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THE TWO-SIDED CLOSED IDEALS OF THE ALGEBRA OF BOUNDED LINEAR OPERATORS OF A HILBERT SPACE

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1. INTRODUCTION

If H is a separable Hilbert space, then it is well known [2] that the ideal $\mathfrak{C}(H)$ of compact linear maps of H is the only non trivial closed two-sided ideal in the algebra $\mathfrak{L}(H)$ of bounded linear maps of H. In this paper we determine the closed two-sided ideals in $\mathfrak{L}(H)$ for an arbitrary Hilbert space H.

We recall that a linear map $\alpha : E \to F$ of two normed real, complex, or quaternionic vector spaces is called compact, if the image $\alpha(B)$ of each bounded subset $B \subset E$ is relatively compact in F. If F is a Banach space, the condition of $\alpha(B)$ to be relatively compact in F, can be replaced by $\alpha(B)$ being totally bounded in F, i.e. for each $\varepsilon > 0$ there exist finitely many open balls in F with radius ε and with centres in $\alpha(B)$, which cover $\alpha(B)$. The concept of totally bounded subsets has a natural generalization. Let ω denote a cardinal number. We call a subset of a metric space ω-bounded, if for each $\varepsilon > 0$ there exists a set of open balls with radius ε and with centres in this subset, such that the cardinal number of the set of open balls is smaller than ω, and such that the open balls cover the subset. If $\omega = \aleph_0$, this is just the definition of a totally bounded subset. We call now a bounded linear map $\alpha : E \to F$ ω-compact, if the image $\alpha(B)$ of each bounded subset $B \subset E$ is ω-bounded. If F is a Banach space and $\omega = \aleph_0$, this is then equivalent to α is compact.

If E is a normed vector space, we denote the set of all ω-compact bounded linear maps of E with $\mathfrak{C}_\omega(E)$. We show that $\mathfrak{C}_\omega(E)$ is a closed two-sided ideal in the algebra $\mathfrak{L}(E)$ of all bounded linear maps of E.

If H and K are Hilbert spaces, we obtain two other characterizations of ω-compact bounded linear maps $\alpha : H \to K$. First, a bounded linear map $\alpha : H \to K$ is ω-compact if and only if each closed linear subspace $T \subset \alpha(H)$ has $\dim(T) < \omega$. Second, a bounded linear map $\alpha : H \to K$ is ω-compact if and only if for each $\varepsilon > 0$ there exists a closed linear subspace $S \subset H$ with $\operatorname{codim}(S) < \omega$, such that the norm of α restricted to S is smaller than ε (Rellich criterion [3]).

We consider further the two-sided ideal $\mathfrak{E}_\omega(H)$ of all bounded linear maps α of the Hilbert space H with $\dim(\mathrm{Cl}(\alpha(H))) < \omega$. We show that $\mathrm{Cl}(\mathfrak{E}_\omega(H)) = \mathfrak{L}_\omega(H)$. If ω is not a limit cardinal number, then even $\mathfrak{E}_\omega(H) = \mathfrak{C}_\omega(H)$.

Let $\mathfrak{J}$ be an arbitrary two-sided ideal in $\mathfrak{L}(H)$. We prove that either $\mathfrak{J} = \mathfrak{C}_\omega(H)$, or there is a limit cardinal number ω with $\mathfrak{E}_\omega(H) \subset \mathfrak{J} \subset \mathfrak{C}_\omega(H)$. In particular, if $\mathfrak{J}$ is a closed two-sided ideal in $\mathfrak{L}(H)$, then there is a cardinal number ω with $\mathfrak{J} = \mathfrak{C}_\omega(H)$. The closed two-sided ideals in $\mathfrak{L}(H)$ form thus the chain

$$\{0\} \subset \mathfrak{C}(H) = \mathfrak{C}_{\aleph_0}(H) \subset \ldots \subset \mathfrak{C}_\omega(H) \subset \ldots \subset \mathfrak{L}(H) = \mathfrak{C}_{\omega_0+1}(H),$$

where $\omega_0 = \dim(H)$.

We would like to thank Dr. E. GERLACH for a usefull conversation.

It was mentioned to us that similar results have been also obtained by B. GRAMSCH (to appear).

2. PRELIMINARIES

If X is a set, let card (X) denote its cardinal number. If ω is a cardinal number, $\omega + 1$ denotes its successor, and if ω has a predecessor it is denoted by $\omega - 1$. A cardinal number without a predecessor is called a limit cardinal number.

If E and F are real, complex, or quaternionic normed vector spaces, let $\mathfrak{L}(E, F)$ be the vector space of continuous (=bounded) linear maps (= operators) from E to F with the norm

$$\|\alpha\| = \sup\{\|\alpha(x)\|;\ x \in H \text{ and } \|x\| = 1\} \text{ for every } \alpha \in \mathfrak{L}(H, K).$$

Instead of $\mathfrak{L}(E, E)$ we write $\mathfrak{L}(E)$.

H denotes in the following a real, complex, or quaternionic Hilbert space of infinite dimension (the finite dimensional case is trivial). If $x, y \in H$, then (x, y) is the inner product, and $\|x\| = \sqrt{(x, x)}$ the norm of x.

If S is a closed linear subspace of H, $\pi_S : H \to H$ denotes the orthogonal projection onto S defined by $\pi_S(x) = x$ for $x \in S$ and $\pi_S(x) = 0$ for $x \in S^\perp$, where $S^\perp$ is the orthogonal complement of S in H. Notice that $\pi_S \in \mathfrak{L}(H)$, $\|\pi_S\| = 1$, and $\pi_S + \pi_{S^\perp} = \mathrm{id}$.

Theorem 2.1. *If $\gamma \in \mathfrak{L}(H, K)$ maps H one-to-one onto K, then $\gamma^{-1} \in \mathfrak{L}(K, H)$.*

Proof. See for example [7] page 18.

