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ORBITS OF TRANSFORMATION GROUPS ON CERTAIN GRASSMANN MANIFOLDS

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(*Continuation*)*)

2. THE MANIFOLD $\Gamma^{[k]}$

Let V be a vector space and $W \subset V$ its subspace. Vectors $X_1, X_2, \ldots, X_k \in V$ will be called *linearly independent over* W if, under the canonical projection $q : V \to V/W$, the vectors $qX_1, qX_2, \ldots, qX_k$ are linearly independent.

Now let us introduce the coordinates we shall use in this section.

Theorem 15. *For any point* $\boldsymbol{x} \in A^2$ *and any admissible* $\mathfrak{R}^\alpha$ *with the origin* $\boldsymbol{x}$, *the set* $U^\alpha_{26} \cap \Gamma^1_4$ *is a* $G_c(\boldsymbol{x})$-*covering set of* Γ^1_4.

Proof. As usual, let us denote by $\mathfrak{t}$ the Lie algebra of all infinitesimal translations of A^2; put $\mathfrak{t}^0 = \mathfrak{t} \dot{-} (0)$. Let $\mathfrak{F}_3$ be the Stiefel manifold of all triplets $\{X_1, X_2, X_3\}$, where $X_1, X_2, X_3 \in \mathfrak{g}$ are linearly independent over $\mathfrak{t}$. Then the manifold $\mathfrak{F}_3 \times \mathfrak{t}^0$ can be regarded as a fibre bundle with a projection $p : \mathfrak{F}_3 \times \mathfrak{t}^0 \to \Gamma^1_4$, where the map p is given as follows: for $\{X_1, X_2, X_3\} \in \mathfrak{F}_3$, $X_0 \in \mathfrak{t}^0$, we put $p\{X_0, \{X_1, X_2, X_3\}\} = (X_0, X_1, X_2, X_3) \in \Gamma^1_4$.

With respect to Proposition IV it suffices to prove the following: for any $\boldsymbol{x} \in A^2$ and any admissible $\mathfrak{R}^\alpha$ with the origin $\boldsymbol{x}$ the set $p^{-1}(U^\alpha_{26} \cap \Gamma^1_4)$ is a $G_c(\boldsymbol{x})$-covering set of $\mathfrak{F}_3 \times \mathfrak{t}^0$.

Let $X_0 \in \mathfrak{t}^0$, $\{X_1, X_2, X_3\} \in \mathfrak{F}_3$; then in arbitrary coordinate system $\mathfrak{R}^\alpha$ we can write

$$(61) \qquad X_0 = u^\alpha \frac{\partial}{\partial x^\alpha} + v^\alpha \frac{\partial}{\partial y^\alpha}$$

$$X_i = u_i^\alpha \frac{\partial}{\partial x^\alpha} + v_i^\alpha \frac{\partial}{\partial y^\alpha} + a_i^\alpha x^\alpha \frac{\partial}{\partial x^\alpha} + b_i^\alpha x^\alpha \frac{\partial}{\partial y^\alpha} + c_i^\alpha y^\alpha \frac{\partial}{\partial x^\alpha} + d_i^\alpha y^\alpha \frac{\partial}{\partial y^\alpha}$$

$$i = 1, 2, 3 .$$

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According to the notation of Part I, Section 4, we have

$$E_{26}^{\alpha} = \left(\frac{\partial}{\partial y^{\alpha}}, y^{\alpha}\frac{\partial}{\partial y^{\alpha}}\right), \qquad U_{26}^{\alpha} = \{\mathscr{P} \in \Gamma_4 \mid \mathscr{P} \cap E_{26}^{\alpha} = 0\}.$$

