}(P) \neq \infty$. Then P is an l-compact space.*

Proof. If P is a compact space the assertion is evident. Let $n > 0$ be an integer and for $1 \leqq \nu \leqq n$ let there exist sets A_ν with the properties introduced in definition 4.2. Let $x \in A_1$ and let U be a neighbourhood of the point x such that sets $A_\nu \cap U$ fail to be relatively compact in P for $i \leqq \nu \leqq n$, where $1 \leqq i \leqq n$. In the other cases, let these sets be relatively compact in P. Then there exists an interior cover $\mathscr{P}$ of the space P from which a finite cover of the set $A_i \cap U$ cannot be chosen.

A neighbourhood X of the point x exists such that $X \in \mathscr{P}$. If the set $A_i - X$ is relatively compact in P, then there exists a finite cover $\mathscr{K} \subseteqq \mathscr{P}$ of the set $A_i - X$. Then, however, $\mathscr{K} \cup \{X\}$ is a finite cover of the set A_i, which is a contradiction. Consequently, the set $X \cap A_i$ is relatively compact in P. Let us put $V = U \cap X$. Then V is a neighbourhood of the point x and, according to 1.9, sets $V \cap A_j$ are relatively compact in P for $1 \leqq j \leqq i$. From 1.9 it follows that for any point $x \in P$ its relatively compact neighbourhood in P exists, Q.E.D.

4.6. *Let Q be a topological space and let n be a non-negative integer. Then the following statements are equivalent*:

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[8]) With respect to the statement 8.6 one could define $\mathrm{l}(P) = \mathrm{card}\,(\mathfrak{h}(P) - P)$; but in this paper there are not studied spaces P with the property $\mathrm{card}\,(\mathfrak{h}(P) - P) \geqq \aleph_0$ and for this reason I have defined $\mathrm{l}(P) = \infty$ for these spaces.

(A) card $(\mathfrak{h}(Q) - Q) = n$,
(B) card $I(Q) = 2^n - 1$,
(C) $\mathfrak{l}(Q) = n$.

Proof. I. If $n = 0$, then the equivalence of the introduced assertions follows from 2.2, 3.5 and from the remark following 2.7.

Thus, let us suppose, in what follows, that $n > 0$.

II. Let (A) hold. Then Q is non-compact according to the remark following 2.7. If we denote N_0 the set of all minimal elements of the set $I(Q)$, we have, by 3.5, $N_0 \in \mathfrak{N}(I(Q))$ and $P(N_0) = \mathfrak{h}(Q)$; consequently $N_0 = \{\mathfrak{n}_1, \ldots, \mathfrak{n}_n\}$, where $\mathfrak{n}_\nu$ are mutually different ultrafilters of the set Q for $1 \leqq \nu \leqq n$. From 2.1 it follows that the assertion (B) holds.

Let us put $\mathfrak{n} = \bigvee_{\nu=1}^{n} \mathfrak{n}_\nu$. According to 2.1, $\mathfrak{n}$ is the largest element in $I(Q)$ and according to 4.3, $\{\mathfrak{n}\} \in \mathfrak{N}(I(Q))$ and $P(\{\mathfrak{n}\})$ is a one-point compactification of the space Q.