For each $\alpha \in \mathfrak{L}(H, K)$ the adjoint $\alpha^* \in \mathfrak{L}(K, H)$ is defined. We recall:

Lemma 2.1. $(\alpha(x,)\ y) = (x, \alpha^*(y))$ *for each $x \in H$ and $y \in K$* (definition), $(\alpha^*)^* = \alpha$.

Definition 2.1. Let $\alpha \in \mathfrak{L}(H, K)$ and let S be a linear subspace of the Hilbert space H. We define

$$c(S, \alpha) = \inf\{\|\alpha(x)\|;\ x \in S \text{ and } \|x\| = 1\}.$$

Lemma 2.2. *Let $\alpha \in \mathfrak{L}(H, K)$ and let S be a closed linear subspace of H with $c(S, \alpha) > 0$. $\alpha(S)$ is then a closed linear subspace of K.*

Proof. Suppose that $y \in K$ such that there is a sequence $\{\alpha(x_n)\}_{n=1}^{\infty}$ with $\lim \alpha(x_n) = y$. The sequence $\{x_n\}_{n=1}^{\infty}$ is then a Cauchy sequence because of

$$\|x_n - x_m\| \leqq (c(S, \alpha))^{-1} . \|\alpha(x_n) - \alpha(x_m)\| .$$

Let $x = \lim x_n$. Then $\alpha(x) = y$. Which proves that $\alpha(S)$ is closed.

Recall that the dimension of the Hilbert space H is defined as $\dim(H) = \operatorname{card}(I)$, where I is the index set of a complete orthonormal system $\mathfrak{B} = \{e_i\}_{i \in I}$ of H.

Lemma 2.3. *Let $A \subset H$ be a subset with $\operatorname{card}(A) \geqq \aleph_0$. We consider the set $R(A) = \{r_1 . x_1 + \ldots + r_n . x_n;\ x_1, \ldots, x_n \in A$ and $r_1, \ldots, r_n \in Q\}$, where Q is the subfield of the field of the Hilbert space formed by the elements with rational real components. Then $\operatorname{card}(R(A)) = \operatorname{card}(A)$.*

Proof. There is an obvious map of the set $\bigcup_{n=1}^{\infty} (Q \times A)^n$ onto the set $R(A)$. We have $\operatorname{card}(Q \times A) = \operatorname{card}(A)$, and hence $\operatorname{card}\left(\bigcup_{n=1}^{\infty} (Q \times A)^n\right) = \operatorname{card}(A)$. Consequently, $\operatorname{card}(R(A)) \leq \operatorname{card}(A)$, and therefore $\operatorname{card}(R(A)) = \operatorname{card}(A)$.

Lemma 2.4. *If $\alpha \in \mathfrak{L}(H, K)$, then $\dim(\mathrm{Cl}(\alpha(H))) \leqq \dim(H)$.*

Proof. Let $\mathfrak{B} = \{e_i\}_{i \in I}$ be a complete orthonormal system of the Hilbert space H. We consider the set $\mathfrak{D} = R(\mathfrak{B})$ of the preceding lemma 2.3. First we observe that $\mathrm{Cl}(\alpha(\mathfrak{D})) = \mathrm{Cl}(\alpha(H))$. Namely suppose that $y \in \mathrm{Cl}(\alpha(H))$ and let an $\alpha > 0$ be given. There is then a $y' \in \alpha(H)$ with $\|y - y'\| < \frac{1}{2}\varepsilon$. Let $x' \in H$ with $\alpha(x') = y'$. There is a $d \in \mathfrak{D}$ with $\|x' - d\| < \varepsilon/(2 . \|\alpha\|)$. Hence $\|\alpha(d) - y\| \leqq \|\alpha(d) - \alpha(x')\| + \|y' - y\| \leqq \|\alpha\| . \|d - x'\| + \|y' - y\| < \varepsilon$.

Let $\mathfrak{C} = \{f_j\}_{j \in J}$ be a complete orthonormal system of the Hilbert space $\mathrm{Cl}(\alpha(H))$. We define a map

$$s : J \to \alpha(\mathfrak{D}) ,$$

$$s(f_j) = g_j , \quad \text{where} \quad g_j \in \alpha(\mathfrak{D}) \quad \text{with} \quad \|g_j - f_j\| < \tfrac{1}{2} .$$

This map is one-to-one. Namely if $f_j \neq f_k$ and $s(f_j) = s(f_k) = g_j = g_k$, then $\sqrt{2} = \|f_j - f_k\| \leqq \|f_j - g_j\| + \|g_k - f_k\| < 1$, which is not possible. Therefore $\operatorname{card}(J) \leqq \operatorname{card}(\alpha(\mathfrak{D})) \leqq \operatorname{card}(\mathfrak{D}) = \operatorname{card}(I)$.

3. ω-BOUNDED SUBSETS OF A METRIC SPACE

Definition 3.1. Let X be a metric space. $O_r(x) = \{y;\ y \in X$ with $d(x, y) < r\}$ is the open ball in X with centre x and radius $r > 0$. Let ω be a cardinal number. A subset

$A \subset X$ is called ω-bounded, if for each $\varepsilon > 0$ there exists a set of points $\{x_m\}_{m \in M}$, $x_m \in A$, with card $(M) < \omega$, and with $A \subset \bigcup_{m \in M} O_\varepsilon(x_m)$.

The special case $\omega = \aleph_0$ coincides with the definition of A to be totally bounded. In this case the following holds:

Theorem 3.1. *Let X be a complete metric space. A subset $A \subset X$ is totally bounded if and only if* Cl (A) *is compact.*

Proof. See for example [5] page 22.

We exhibit a few properties of the concept "ω-bounded", which will be used later on.

Lemma 3.1. *If $A \subset X$ is ω-bounded and if ω' is a cardinal number with $\omega' > \omega$, then A is also ω'-bounded.*

Proof. Trivial.