Because the vectors X_1, X_2, X_3 are linearly independent over $\mathfrak{t}$, the matrix

$$\begin{pmatrix} a_1^{\alpha} & b_1^{\alpha} & c_1^{\alpha} & d_1^{\alpha} \\ a_2^{\alpha} & b_2^{\alpha} & c_2^{\alpha} & d_2^{\alpha} \\ a_3^{\alpha} & b_3^{\alpha} & c_3^{\alpha} & d_3^{\alpha} \end{pmatrix}$$

is of rank 3. Let us denote by p_i^{α} the determinant which arises by dropping the i-th column of the matrix. Then we see easily that

$$\{X_0, \{X_1, X_2, X_3\}\} \in p^{-1}(\Gamma_4^1 \cap U_{26}^{\alpha}) \quad \text{if and only if} \quad u^{\alpha} \neq 0\,, \quad p_4^{\alpha} \neq 0\,.$$

Let us introduce the following notation: if f is a local function on a manifold $\mathfrak{M}$, put

$$E(f) = \{q \in \mathfrak{M}; f(q) \neq 0\}\,. \tag{62}$$

In this notation we can write $p^{-1}(\Gamma_4^1 \cap U_{26}^{\alpha}) = E(u^{\alpha}) \times E'(p_4^{\alpha})$, where, of course, $E(u^{\alpha}) \subset \mathfrak{t}^0$, $E'(p_4^{\alpha}) \subset \mathfrak{F}_3$. With respect to Proposition III it remains to prove the following assertions:

For any $\mathfrak{R}^{\alpha}$ with the origin $\mathbf{x} \in A^2$

a) $E(u^{\alpha})$ is a $G_c(\mathbf{x})$-covering set of $\mathfrak{t}^0$,

b) $E'(p_4^{\alpha})$ is a $G_c(\mathbf{x})$-covering set of $\mathfrak{F}_3$.

Let $\mathbf{x} \in A^2$ and let $\mathfrak{R}^{\alpha}$ be a coordinate system with the origin $\mathbf{x}$; then we have obviously

$$\mathfrak{g}_c(\mathbf{x}) = \left(x^{\alpha}\frac{\partial}{\partial x^{\alpha}}, x^{\alpha}\frac{\partial}{\partial y^{\alpha}}, y^{\alpha}\frac{\partial}{\partial x^{\alpha}}, y^{\alpha}\frac{\partial}{\partial y^{\alpha}}\right).$$

From the formulae (2) follows immediately ad $y^{\alpha}(\partial/\partial x^{\alpha})(u^{\alpha}) = v^{\alpha}$ and if $X_0(u^{\alpha}, v^{\alpha}) \in \mathfrak{t}^0$, $u^{\alpha} = 0$, we have $v^{\alpha} \neq 0$. According to Proposition V the assertion *a*) holds.

Further let us denote $\varphi : \mathfrak{g}_c(\mathbf{x}) \to \mathrm{X}(\mathfrak{F}_3)$ the Lie algebra homomorphism induced by the action $G_c(\mathbf{x}) \times \mathfrak{F}_3 \to \mathfrak{F}_3$. Using (2) we obtain easily the following table for the infinitesimal transformations of the functions p_i^{α} on $\mathfrak{F}_3$:

	$\varphi\left(x^{\alpha}\frac{\partial}{\partial x^{\alpha}}\right)$	$\varphi\left(x^{\alpha}\frac{\partial}{\partial y^{\alpha}}\right)$	$\varphi\left(y^{\alpha}\frac{\partial}{\partial x^{\alpha}}\right)$	$\varphi\left(y^{\alpha}\frac{\partial}{\partial y^{\alpha}}\right)$
p_1^{α}	0	p_2^{α}	p_3^{α}	0
p_2^{α}	p_2^{α}	0	$p_1^{\alpha} + p_4^{\alpha}$	$-p_2^{\alpha}$
p_3^{α}	$-p_3^{\alpha}$	$p_1^{\alpha} + p_4^{\alpha}$	0	p_3^{α}
p_4^{α}	0	p_2^{α}	p_3^{α}	0

According to Proposition V we deduce from the table that $E'(p_2^\alpha) \cup E'(p_3^\alpha) \cup E'(p_4^\alpha)$ is a $G_c(\mathbf{x})$-covering set of $\mathfrak{F}_3$ (see (62)) and that $E'(p_4^\alpha)$ is a $G_c(\mathbf{x})$-covering set of $E'(p_2^\alpha) \cup E'(p_3^\alpha) \cup E'(p_4^\alpha)$. Now the assertion $b)$ follows from Proposition I, $b)$. This completes the proof of Theorem 15.

