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ON CLOSED MAPS, INCREASING DIMENSION

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The history of the question one can find in the survey [1], written by P. S. ALEXANDROFF. Throughout the present article $f : X \to Y$ will be a closed continuous map from one normal space onto the other with $\dim X < \dim Y < \infty$.[1]) The main object of our interest will be the set $NT = \{y \in Y : f^{-1}y \textit{ is not a single point}\}$. The points of the set NT will be said to be the *points of non-triviality* of the map. The question is: how many points of non-triviality one can find in Y in the situation just described? Recently G. SCORDEV has proved that if X is a paracompact space and $\dim f^{-1}y = 0$ for all $y \in Y$, then $\mathrm{rd}_Y NT \geqq \dim Y - 1$. Recall that $\mathrm{rd}_Y A$, where A is a subset of Y, is defined as $\sup \{\dim A' : A' \subseteq A$ and A' is closed in $Y\}$. The proof of this assertion very essentially depends on some algebraic constructions based on the spectral sequence of the map[2]). Here is the point, where paracompactness of X and the second condition are very essential. We drop both these conditions and get a final theorem, using some purely topological and quite understandable technics.

Theorem. *Let $f : X \to Y$ be a closed continuous map from a normal space X onto some normal space Y. Suppose further that* $\dim X < \infty$, $\dim Y < \infty$. *Then* $\mathrm{rd}_Y NT \geqq \dim Y - \dim X - 1$.[3])

When $\dim X = 0$, this result is better than the Scordev's one: we need not suppose that the space X is paracompact.

Now we shall clarify some notation which will be used in the proof.

Let η be a finite covering of the space X and let y be any point of Y. The least number of elements of the covering η the union of which contains $f^{-1}y$ is said to be the *index of f in y relative to η*, and it is written as $I(f, \eta, y)$, or, briefly, $I(y)$.

Put $Y(\eta) = \{y \in Y : I(f, \eta, y) \geqq 2\}$. It is easy to see that $NT \supseteq Y(\eta)$. As the map f

[1]) Throughout the paper dim is to be understood as the covering dimension of the space under consideration.

[2]) The proof has not been published yet.

[3]) The well known map of the Cantor set onto the unite segment shows that the inequality cannot be strengthened.

is closed and continuous, the set $Y(\eta)$ is closed in Y. Now we are in a position to prove the theorem.

Proof. By the definition of dim, there exists an open finite covering ξ of the space Y such that there is no open finite refinement of ξ the order of which is less than $(\dim Y + 1)$. Put $f^{-1}\xi = \{f^{-1}A : A \in \xi\}$.

Clearly, $f^{-1}\xi$ is an open finite covering of X. Let η be any finite open refinement of $f^{-1}\xi$, the order of which is less or equal to $(\dim X + 1)$. Let us take any open subset U of the space X and put $f^*U = Y \setminus f(X \setminus U)$. The set f^*U is open in Y and $f^{-1}(f^*U) \subset U$. Consider the family $f^*\eta = \{f^*U : U \in \eta\}$. Obviously, each element of the family is contained in some member of ξ. If $y \in \bigcap_{i=1}^{k} f^*U_i$, then

$$f^{-1}y \subset f^{-1}\Big(\bigcap_{i=1}^{k} f^*U_i\Big) = \bigcap_{i=1}^{k} f^{-1}(f^*U_i) \subset \bigcap_{i=1}^{k} U_i\,.$$

Hence, the order of the family $f^*\eta$ is not bigger than the order of η on X, and the last is less or equal to $(\dim X + 1)$. It follows from the definitions that the set of all points of Y which are not covered by $f^*\eta$ coincides with $Y(\eta)$. Suppose that

$$\dim Y(\eta) < \dim Y - \dim X - 1$$

and let us find a contradiction. Pick a closed finite covering λ of $Y(\eta)$ with the order $\leqq \dim Y - \dim X - 1$ the elements of which are contained in elements of ξ. Now, using the normality of Y we can extend λ into a family $\tilde{\lambda}$ of open subsets of Y, each of which is smaller than some element of ξ, such that the order of $\tilde{\lambda}$ is not bigger than the order of λ. Then $\xi^* = f^*\eta \cup \tilde{\lambda}$ is an open finite covering of Y refining ξ, and, clearly,

$$\begin{aligned} &\text{order } \xi^* \leqq \text{order } f^*\eta + \text{order } \tilde{\lambda} \leqq \\ &\leqq \dim X + 1 + \dim Y - \dim X - 1 = \dim Y < \dim Y + 1\,. \end{aligned}$$

Here is the contradiction, which completes the proof.

A problem. Let $f : X \to Y$ be a closed continuous map of a normal zerodimensional (finite-dimensional, countably dimensional) space onto an uncountably dimensional normal space Y. Is it true that $\mathrm{rd}_Y NT$ is not countable? A weaker question: in the situation just described, is it true that $\mathrm{rd}_Y NT$ is uncountably dimensional? I have proved that the answer on the first question is in affirmative when X is a metric space or when X is a continuous image of a separable metric space [2].

Another problem. Put $NT_2 = \{y \in Y : f^{-1}y$ contains more than two points$\}$, and define NT_k for any positive integer k in the obvious way. Is it true that in the situation under consideration

$$\mathrm{rd}_Y NT_k \geqq \dim Y - \dim X - k\,?$$

References

[1] *П. С. Александров:* О некоторых результатах в теории топологических пространств, полученных за последние 25 лет, УМН *15*, (1960), 25—95.

[2] *А. В. Архангельский:* О замкнутых отображениях, бикомпактных множествах и одной задаче П. С. Александрова, Матем. сб. *69* (*111*) (1966), 13—34.

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