Lemma 3.2. *If $A \subset X$ is ω-bounded, each subset $B \subset A$ is ω-bounded.*

Proof. Let an $\varepsilon > 0$ be given. There exists a set of points $\{y_m\}_{m \in M}$, $y_m \in A$, with card $(M) < \omega$, and with $A \subset \bigcup_{m \in M} O_{\varepsilon/2}(y_m)$. Let $M' = \{m;\ m \in M$ with $O_{\varepsilon/2}(y_m) \cap \cap B \neq \emptyset\}$. For each $m \in M'$ we choose a $x_m \in O_{\varepsilon/2}(y_m)$. We conclude that $B \subset \subset \bigcup_{m \in M'} O_\varepsilon(x_m)$.

Lemma 3.3. *Let X and Y be matric spaces, and let $f: X \to Y$ be a uniformly continuous map. If $A \subset X$ is ω-bounded, then also $f(A)$ is ω-bounded.*

Proof. Let an $\varepsilon > 0$ be given. There is a $\delta > 0$ with $d(f(x_1), f(x_2)) < \varepsilon$ for $d(x_1, x_2) < \delta$. Since A is ω-bounded, there exists a set of points $\{x_m\}_{m \in M}$, $x_m \in A$, with card $(M) < \omega$, and with $A \subset \bigcup_{m \in M} O_\delta(x_m)$. Consequently, $f(A) \subset \bigcup_{m \in M} O_\varepsilon(f(x_m))$.

4. ω-COMPACT LINEAR MAPS OF A NORMED VECTOR SPACE

Definition 4.1. Let E and F be real, complex, or quaternionic vector spaces, and let ω be a cardinal number. A linear map $\alpha \in \mathfrak{L}(E, F)$ is called ω-compact, if the image $\alpha(B)$ of each bounded subset $B \subset E$ is ω-bounded.

The special case $\omega = \aleph_0$ is in view of theorem 3.1 equivalent to the definition of α to be compact or completely continuous. In this case it is not necessary to require the continuity of the linear map in the preceding definition, because it is a consequence of the remainder. If $\omega < \aleph_0$, necessarily $\alpha = 0$.

Lemma 4.1. *Let E, F, and G be normed vector spaces, and assume that $\alpha \in \mathfrak{L}(E, F)$ and $\beta \in \mathfrak{L}(F, G)$. If α or β is ω-compact, then $\beta \,.\, \alpha$ is ω-compact.*

Proof. Suppose that α is ω-compact. Let $B \subset E$ be a bounded subset. Then $\alpha(B)$ is ω-bounded. Since β is uniformly continuous, $\beta(\alpha(B))$ is ω-bounded by lemma 3.3.

Next, let β be ω-compact. If $B \subset E$ is bounded, then also $\alpha(B)$. Hence $\beta(\alpha(B))$ is ω-bounded.

Lemma 4.2. *Let E and F be normed vector spaces, and assume that $\alpha, \beta \in \mathfrak{L}(E, F)$ are ω-compact. Then $\alpha + \beta$ is ω-compact. If c is a constant, then $c \,.\, \alpha$ is ω-compact.*

Proof. We may suppose that $\omega \geqq \aleph_0$. We consider the maps

$$H \xrightarrow{\alpha \times \beta} H \times H \xrightarrow{\sigma} H,$$

where $(\alpha \times \beta)(x) = (\alpha(x), \beta(x))$ and $\sigma(x, y) = x + y$. Then $\alpha + \beta = \sigma \,.\, (\alpha \times \beta)$. We show that $\alpha \times \beta$ is ω-compact. Let $B \subset E$ be a bounded subset, and let an $\varepsilon > 0$ be given. There exist points $\{y_m\}_{m \in M}$, $y_m \in \alpha(B)$, and points $\{z_n\}_{n \in N}$, $z_n \in \beta(B)$, with $\operatorname{card}(M) < \omega$, $\operatorname{card}(N) < \omega$, $\alpha(B) \subset \bigcup_{m \in M} O_{\varepsilon/2}(y_m)$, and $\beta(B) \subset \bigcup_{n \in N} O_{\varepsilon/2}(z_n)$. We conclude that $(\alpha \times \beta)(B) \subset \bigcup_{(m,n) \in M \times N} O_\varepsilon((y_m, z_n))$. Notice that $\operatorname{card}(M \times N) < \omega$. Lemma 4.1 implies then that $\alpha + \beta$ is ω-compact.

If c is a constant, $\lambda_c : E \to E$, $\lambda_c(x) = c \,.\, x$, is a continuous linear map. Lemma 4.1 proves again that $c \,.\, \alpha$ is ω-compact.

Lemma 4.3. *Let E and F be normed vector spaces, let $\{\alpha_n\}_{n=1}^{\infty}$ be a convergent sequence with $\alpha_n \in \mathfrak{L}(E, F)$ is ω-compact, and assume that $\alpha = \lim \alpha_n$. α is then also ω-compact.*

Proof. Let $B \subset E$ be a bounded subset. We show that $\alpha(B)$ is ω-bounded. Let an $\varepsilon > 0$ be given. There is a constant b with $\|x\| < b$ for all $x \in B$. We choose an n_0 with $\|\alpha - \alpha_{n_0}\| < \varepsilon/3 \,.\, b$. Since $\alpha_{n_0}(B)$ is ω-bounded, there exists a set of points $\{y_m\}_{m \in M}$, $y_m \in \alpha_{n_0}(B)$, with $\operatorname{card}(M) < \omega$, and with $\alpha_{n_0}(B) \subset \bigcup_{m \in M} O_{\varepsilon/3}(y_m)$. For each $m \in M$ we select a $x_m \in B$ with $\alpha_{n_0}(x_m) = y_m$. Now we consider the points $\{z_m\}_{m \in M}$, $z_m = \alpha(x_m) \in \alpha(B)$. We claim that $\alpha(B) \subset \bigcup_{m \in M} O_\varepsilon(z_m)$. Namely let $x \in B$ be given. There is a y_m with $\|\alpha_{n_0}(x) - y_m\| < \varepsilon/3$. We compute

$$\|\alpha(x) - z_m\| \leqq \|\alpha(x) - \alpha_{n_0}(x)\| + \|\alpha_{n_0}(x) - \alpha_{n_0}(x_m)\| + \|\alpha_{n_0}(x_m) - \alpha(x_m)\| < \varepsilon .$$