According to Section 7 of the first Part, we have the following practical consequence of Theorem 15: *whenever the one-to-one property of a map* $\mathscr{P} \to \{O_1(\mathscr{P}), \ldots \ldots, O_m(\mathscr{P})\}$, $\mathscr{P} \in \mathfrak{M} \subset \Gamma_4^1$, *is to be proved, we can limit ourselves to coordinate systems with a fixed origin, when expressing all the necessary relations. We can even limit ourselves to an open subset of a coordinate* $G_c(\mathbf{x})$*-type for any* $\mathbf{x} \in A^2$. Let now $\mathscr{P} \in \Gamma_4^1$ and $\mathbf{x} \in A^2$ be arbitrary. According to Theorem 15 there is an $\mathfrak{R}^\alpha$ with the origin $\mathbf{x}$ such that $\mathscr{P} \in U_{26}^\alpha \cap \Gamma_4^1$. If we express each vector $X \in \mathscr{P}$ in the form (1), (we shall omit the index α again) then, in the corresponding coordinate system $\mathfrak{S}_\alpha$ of $\mathfrak{g}$, the block $\mathscr{P}$ is given by equations of the form

$$(63) \qquad d = au_1 + bu_2 + cu_3\,, \quad v = av_1 + bv_2 + cv_3 + uv_0\,.$$

Here $u_1, u_2, u_3, v_0, v_1, v_2, v_3$ *are the coordinates of the block* $\mathscr{P}$ *with respect to the local coordinate system* $\mathfrak{S}_{26}^\alpha$ *or, more briefly, with respect to the coordinate system* $\mathfrak{R}^\alpha$. (The last abbreviation is possible thanks to the fact that only the coordinates $\mathfrak{S}_{26}^\alpha$ will be used on Γ_4^1). Another form of (63) is the following one: for any $\mathscr{P} \in \Gamma_4^1$ and any $\mathfrak{R}^\alpha$ such that $\mathscr{P} \in U_{26}^\alpha$, the block $\mathscr{P}$ is determined by its basis

$$(64) \qquad X_1^\alpha = x\frac{\partial}{\partial x} + u_1 y\frac{\partial}{\partial y} + v_1\frac{\partial}{\partial y}\,, \quad X_2^\alpha = x\frac{\partial}{\partial y} + u_2 y\frac{\partial}{\partial y} + v_2\frac{\partial}{\partial y}\,,$$

$$X_3^\alpha = y\frac{\partial}{\partial x} + u_3 y\frac{\partial}{\partial y} + v_3\frac{\partial}{\partial y}\,, \quad X_\tau^\alpha = \frac{\partial}{\partial x} + v_0\frac{\partial}{\partial y}\,.$$

Let $\mathscr{P} \in \Gamma_4^1$, $X \in \mathscr{P} \dot{-} \mathfrak{t}$. *The set of all d-elements of the form* $\xi = (X + \lambda X_0)$, *where* $X_0 \in \mathscr{P} \cap \mathfrak{t}$ *and* λ *is a real number, will be called a d-line determined in* $\mathscr{P}$ *by the vector* X, *or else, by the directional element* $\eta = (X)$. A vector Y will be referred to as belonging to a d-line ξ if $(Y) \in \xi$. It is obvious that two vectors $X, Y \in (\mathscr{P} \dot{-} (\mathscr{P} \cap \mathfrak{t}))$ belong to the same d-line if and only if they are linearly dependent over $\mathfrak{t}$. If X belongs to $\mathfrak{g}_e$ or to $\mathfrak{g}^0$, then so does any vector Y belonging to the d-line ξ determined by X.

Let us introduce

$$(64a) \qquad \xi_\tau(\mathscr{P}) = \mathscr{P} \cap \mathfrak{t} \quad for \quad \mathscr{P} \in \Gamma_4^1\,.$$

By (64), in any admissible $\mathfrak{R}^\alpha$, the d-element $\xi_\tau(\mathscr{P})$ is generated by the vector X_τ^α.