In $\mathfrak{h}(Q)$ there exist mutually disjoint neighbourhoods $U_1, \ldots, U_n$ of points $\mathfrak{n}_1, \ldots \ldots, \mathfrak{n}_n$. Let us put $A_\nu = U_\nu \cap Q$ for $1 \leqq \nu \leqq n - 1$, $A_n = Q - \bigcup_{\nu=1}^{n-1} A_\nu$. Then $A_\nu \in \mathfrak{n}_\nu$ for $\nu = 1, 2, \ldots, n$. The set A_ν is not relatively compact in Q $(1 \leqq \nu \leqq n)$. Actually, A_ν were relatively compact in Q then by 4.2 we would have $Q - A_\nu \in \mathfrak{n}$; thus $Q - A_\nu \in \mathfrak{n}_\nu$, which is a contradiction. Let $M \subseteqq A_\nu$. If $U \in \mathfrak{n}_\nu$ exists such that $U \cap M = \emptyset$, then $U \cap \bigcup_{\substack{i=1 \\ i \neq \nu}}^{n} A_i \in \mathfrak{n}$; so there exists, according to 4.2, a relatively compact set X in Q such that $Q - X \subseteqq U \cap \bigcup_{\substack{i=1 \\ i \neq \nu}}^{n} A_i$. Then we have $X \supseteqq M$ and, according to 1.9, M is relatively compact in Q. If $U \cap M \neq \emptyset$ for all sets $U \in \mathfrak{n}_\nu$, then the system of sets $\{U \cap M \mid U \in \mathfrak{n}_\nu\}$ forms a basis of a filter $\mathfrak{n}' \in I(Q)$, $\mathfrak{n}' \leqq \mathfrak{n}_\nu$; thus $\mathfrak{n} = \mathfrak{n}_\nu$. But then $M \in \mathfrak{n}_\nu$, $M \cap (A_\nu - M) = \emptyset$ and, according to what precedes, $A_\nu - M$ is a relatively compact set in Q. Consequently, (C) is valid.

III. Let (C) hold. Let A_ν be, for $\nu = 1, \ldots, n$, the sets described in the definition 4.2. According to 1.9, the system of sets $\{A_\nu - X \mid X$ is relatively compact in $Q\}$ forms a basis of a filter $\mathfrak{n}_\nu \in F(Q)$. Since A_ν is not relatively compact in Q, then, according to 1.9, we have $\mathfrak{n}_\nu > \mathfrak{o}$. From 4.5 it follows that $\mathfrak{n}_\nu \in I(Q)$.

If $\mathfrak{n}_0 \in F(Q)$, $\mathfrak{n}_0 < \mathfrak{n}_\nu$, then there exists $M \in \mathfrak{n}_0 - \mathfrak{n}_\nu$, $M \subseteqq A_\nu$. Then $A_\nu - M$ fails to be relatively compact in Q and consequently, M is relatively compact in Q; thus $A_\nu - M \in \mathfrak{n}_\nu$, from whence $\mathfrak{n}_0 = \mathfrak{o}$ follows. Therefore, $\mathfrak{n}_\nu$ is an ultrafilter on Q.

Since the sets A_ν are mutually disjoint, ultrafilters $\mathfrak{n}_1, \ldots, \mathfrak{n}_n$ are mutually different.

If there exists an ultrafilter $\mathfrak{n}' \in I(Q)$, $\mathfrak{n}' \neq \mathfrak{n}_\nu$ for $1 \leqq \nu \leqq n$, then we have $M \in \mathfrak{n}'$, $M \notin \mathfrak{n}_\nu$ for $1 \leqq \nu \leqq n$. Then $A_\nu - M \in \mathfrak{n}_\nu$, consequently a relatively compact set X_ν exists in Q such that $A_\nu - X_\nu \subseteqq A_\nu - M$ from whence $X_\nu \supseteqq M \cap A_\nu$ follows.

According to 1.9, $M = \bigcup_{\nu=1}^{n} (M \cap A_\nu)$ is relatively compact in Q. From 4.5 and 4.3 it follows that $I(Q)$ has the largest element $\mathfrak{m}$; from 4.3 and 4.2 we then get that $Q - M \in \mathfrak{m}$. Thus, $Q - M \in \mathfrak{n}'$ which is a contradiction.

From 3.5 it follows that card $(\mathfrak{h}(Q) - Q) = n$. Thus, (A) is valid.

IV. If (B) holds the set $I(Q)$, according to 2.1, possesses n minimal elements and, according to 3.5, we have card $(\mathfrak{h}(Q) - Q) = n$. Thus, (A) is valid.

In this way the assertion is proved.

4.7. *The function* $\mathfrak{l}(P)$ *defined in definition* 4.2 *is a one-valued function.*

The proof follows directly from 4.6.