Definition 4.2. Let E be a normed vector space, ω a cardinal number. We define

$$\mathfrak{C}_\omega(E) = \{\alpha;\ \alpha \in \mathfrak{L}(E) \text{ and } \alpha \text{ is } \omega\text{-compact}\}.$$

Theorem 4.1. $\mathfrak{C}_\omega(E)$ *is a closed two-sided ideal in the algebra* $\mathfrak{L}(E)$. *If* ω_1 *and* ω_2 *are two cardinal numbers with* $\omega_1 < \omega_2$, *then* $\mathfrak{C}_{\omega_1}(E) \subset \mathfrak{C}_{\omega_2}(E)$.

Proof. Lemmas 4.1, 4.2, and 4.3.

Notice that if $\omega < \aleph_0$, then $\mathfrak{C}_\omega(E) = \{0\}$.

5. ω-COMPACT LINEAR MAPS OF HILBERT SPACES

Lemma 5.1. *Let H and K be Hilbert spaces.* $\alpha \in \mathfrak{L}(H, K)$ *is* ω*-compact if and only if* $\alpha^* . \alpha$ *is* ω*-compact.*

Proof. If $\alpha \in \mathfrak{L}(H, K)$ is ω-compact, then $\alpha^* . \alpha$ is ω-compact by lemma 4.1. Suppose now that $\alpha^* . \alpha \in \mathfrak{L}(H)$ is ω-compact. We prove that α is ω-compact. Let $B \subset H$ be bounded, and let an $\varepsilon > 0$ be given. There is a constant b with $\|x\| < b$ for all $x \in B$. Since $\alpha^* . \alpha(B)$ is ω-bounded, there exists a set of points $\{y_m\}_{m \in M}$, $y_m \in \alpha^* . \alpha(B)$, with card $(M) < \omega$, and with $\alpha^* . \alpha(B) \subset \bigcup_{m \in M} O_{\varepsilon^2/2b}(y_m)$. For each $m \in M$ we choose a $x_m \in B$ with $\alpha^* . \alpha(x_m) = y_m$. We consider then the set of points $\{z_m\}_{m \in M}$, where $z_m = \alpha(x_m) \in \alpha(B)$. We claim that $\alpha(B) \subset \bigcup_{m \in M} O_\varepsilon(z_m)$. Namely suppose that $x \in B$. There is an m with $\|y_m - \alpha^* . \alpha(x)\| < \varepsilon^2/2b$. We compute

$$\|\alpha(x) - z_m\|^2 = (\alpha(x - x_m), \ \alpha(x - x_m)) = (\alpha^* . \alpha(x - x_m), x - x_m) \leqq$$
$$\leqq \|\alpha^* . \alpha(x) - y_m\| . \|x - x_m\| < \frac{\varepsilon^2}{2b} . 2b = \varepsilon^2 .$$

Hence $x \in O_\varepsilon(z_m)$.

Corollary 5.1. $\alpha \in \mathfrak{L}(H, K)$ *is* ω*-compact if and only if* $\alpha^* \in \mathfrak{L}(K, H)$ *is* ω*-compact.*

Proof. If $\alpha \in \mathfrak{L}(H, K)$ is ω-compact, then $\alpha . \alpha^* = (\alpha^*)^* . \alpha^*$ is ω-compact by lemma 4.1. Lemma 5.1 implies that α^* is ω-compact. If α^* is ω-compact, then $\alpha = (\alpha^*)^*$ is ω-compact by the preceding.

Corollary 5.2. $\mathfrak{C}_\omega(H)$ *is a closed two-sided *-ideal of* $\mathfrak{L}(H)$.

Proof. $\mathfrak{C}_\omega(H)$ is a *-ideal by corollary 5.1. Actually any two-sided ideal in $\mathfrak{L}(H)$ is automatically a *-ideal. Compare for example theorem 1.2 of [2].

Lemma 5.2. *Let* $S \subset H$ *be a closed linear subspace of the Hilbert space H, and assume that* dim $(S) = \omega \geqq \aleph_0$. *The projection* π_S *is then* $(\omega + 1)$*-compact, but not* ω*-compact.*

Proof. We show that S is $(\omega + 1)$-bounded. Let $\mathfrak{B} = \{e_i\}_{i \in I}$ be a complete orthonormal system of S. Then card $(I) = \omega$. We consider the set $\mathfrak{D} = R(\mathfrak{B})$ of lemma 2.3.

Thus card $(\mathfrak{D}) = \text{card}(I) = \omega$. Let an $\varepsilon > 0$ be given. Then $S \subset \bigcup_{d \in \mathfrak{D}} O_\varepsilon(d)$. Lemma 3.2 implies that π_S is $(\omega + 1)$-compact.

Suppose now that π_S is ω-compact. Since the set $\mathfrak{B}$ is bounded, $\pi_S(\mathfrak{B}) = \mathfrak{B}$ must be ω-bounded. Let $\varepsilon = \frac{1}{2}$. There exists then a set of points $\{x_m\}_{m \in M}$, $x_m \in \mathfrak{B}$, with card $(M) < \omega$, and with $\mathfrak{B} \subset \bigcup_{m \in M} O_{1/2}(x_m)$. We define the map

$$s : I \to M\,, \quad s(i) = m \quad \text{with} \quad e_i \in O_{1/2}(x_m)\,.$$

The map s is one-to-one. Namely if $s(i) = s(j)$, $i \neq j$, then $\|e_i - x_m\| < \frac{1}{2}$ and $\|e_j - x_k\| < \frac{1}{2}$, and hence $\sqrt{2} = \|e_i - e_j\| \leqq \|e_i - x_m\| + \|x_m - e_j\| < 1$, which is not possible. Therefore $\omega = \text{card}(I) \leqq \text{card}(M) < \omega$, which is a contradiction. Thus π_S is not ω-compact.