Let ξ be a d-line of $\mathscr{P}$ and η_0 a d-element of $\mathfrak{t}$. Let be given two vectors X_1, X_2 belonging to ξ and other vectors $Y_1, Y_2 \in \eta_0$. Assume all these vectors to be non-zero. Then either $[X_1, Y_1] = [X_2, Y_2] = 0$, or both $[X_1, Y_1]$ and $[X_2, Y_2]$ are non-zero and they determine the same d-element $\zeta \subset \mathfrak{t}$. In the former case we write $[\eta_0, \hat{\xi}] = 0$ and in the latter $[\eta_0, \hat{\xi}] = \zeta$. Let $X \in \mathfrak{g}$ be given, $X = u\, \partial/\partial x + v\, \partial/\partial y + ax\, \partial/\partial x + bx(\partial/\partial y) + cy(\partial/\partial x) + dy(\partial/\partial y)$. Remind that all singularities of X are given by the system

$$(12) \qquad u + ax + cy = 0\,, \quad v + bx + dy = 0\,,$$

and the same holds when $X \in C\mathfrak{g}$ has imaginary coordinates $a, b, \ldots, v$. Let $X \in \mathscr{P}$ and let $\hat{\xi}$ be the corresponding d-line in $\mathscr{P}$. By eliminating the coordinates u, v from (12) and from the second relation (63), we obtain the equation

$$(65) \qquad a(v_1 - xv_0) + b(x + v_2) + c(v_3 - v_0 y) + dy = 0\,.$$

This equation expresses *the union of singularities of all vectors* $Y \in \hat{\xi}$. We shall call the last set *the set of singularities of the d-line* $\hat{\xi}$, when there is no risk of confusion.

Proposition 16. Let $\hat{\xi}$ be a d-line in $\mathscr{P} \in \Gamma_4^1$. *Then the following cases are possible*:

a) *The equation* (65) *has not any solution.*

b) *The equation* (65) *is fulfilled identically.*

c) (65) *determines a line*; *each point of that line is a single singularity of a single d-element from* $\hat{\xi}$.

d) (65) *determines the pointwise singular line of a unique d-element from* $\hat{\xi}$; *the other d-elements from* $\hat{\xi}$ *do not admit any singularity.*

The cases *a*), *b*), *d*) will occur if and only if $\hat{\xi} \subset \mathfrak{g}^0$.

Proof is obvious from the way we have obtained the equation (65).

We shall express ourselves as follows: in the case *a*): *the d-line* $\hat{\xi}$ *does not admit any singularity*; in the case *b*): *the d-line* $\hat{\xi}$ *has a pointwise singular plane*; in the cases *c*) and *d*): *the d-line* $\hat{\xi}$ *has a pointwise singular line.*

Let us remark that if two d-elements $\xi_1, \xi_2 \subset \mathscr{P} \dotdiv \xi_\tau$ belong to the same d-line $\hat{\xi}$, they have the same homogeneous coordinates a, b, c, d; and inversely. For this reason the homogeneous coordinates a, b, c, d of an d-element $\xi \subset \hat{\xi}$ will be referred to as *the homogeneous coordinates of the d-line* $\hat{\xi}$ (with respect to the corresponding coordinate system).

If a d-line $\hat{\xi} \subset \mathscr{P}$ satisfies the inclusion $\hat{\xi} \subset \mathfrak{g}_e$, then we have the relation $a + d = 0$ for its homogeneous coordinates. From (63) follows

$$(66) \qquad (u_1 + 1)\, a + u_2 b + u_3 c = 0\,.$$

The set of singularities of the d-line is given by the equation

$$(67) \qquad a(v_1 - y - xv_0) + b(x + v_2) + c(v_3 - v_0 y) = 0\,.$$

Consider an invariant decomposition

$$(68) \qquad \Gamma_4^1 = \mathfrak{M}_r \cup \mathfrak{M}_e ,$$

where $\mathfrak{M}_e = \{\mathscr{P} \in \Gamma_4^1 \mid \mathscr{P} \subset \mathfrak{g}_e\}$.