4.8. *Let* Q_1, Q_2 *be disjoint topological spaces;* $Q_1 + Q_2$ *denotes a topological sum of spaces* Q_1 *and* Q_2 *and expressions* $a + \infty$, $\infty + a$ *stand for* ∞ *for* $a = 0, 1, 2, \ldots, \infty$. *Then* $\mathfrak{l}(Q_1 + Q_2) = \mathfrak{l}(Q_1) + \mathfrak{l}(Q_2)$.

Proof. I. If $\mathfrak{l}(Q_1) = \mathfrak{l}(Q_2) = 0$, then evidently $\mathfrak{l}(Q_1 + Q_2) = 0$.

II. Let $\mathfrak{l}(Q_1) = n_1$ where $n_1 > 0$ is an integer and let A^1_ν, $\nu = 1, 2, \ldots, n_1$ be sets possessing the properties mentioned in definition 4.2 with respect to the space Q_1. If $\mathfrak{l}(Q_2) = n_2$, where $n_2 > 0$ is an integer, then let A^2_ν, $\nu = 1, \ldots, n_2$ be sets having the properties introduced in definition 4.2 with respect to the space Q_2. Let us put $n = n_1 + n_2$, $B_1 = A^1_1, \ldots, B_{n_1} = A^1_{n_1}$, $B_{n_1+1} = A^2_1, \ldots, B_{n_1+n_2} = B_n = A^2_{n_2}$. If $\mathfrak{l}(Q_2) = 0$, let us put $n = n_1$, $B_1 = A^1_1 \cup Q_2$, $B_2 = A^1_2, \ldots, B_{n_1} = B_n = A^1_{n_1}$. According to 1.9 the sets $B_1, \ldots, B_n$ have the property introduced in definition 4.2 with respect to the space $Q_1 + Q_2$. Consequently $\mathfrak{l}(Q_1 + Q_2) = n = \mathfrak{l}(Q_1) + \mathfrak{l}(Q_2)$.

III. Let $\mathfrak{l}(Q_1) = \infty$. According to 1.6 we have $\mathfrak{l}(Q_1 + Q_2) \neq 0$. If $\mathfrak{l}(Q_1 + Q_2) = n$, where $n > 0$ is an integer, then let $A_1, \ldots, A_n$ denote sets mentioned in definition 4.2 with respect to the space $Q_1 + Q_2$. Since Q_1 cannot be relatively compact in Q_1 (then Q_1 would be a compact space), then according to 1.9, ν_0 $(1 \leqq \nu_0 \leqq n)$ exists such that $A_{\nu_0} \cap Q_1$ is not relatively compact in Q. Let us assume that sets $A_\nu \cap Q_1$ are not relatively compact in Q_1 for $1 \leqq \nu \leqq \nu_1$, where $1 \leqq \nu_1 \leqq n$ and for $\nu_1 < \nu \leqq n$ are relatively compact in Q. Let us put $B_1 = (A_1 \cup \bigcup_{\nu_1 < \nu \leqq n} A_\nu) \cap Q_1$, $B_\nu = A_\nu \cap Q_1$ for $2 \leqq \nu \leqq \nu_1$. From 1.9 it follows that sets $B_1, \ldots, B_{\nu_1}$ have the properties mentioned in definition 4.2 with respect to the space Q_1. Consequently $\mathfrak{l}(Q_1) = \nu_1$, which is a contradiction.

Since $Q_1 + Q_2 = Q_2 + Q_1$, the assertion is proved.

Example 4.2. Let Q be a set, card $Q \geqq \exp\exp \aleph_0$, and denote $\mathfrak{A}$ the system of all free ultrafilters on Q. Let us choose $M \subset Q$, card $M = \aleph_0$ and let us denote $\mathfrak{B}$ the system of all free ultrafilters $\mathfrak{v}$ on Q with the property: $V \in \mathfrak{v} \Rightarrow V \cap M \neq \emptyset$. Since for $\mathfrak{v}_1, \mathfrak{v}_2 \in \mathfrak{B}$, $\mathfrak{v}_1 \neq \mathfrak{v}_2$, $Sp_M\, \mathfrak{v}_1$, $Sp_M\, \mathfrak{v}_2$ are different free ultrafilters on M we have, according to 1.2, card $\mathfrak{B} = \exp\exp \aleph_0$.