Corollary 5.3. *Let H be a Hilbert space, and let ω_1 and ω_2 be two cardinal numbers with $\aleph_0 \leqq \omega_1 < \omega_2 \leqq \dim(H) + 1$. Then $\mathfrak{C}_{\omega_1}(H) \subset \mathfrak{C}_{\omega_2}(H)$, and $\mathfrak{C}_{\omega_1}(H) \neq \mathfrak{C}_{\omega_2}(H)$.*

Proof. Let $S \subset H$ be a closed linear subspace with $\dim(S) = \omega_1$. Then $\pi_S \in \mathfrak{C}_{\omega_2}(H)$, but $\pi_S \notin \mathfrak{C}_{\omega_1}(H)$.

Lemma 5.3. *Let H and K be Hilbert spaces, and let $\alpha \in \mathfrak{L}(H, K)$. For each $\varepsilon > 0$ there exists a closed linear subspace $S \subset H$ with*

$$c(S, \alpha) \geqq \varepsilon \quad \textit{and} \quad \|\alpha|_{S^\perp}\| < \varepsilon\,.$$

Proof. We consider the map $\beta = \alpha^* . \alpha \in \mathfrak{L}(H)$ and its spectral representation (see for example [8], page 25). It follows that $\beta = \int_0^\infty \lambda \, de_\lambda$, where $\{e_\lambda\}_{\lambda \geqq 0}$ the spectral family defined by β. We define $S = (\mathrm{Cl}([\bigcup_{0 \leqq \lambda < \varepsilon} e_\lambda(H)]))^\perp$. ($[A]$ denotes the linear subspace spanned by the subset $A \subset H$). From the definition of the spectral representation we conclude that

$$(\alpha(x), \alpha(x)) = (\beta(x), x) \geqq \varepsilon \,.\, \|x\|^2 \quad \text{for} \quad x \in S\,,$$

and

$$(\alpha(x), \alpha(x)) = (\beta(x), x) \leqq \varepsilon \,.\, \|x\|^2 \quad \text{for} \quad x \in S^\perp\,.$$

This shows that $c(S, \alpha) \geqq \varepsilon$ and $\|\alpha|_S\| \leqq \varepsilon$. Replacing ε by $\varepsilon/2$ proves the lemma.

Remark. It would be nice to have a more direct proof of lemma 5.3 without resorting to spectral theory.

Theorem 5.1. *Let H and K be Hilbert spaces, let $\alpha \in \mathfrak{L}(H, K)$, and let $\omega \geqq \aleph_0$ be a cardinal number. α is ω-compact if and only if each closed linear subspace $T \subset \alpha(H)$ has $\dim(T) < \omega$.*

Proof. Suppose that α is ω-compact. Let us assume that there is a closed linear subspace $T_0 \subset \alpha(H)$ with $\dim(T_0) \geqq \omega$. Then $S_0' = \alpha^{-1}(T_0)$ is a closed linear subspace of H with $\alpha(S_0') = T_0$. Let $S_0 = (\text{kernel}(\alpha))^\perp \cap S_0'$. $\alpha|_{S_0} : S_0 \to T_0$ is a continuous isomorphism onto T_0, and $(\alpha|_{S_0})^{-1} \in \mathfrak{L}(T_0, S_0)$ by theorem 2.1. Let $c = \|(\alpha|_{S_0})^{-1}\|$, and let $B_0 = \{x; x \in H \text{ with } \|x\| \leqq c\}$. Consequently, $\{y; y \in T_0 \text{ and } \|y\| \leqq 1\} \subset \alpha(B_0)$. Because B_0 is bounded, $\alpha(B_0)$ is ω-compact. Hence there exists a set of points $\{y_m\}_{m\in M}$, $y_m \in \alpha(B_0)$, with $\text{card}(M) < \omega$, and with $\alpha(B_0) \subset \bigcup_{m\in M} O_{1/2}(y_m)$. We consider now a complete orthonormal system $\mathfrak{B} = \{e_i\}_{i\in I}$ of T_0. Then $\mathfrak{B} \subset \alpha(B_0)$ and $\text{card}(I) \geqq \omega$. We construct a map

$$s : I \to M\,, \quad s(i) = m \quad \text{with} \quad e_i \in O_{1/2}(y_m)\,.$$

The map s is one-to-one by the same argument as in the proof of lemma 5.2. Thus $\text{card}(I) = \text{card}(M) < \omega \leqq \text{card}(I)$, which is a contradiction. Therefore all closed linear subspaces $T \subset \alpha(H)$ have $\dim(T) < \omega$.

Now we assume that each closed linear subspace $T \subset \alpha(H)$ has $\dim(T) < \omega$. We show that α is ω-compact. Suppose that $\alpha \neq 0$. Let $B \subset H$ be a bounded subset, and let an $\varepsilon > 0$ be given. There is a constant b with $\|x\| < b$ for all $x \in B$. We apply lemma 5.3. There exists a closed linear subspace $S \subset H$ with $c(S, \alpha) < \varepsilon/(2b \,.\, \|\alpha\|)$ and $\|\alpha|_{S^\perp}\| < \varepsilon/(2b \,.\, \|\alpha\|)$. By lemma 2.2 $\alpha(S)$ is a closed linear subspace, and by hypothesis $\dim(\alpha(S)) < \omega$. As shown in the proof of lemma 5.2, $\alpha(S)$ is ω-bounded. There exists a set of points $\{y_m\}_{m\in M}$, $y_m \in \alpha(S)$, with $\text{card}(M) < \omega$, and with $\alpha(S) \subset \bigcup_{m\in M} O_{\varepsilon/2}(y_m)$. We conclude that $\alpha(B) \subset \bigcup_{m\in M} O_\varepsilon(y_m)$. Namely if $x \in B$, then $x = x_1 + x_2$, $x_1 \in S$, $x_2 \in S^\perp$, and $\|x_1\|, \|x_2\| < b$. Thus $\alpha(x_1) \in O_{\varepsilon/2}(y_m)$. Therefore $\|\alpha(x) - y_m\| \leqq \|\alpha(x_1) - y_m\| + \|\alpha(x_2)\| < \varepsilon$. Hence $\alpha(x) \in O_\varepsilon(y_m)$. This proves that $\alpha(B)$ is ω-bounded.