Let us start with the open submanifold $\mathfrak{M}_r$. Obviously $\mathscr{P} \in \mathfrak{M}_r$ if and only if the equation (66) does not vanish identically, i.e., if and only if at least one of the numbers $u_1 + 1, u_2, u_3$ is non-zero. If this is the case we obviously have $\dim [\mathscr{P} \cap \mathfrak{g}_e/\xi_\tau(\mathscr{P})] = 2$. If we denote $d_e\mathscr{P} = [\mathscr{P} \cap \mathfrak{g}_e, \mathscr{P} \cap \mathfrak{g}_e]$ (see Note 2), then $\dim (d_e\mathscr{P}/(d_e\mathscr{P} \cap \mathfrak{t})) = 1$.

Using (64) we see easily that any d-element $\xi \subset d_e\mathscr{P}$ is given by a vector of the form

$$(69) \quad X_d^\alpha = (u_1 + 1)\left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}\right) + 2u_3 x \frac{\partial}{\partial y} + 2u_2 y \frac{\partial}{\partial x} + m \frac{\partial}{\partial x} + n \frac{\partial}{\partial y}$$

in any coordinate system $\mathfrak{R}^\alpha$.

The d-element ξ admits a couple of singular lines (real different, imaginary conjugate, or real coincident) with the common equation

$$(70) \qquad u_3(x - x_0)^2 - (u_1 + 1)(x - x_0)(y - y_0) - u_2(y - y_0)^2 = 0 ,$$

where only the point (x_0, y_0) depends on the choice of ξ in $d_e\mathscr{P}$. (Cf. the formula (3)). Consequently, *with the subspace $d_e\mathscr{P}$ (and thus with the block $\mathscr{P} \in \mathfrak{M}_r$), we can join a couple $k^\infty(\mathscr{P})$ of improper points (real or imaginary conjugate) of the plane CA^2.* Using the homogeneous coordinates v_x, v_y in CA^1_∞ (see Part I, Section 6) we obtain

$$(71) \qquad k^\infty(\mathscr{P}) \equiv u_3 v_x^2 - (u_1 + 1) v_x v_y - u_2 v_y^2 = 0 .$$

Hence $k^\infty(\mathscr{P})$ is an equivariant object from $\mathfrak{M}_r$ into CA^1_∞. Now using (64) again we find easily that $\dim (d_e\mathscr{P} \cap \mathfrak{t}) = \dim [d_e\mathscr{P}, \xi_\tau(\mathscr{P})] \geqq 1$. Consider another invariant decomposition

$$(72) \qquad \mathfrak{M}_r = \mathfrak{M}_r^1 \cup \mathfrak{M}_r^2 ,$$

where $\mathfrak{M}_r^1 = \{\mathscr{P} \in \mathfrak{M}_r \mid \dim (d_e\mathscr{P} \cap \mathfrak{t}) = 1\}$, $\mathfrak{M}_r^2 = \{\mathscr{P} \in \mathfrak{M}_r \mid d_e\mathscr{P} \supset \mathfrak{t}\}$. We deduce easily from the definition that $\mathscr{P} \in \mathfrak{M}_r^2$ if and only if, in any coordinate system, we have

$$(73) \qquad R = u_2 v_0^2 + (u_1 + 1) v_0 - u_3 \neq 0 .$$

Let us denote by

$$(74) \qquad \xi^\infty(\mathscr{P}) \equiv v_y - v_0 v_x = 0 ,$$

the improper singularity of the d-element $\xi_\tau(\mathscr{P}) = (\partial/\partial x + v_0(\partial/\partial y))$. Then the relation (73) possesses an additional geometrical signification: $R \neq 0 \Leftrightarrow \xi^\infty(\mathscr{P}) \notin k^\infty(\mathscr{P})$.

Let now $\mathscr{P} \in \mathfrak{M}_r^2$ be given.

Proposition 17. *There is exactly one point $Q(\mathscr{P})$ in A^2 which is a common singularity for all d-lines $\xi \subset \mathscr{P} \cap \mathfrak{g}_e$. This point is given by*

$$(75) \qquad x = \frac{A - v_0 C}{R}, \quad y = \frac{B - v_0 A}{R},$$

where $A = u_3 v_2 - u_2 v_3$, $B = (1 + u_1) v_3 - u_3 v_1$, $C = (1 + u_1) v_2 - u_2 v_1$ and R is given by (73).