For any $\mathfrak{a} \in \mathfrak{A} - \mathfrak{B}$ there exists $A \in \mathfrak{a}$, $A \cap M = \emptyset$; thus $Q - M \in \mathfrak{a}$. Let us put $\mathfrak{b} = \bigvee \mathfrak{a}$ $(\mathfrak{a} \in \mathfrak{A} - \mathfrak{B})$. According to 1.3 $\mathfrak{b}$ is a free filter and $Q - M \in \mathfrak{b}$. Consequently, according to 1.3, $\mathfrak{b} \wedge \mathfrak{v} = \mathfrak{o}$ for any $\mathfrak{v} \in \mathfrak{B}$, because $M \in \mathfrak{v}$ for $\mathfrak{v} \in \mathfrak{B}$.

Let us choose $\mathfrak{v}_0 \in \mathfrak{B}$ and a one-to-one mapping f of a set $\mathfrak{B}_1 = (\mathfrak{B} - \{\mathfrak{v}_0\}) \cup (\mathfrak{b})$ into Q. Let us put $\tau q = \{X \cup (q) | X \in f^{-1}(q)\}$ for $q \in f^1(\mathfrak{B}_1)$, $\tau q = \{X \mid q \in X \subseteqq Q\}$ for $q \in Q - f^1(\mathfrak{B}_1)$. For $q \in Q$ we have evidently $\tau q \in F(Q)$ and for $q_1, q_2 \in Q$, $q_1 \neq q_2$, according to 1.3, $\tau q_1 \wedge \tau q_2 = \mathfrak{o}$; thus, (Q, τ) is a topological space.

Evidently, $I(Q, \tau) \cap \mathfrak{A} = (\mathfrak{v}_0)$ and if $\mathfrak{a}_1$ is a free filter on Q, $\mathfrak{v}_0 \neq \mathfrak{a}_1 \neq \mathfrak{o}$, then according to 1.1, there exists $\mathfrak{a}_2 \in \mathfrak{A} - (\mathfrak{v}_0)$, $\mathfrak{a}_1 \geqq \mathfrak{a}_2$. From this $\mathfrak{a}_1 \notin I(Q, \tau)$ easily follows. Consequently, card $I(Q, \tau) = 1$ and, according to 4.6, we have $\mathfrak{l}(Q, \tau) = 1$.

4.9. *Let Q be a topological space. The ordered sets $\mathscr{K}(Q)$ and $\mathscr{R}(Q)$ are modular (distributive) lattices exactly when $\mathfrak{l}(Q) \leqq 3(\mathfrak{l}(Q) \leqq 2)$.*[9])

Proof. If Q is a compact space, then $\mathfrak{l}(Q) = 0$ and, according to the remark following 2.7, we have $\mathscr{K}(Q) = \mathscr{R}(Q) = \{Q\}$, which is a distributive lattice. If Q is a non-compact space, then $I(Q) \neq \emptyset$ by 2.2 and $\mathrm{r}(I(Q)) = \text{card } N_0 = \text{card } (\mathfrak{h}(Q) - (Q))$ by 3.5, where the mapping r is defined in definition 2.4, [9] and N_0 is the set of minimal elements in $I(Q)$. Now, the assertion follows easily from 4.7, [9] and 4.6.

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[9]) Q being an l-compact space, one can easily show that $\mathscr{K}(Q)$ and $\mathscr{R}(Q)$ are lattices. I do not know necessary and sufficient conditions on a topological space Q for $\mathscr{K}(Q)$ and $\mathscr{R}(Q)$ to be lattices.