Theorem 5.2. (Rellich criterion). *Let H and K be Hilbert spaces, let $\alpha \in \mathfrak{L}(H, K)$, and let $\omega \geqq \aleph_0$ be a cardinal number. α is ω-compact if and only if for each $\varepsilon > 0$ there exists a closed linear subspace $T \subset H$ with* codim $(T) < \omega$ *and with* $\|\alpha|_T\| < \varepsilon$.

Proof. Suppose that α is ω-compact. Let $\varepsilon > 0$ be given. We apply lemma 5.3. There exists a closed linear subspace $S \subset H$ with $c(S, \alpha) > \varepsilon$ and $\|\alpha|_{S^\perp}\| < \varepsilon$. By lemma 2.2 $\alpha(S)$ is a closed linear subspace, and by theorem 5.1 $\dim(\alpha(S)) < \omega$. Since $\pi_S : S \to \alpha(S)$ is one-to-one, it is by theorem 2.1 a continuous isomorphism onto the Hilbert space $\alpha(S)$. Therefore $\dim(S) < \omega$. Let $T = S^\perp$.

Next we assume that for each $\varepsilon > 0$ there exists a closed linear subspace $T \subset H$ with codim $(T) < \omega$ and with $\|\alpha|_T\| < \varepsilon$. We have to show that α is ω-compact. Let $S = T^\perp$. Then $\dim(\text{Cl}(\alpha(S)) \leqq \dim(S) < \omega$ by lemma 2.4. Carrying out the same construction as in the second part of the proof of theorem 5.1, we obtain that α is ω-compact.

Lemma 5.4. *Let H and K be Hilbert spaces, and let $\omega \geqq \aleph_0$ be a cardinal number. If $\alpha \in \mathfrak{L}(H, K)$ is ω-compact, then* $\dim(\mathrm{Cl}(\alpha(H)) \leqq \omega$, *and if ω is not a limit cardinal number even* $\dim(\mathrm{Cl}(\alpha(H)) \leqq \omega - 1$.

Proof. Let $B = \{x; x \in H \text{ and } \|x\| \leqq 1\}$. Then $\alpha(B)$ is ω-bounded, and $\alpha(H) = [\alpha(B)]$. For each $n = 1, 2, \ldots$ there exists a set of points $\{y_{mn}\}_{m \in M_n}$, $y_{mn} \in \alpha(B)$, with $\operatorname{card}(M_n) < \omega$, and with $\alpha(B) \subset \bigcup_{m \in M_n} O_{1/n}(y_{mn})$. Let $Y = \bigcup_{n=1}^{\infty} \{y_{mn}\}_{m \in M_n}$. Then $\operatorname{card}(Y) \leqq \omega$, and if ω is not a limit cardinal number, then $\operatorname{card}(Y) \leqq \omega - 1$. We consider the set $\mathfrak{D} = R(Y) = \{r_1 . y_1 + \ldots + r_n . y_n; y_1, \ldots, y_n \in Y \text{ and } r_1, \ldots, r_n \in Q\}$ as defined in lemma 2.3. It follows from lemma 2.3 that $\operatorname{card}(\mathfrak{D}) \leqq \omega$, and if ω is not a limit cardinal number $\operatorname{card}(\mathfrak{D}) \leqq \omega - 1$.

We claim that $\mathrm{Cl}(\mathfrak{D}) = \mathrm{Cl}(\alpha(H))$. Namely let $x \in \mathrm{Cl}(\alpha(H))$, and let an $\varepsilon > 0$ be given. There is a $x' \in \alpha(H)$ with $\|x - x'\| < \varepsilon/2$. Because of $\alpha(H) = [\alpha(B)]$, $x' = \lambda_1 . z_1 + \ldots + \lambda_n . z_n$, where $z_1, \ldots, z_n \in \alpha(B)$, and $\lambda_1, \ldots, \lambda_n$ are scalars. Let $c = \max\{|\lambda_1|, \ldots, |\lambda_n|, \|z_1\|, \ldots, \|z_n\|, 1\}$. We choose $y_1, \ldots, y_n \in Y$ with $\|z_i - y_i\| < \varepsilon/(4 . c . n)$, $i = 1, \ldots, n$, and $r_1, \ldots, r_n \in Q$ with $|\lambda_i - r_i| < \varepsilon/(4 . c . n)$, $i = 1, \ldots, n$. Let $d = r_1 . y_1 + \ldots + r_n . y_n \in \mathfrak{D}$. It follows that $\|x' - d\| < \varepsilon/2$, and therefore $\|x - d\| < \varepsilon$.

Let $\mathfrak{B} = \{e_i\}_{i \in I}$ be a complete orthonormal system of the Hilbert space $\mathrm{Cl}(\alpha(H))$. We define a map

$$s : I \to \mathfrak{D}, \quad s(i) = y_i \quad \text{such that} \quad \|e_i - y_i\| < \tfrac{1}{2}.$$

The map s is one-to-one by the same argument as used in the proof of lemma 5.2. Therefore $\dim(\mathrm{Cl}(\alpha(H)) = \operatorname{card}(I) \leqq \operatorname{card}(\mathfrak{D})$, which proves the lemma.