Proof. It suffices to find a point $[x, y]$ satisfying (67), whenever the homogeneous coordinates a, b, c of a d-line fulfil (66). For this it is necessary and sufficient that the matrix

$$\begin{pmatrix} 1 + u_1 & u_2 & u_3 \\ v_1 - y - xv_0 & x + v_2 & v_3 - v_0 y \end{pmatrix}$$

be of rank 1. Denote by $D_1(x, y)$, $-D_2(x, y)$, $D_3(x, y)$ the determinants of that matrix which arise by dropping the first, second or third column. Then we obtain a system of equations

$$D_1(x, y) = D_2(x, y) = D_3(x, y) = 0 .$$

The wanted point must satisfy, in particular, the system

$$D_1(x, y) + v_0 D_3(x, y) = 0 , \quad D_2(x, y) - v_0 D_1(x, y) = 0 ,$$

whence (75) follows. On the other hand, a direct calculation shows that the values (75) actually are solutions of the original system $D_1 = D_2 = D_3$, q.e.d.

Let us denote by $k(\mathscr{P})$ the line couple determined by the couple $k^\infty(\mathscr{P})$ of improper points and by the proper point $Q(\mathscr{P})$. Then $k(\mathscr{P})$ is an equivariant object on $\mathfrak{M}_r^2$. In each coordinate system with the origin $Q(\mathscr{P})$ we have

$$(76) \qquad k(\mathscr{P}) \equiv u_3 x^2 - (u_1 + 1)\, xy - u_2 y^2 = 0 ,$$

$$(77) \qquad A = D_1(0, 0) = 0 , \quad B = -D_2(0, 0) = 0 , \quad C = D_3(0, 0) = 0 .$$

Proposition 18. *Let $\mathscr{P} \in \mathfrak{M}_r^2$. Then there is exactly one d-line $\xi^1 \subset \mathscr{P}$ such that $[\xi^1, \xi_\tau(\mathscr{P})] = 0$. This* d-line *admits a pointwise singular line*

$$(78) \qquad q(\mathscr{P}) \equiv R(y - v_0 x) + v_0 A - (v_0)^2\, C + (v_0)^2\, v_2 + v_0 v_1 - v_3 = 0 .$$

In any coordinate system with the origin $Q(\mathscr{P})$ we have

$$(78') \qquad q(\mathscr{P}) \equiv R(y - v_0 x) + v_0{}^2 v_2 + v_0 v_1 - v_3 = 0 .$$

Proof. We obtain easily the relations $a + v_0 c = 0$, $b + v_0 d = 0$, $d = u_1 a +$

$+ u_2 b + u_3 c$, binding the homogeneous coordinates of $\hat{\xi}^1$. Hence, exact up to a proportionality factor,

$$(79) \qquad a = v_0(1 + u_2 v_0)\,, \quad b = v_0(u_3 - u_1 v_0)$$
$$c = -(1 + u_2 v_0)\,, \quad d = u_1 v_0 - u_3\,.$$

The equation (78) follows from (65), and (78′) will be obtained from (78) and (77), q.e.d.

Because $q(\mathscr{P})$ passes through the improper point $\xi^\infty(\mathscr{P})$ (see (74)), it is not parallel to any line of the couple $k(\mathscr{P})$. It is obvious that the d-line $\hat{\xi}^1$ with coordinates (79) belongs to $\mathfrak{g}^0$ and according to (78) we have the case d) of Proposition 16. Thus there is exactly one d-element $\tilde{\xi} \in \hat{\xi}^1$ having (78) as its pointwise singular line. With regard to $R \neq 0$ we have $a + d \neq 0$ in (79). Thus the case **f**) of Theorem 1 holds. According to the point **k**) of our Theorem there is a singulr line of $\tilde{\xi}$, different from (78), passing through any prescribed point of the plane A^2. One of those singular lines contains $Q(\mathscr{P})$ and it will be denoted by $\alpha(\mathscr{P})$. In any coordinate system $\mathfrak{R}^\alpha$ with the origin $Q(\mathscr{P})$ we have, taking in account (5) and (79),

$$(80) \qquad \alpha(\mathscr{P}) \equiv (1 + u_2 v_0)\, y + (u_1 v_0 - u_3)\, x = 0\,.$$

Now we have constructed a sufficient number of equivariant objects for a representation of $\mathfrak{M}_r^2$, and we can prove the following.