6. TWO SIDED IDEALS AND THE CLOSED TWO-SIDED IDEALS OF THE ALGEBRA OF BOUNDED LINEAR OPERATORS OF A HILBERT SPACE

Definition 6.1. Let $\omega \geqq \aleph_0$ be a cardinal number. We define

$$\mathfrak{E}_\omega(H) = \{\alpha; \alpha \in \mathfrak{L}(H) \text{ and } \dim(\mathrm{Cl}(\alpha(H))) < \omega\}.$$

Theorem 6.1. *$\mathfrak{E}_\omega(H)$ is a two-sided *-ideal of $\mathfrak{L}(H)$, and $\mathfrak{E}_\omega(H) \subset \mathfrak{L}_\omega(H)$.*

Proof. Let $\alpha \in \mathfrak{E}_\omega(H)$ and $\beta \in \mathfrak{L}(H)$. $\mathrm{Cl}(\alpha . \beta(H)) \subset \mathrm{Cl}(\alpha(H))$ implies that $\alpha . \beta \in \mathfrak{E}_\omega(H)$, and $\dim(\mathrm{Cl}(\beta . \alpha(H)) \leqq \dim(\mathrm{Cl}(\alpha(H)))$ (lemma 1.4) shows that $\beta . \alpha \in \mathfrak{E}_\omega(H)$.

If $\alpha, \beta \in \mathfrak{E}_\omega(H)$, then $\alpha + \beta \in \mathfrak{E}_\omega(H)$. Namely, consider the maps

$$H \xrightarrow{\alpha \times \beta} H \times H \xrightarrow{\sigma} H,$$

where $(\alpha \times \beta)(x) = (\alpha(x), \beta(x))$ and $\sigma(x, y) = x + y$. Then $\alpha + \beta = \sigma . (\alpha \times \beta)$.

We notice that $\mathrm{Cl}((\alpha \times \beta)(H)) = \mathrm{Cl}(\alpha(H)) \times \mathrm{Cl}(\beta(H))$. Therefore $\dim(\mathrm{Cl}(\alpha \times \times \beta)(H))) = \dim(\mathrm{Cl}(\alpha(H))) + \dim(\mathrm{Cl}(\beta(H))) < \omega$. Thus $\dim(\mathrm{Cl}((\alpha + \beta)(H))) = = \dim(\mathrm{Cl}(\sigma . (\alpha \times \beta)(H))) \leqq \dim(\mathrm{Cl}((\alpha \times \beta)(H))) < \omega$, and therefore $\alpha + \beta \in \in \mathfrak{E}_\omega(H)$.

If $\alpha \in \mathfrak{E}_\omega(H)$ and c is a constant, obviously $c . \alpha \in \mathfrak{E}_\omega(H)$.

If $\alpha \in \mathfrak{E}_\omega(H)$, then also $\alpha^* \in \mathfrak{E}_\omega(H)$. Namely kernel $(\alpha^*) = (\alpha(H))^\perp$, and therefore $\alpha^*(H) = \alpha^*(\mathrm{Cl}(\alpha(H)))$. Since $\dim(\mathrm{Cl}(\alpha(H))) < \omega$, $\dim(\mathrm{Cl}(\alpha^*(H))) < \omega$ by lemma 1.4.

Finally, if $\alpha \in \mathfrak{E}_\omega(H)$ let $S = \mathrm{Cl}(\alpha(H))$ and let $\omega' = \dim(S)$. Then $\omega' < \omega$. As shown in the proof of lemma 5.2, S is $(\omega' + 1)$-bounded. Therefore α is $(\omega' + 1)$-compact. Hence $\alpha \in \mathfrak{C}_\omega(H)$.

Definition 6.2. Let $\mathfrak{J} \subset \mathfrak{L}(H)$ be a two-sided ideal. The height $h(\mathfrak{J})$ of the ideal $\mathfrak{J}$ is defined as

$$h(\mathfrak{J}) = \sup\{\dim(T);\ T \subset \alpha(H) \text{ a closed linear subspace, where } \alpha \in \mathfrak{J}\}\ .$$

We call the height $h(\mathfrak{J})$ accessible, if there exists an $\alpha_0 \in \mathfrak{J}$ and a closed linear subspace $T_0 \subset \alpha_0(H)$ with $\dim(T_0) = h(\mathfrak{J})$. Otherwise the height $h(\mathfrak{J})$ is called inaccessible.

We observe that if the height $h(\mathfrak{J})$ is inaccessible, it is necessarily a limit cardinal number.

Lemma 6.1. *Let $\mathfrak{J}$ be a two-sided ideal in $\mathfrak{L}(H)$, let $\alpha \in \mathfrak{J}$, and suppose that $T \subset \subset \alpha(H)$ is a closed linear subspace. Then $\pi_S \in \mathfrak{J}$ for all closed linear subspaces $S \subset H$ with* $\dim(S) \leqq \dim(T)$.

Proof. First, we show that $\pi_T \in \mathfrak{J}$. We consider the map $\pi_T . \alpha \in \mathfrak{J}$. Let $K = = \text{kernel}(\pi_T . \alpha)$. Then $(\pi_T . \alpha)|_{K^\perp} : K^\perp \to T$ is a continuous isomorphism onto the Hilbert space T. We conclude that $\pi_T = (\pi_T . \alpha) . (\iota_{K^\perp}, ((\pi_T . \alpha)|_{K^\perp})^{-1} . \pi'_T) \in \mathfrak{J}$, where $\pi'_T : H \to T$ the orthogonal projection, and $\iota_{K^\perp} : K^\perp \to H$ the natural inclusion.

Next, let $S \subset H$ be a closed linear subspace with $\dim(S) = \dim(T)$. We consider an isomorphism $\gamma : T \to S$. Then $\beta = (\iota_S . \gamma . \pi'_T) . \pi_T \in \mathfrak{J}$, and $\beta(H) = S$. By the preceding argument $\pi_S \in \mathfrak{J}$.

Finally, let $S \subset H$ be a closed linear subspace with $\dim(S) \leqq \dim(T)$. There is a closed linear subspace $S' \subset H$ with $S \subset S'$ and with $\dim(S') = \dim(T)$. As already shown, $\pi_{S'} \in \mathfrak{J}$, and hence $\pi_S = \pi_S . \pi_{S'} \in \mathfrak{J}$.

Theorem 6.2. *Let $\mathfrak{J} \subset \mathfrak{L}(H)$ be a two-sided ideal, and let $\omega = h(\mathfrak{J})$. If the height $h(\mathfrak{J})$ is accessible, $\mathfrak{J} = \mathfrak{C}_{\omega+1}(H)$.*

Proof. Let $\alpha \in \mathfrak{J}$. All closed linear subspaces $T \subset \alpha(H)$ have $\dim(T) = \omega + 1$. Theorem 5.1 implies that α is $(\omega + 1)$-compact, and therefore $\alpha \in \mathfrak{C}_{\omega+1}(H)$.

If $\alpha \in \mathfrak{C}_{\omega+1}(H)$, $\dim(\mathrm{Cl}(\alpha(H))) \leqq \omega$ by lemma 5.4. Let $S = \mathrm{Cl}(\alpha(H))$. Because $h(\mathfrak{J})$ is accessible, lemma 6.1 proves that $\pi_S \in \mathfrak{J}$. Consequently $\alpha = \pi_S . \alpha \in \mathfrak{J}$.

Theorem 6.3. *Let $\mathfrak{J} \subset \mathfrak{L}(H)$ be a two-sided ideal, and let $\omega = h(\mathfrak{J})$. If the height $h(\mathfrak{J})$ is inaccessible, then ω is a limit cardinal number, and $\mathfrak{E}_\omega(H) \subset \mathfrak{J} \subset \mathfrak{C}_\omega(H)$.*

Proof. If $\alpha \in \mathfrak{E}_\omega(H)$, $\dim(\mathrm{Cl}(\alpha(H))) < \omega$. Let $S = \mathrm{Cl}(\alpha(H))$. Since $h(\mathfrak{J})$ is inaccessible, $\pi_S \in \mathfrak{J}$ by lemma 6.1. Therefore $\alpha = \pi_S . \alpha \in \mathfrak{J}$.

If $\alpha \in \mathfrak{J}$, all closed linear subspaces $T \subset \alpha(H)$ have $\dim(T) < \omega$. By theorem 5.1 α is ω-compact, and hence $\alpha \in \mathfrak{C}_\omega(H)$.

Theorem 6.4. *Let $\omega \geqq \aleph_0$ be a cardinal number. Then $\mathrm{Cl}(\mathfrak{E}_\omega(H)) = \mathfrak{C}_\omega(H)$. If ω is not a limit cardinal number, then $\mathfrak{E}_\omega(H) = \mathfrak{C}_\omega(H)$.*

Proof. Let $\alpha \in \mathfrak{C}_\omega(H)$, and let an $\varepsilon > 0$ be given. We apply lemma 5.3. There exists a closed linear subspace $S \subset H$ with $c(S, \alpha) > \varepsilon$ and with $\|\alpha|_{S^\perp}\| < \varepsilon$. By lemma 2.2 $\alpha(S)$ is closed, and by theorem 5.1 $\dim(\alpha(S)) < \omega$. Therefore $\alpha . \pi_S \in \mathfrak{E}_\omega(H)$. We compute $\|\alpha - \alpha . \pi_S\| = \|\alpha . \pi_{S^\perp}\| < \varepsilon$. Thus $\mathrm{Cl}(\mathfrak{E}_\omega(H)) = \mathfrak{C}_\omega(H)$.

If ω is not a limit cardinal number, $h(\mathfrak{E}_\omega(H)) = \omega - 1$, and it is accessible. Theorem 6.2 implies that $\mathfrak{E}_\omega(H) = \mathfrak{C}_\omega(H)$.

Corollary 6.1. *If $\mathfrak{J} \subset \mathfrak{L}(H)$ is a closed two-sided ideal, then there exists a cardinal number ω with $\mathfrak{J} = \mathfrak{C}_\omega(H)$.*

Proof. Theorems 6.2, 6.3, and 6.4.

Corollary 6.2. *The closed two-sided ideals of $\mathfrak{L}(H)$ form the chain*

$$\{0\} \subset \mathfrak{C}(H) = \mathfrak{C}_{\aleph_0}(H) \subset \ldots \subset \mathfrak{C}_\omega(H) \subset \ldots \subset \mathfrak{C}_{\dim(H)+1}(H) = \mathfrak{L}(H),$$

where $\aleph_0 \leqq \omega \leqq \dim(H) + 1$.

References

[1] *N. I. Achieser* and *I. M. Glasmann:* Theorie der linearen Operatoren im Hilbert-Raum, Akademie-Verlag, Berlin, 1954.
[2] *J. W. Calkin:* Twosided ideals and congruences in the ring of bounded operators in Hilbert space, Ann. of Math. *42* (1941), 839—873.
[3] *H. O. Cordes:* On a class of *C*-algebras, Math. Annalen *170* (1967), 283—313.
[4] *J. Dieudonné:* Foundations of modern analysis, Academic Press, 1960.
[5] *N. Dunford* and *J. T. Schwartz:* Linear operators, part I, Interscience Publishers, New York, 1958.
[6] *E. Kamke:* Mengenlehre, Sammlung Göschen Band 999, Berlin 1955.
[7] *L. H. Loomis:* An introduction to abstract harmonic analysis, D. van Nostrand comp., 1953.
[8] *B. S. Nagy:* Spektraldarstellung linearer Transformationen des Hilbertschen Raumes, Springer-Verlag, Berlin, 1942.
[9] *F. Rellich:* Störungstheorie der Spektralzerlegung, Math. Annalen *118* (1942), 462—484.

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