Proposition 19. *Let be given: a point $Q_0 \in A^2$, a couple k_0 of lines (real and different, or imaginary conjugate, or real and coincident) having a double point at Q_0, a real line α_0 passing through Q_0, and another real line q_0, which is non-parallel to α_0 and also to the lines of the couple k_0.*

Let $\mathfrak{R}^\alpha$ be a coordinate system with the following properties:

a) The origin of $\mathfrak{R}^\alpha$ lies at Q_0.

b) The coordinate axes $\vec{x}$, $\vec{y}$ are both non-parallel to q_0, α_0, and to the lines of the couple k_0.

c) If we denote the lines of k_0, taken in any order, by k', k'', then the following relation for the cross-ratia holds:

$$(81) \qquad R(\vec{x}, k', \vec{y}, q_0)\, R(\vec{x}, k'', \vec{y}, q_0) \neq R(\vec{x}, \alpha_0, \vec{y}, q_0)\,.$$

Under these assumptions there is exactly one block $\mathscr{P} \in \mathfrak{M}_r^2 \cap U_{26}^\alpha$ such that $Q(\mathscr{P}) \equiv Q_0$, $k(\mathscr{P}) \equiv k_0$, $q(\mathscr{P}) \equiv q_0$, $\alpha(\mathscr{P}) \equiv \alpha_0$.

Proof. In the prescribed coordinates we can write $Q_0 = [0, 0]$, $k_0 \equiv ax^2 - bxy - y^2 = 0$, $q_0 \equiv y - \beta_0 x + m = 0$, $\alpha_0 \equiv y - nx = 0$ where $n \neq \beta_0$, $a - b\beta_0 - \beta_0^2 \neq 0$. From the condition $Q(\mathscr{P}) \equiv Q_0$ follows (77) and from $k(\mathscr{P}) \equiv k_0$ we obtain $u_3 = au_2$, $1 + u_1 = bu_2$. Now we deduce from (77) $v_3 = av_2$, $v_1 = bv_2$. The condition $q(\mathscr{P}) \equiv q_0$ yields $v_0 = \beta_0$, $(v_0^2 v_2 + v_0 v_1 - v_3)/R = m$. Because

$a - b\beta_0 - \beta_0^2 \neq 0$, it follows from the preceding relations that $v_2 = mu_2$. Finally, from the condition $\alpha(\mathscr{P}) \equiv \alpha$ we deduce $u_2(b\beta_0 + n\beta_0 - a) = \beta_0 - n$. *As a consequence of the assumption* (81), the term in the parentheses is non-zero. Thus $u_2 = (\beta_0 - n)/(b\beta_0 + n\beta_0 - a)$, and $u_2 \neq 0$ since $\beta_0 \neq n$. We have determined certain block $\mathscr{P} \in U_{26}^{\alpha} \cap \Gamma_4^1$. For its coordinates we have $R = u_2(v_0)^2 + (u_1 + 1) v_0 - - u_3 = u_2(\beta_0^2 + b\beta_0 - a) \neq 0$. Hence $\mathscr{P} \in \mathfrak{M}_r^2$ and all objects $Q(\mathscr{P})$, $k(\mathscr{P})$, $q(\mathscr{P})$, $\alpha(\mathscr{P})$ actually exist. Thus the conditions of our Proposition are geometrically satisfied, q.e.d.

Theorem 16. *The objects* $Q(\mathscr{P})$, $k(\mathscr{P})$, $q(\mathscr{P})$, $\alpha(\mathscr{P})$ *form a representing frame on the manifold* $\mathfrak{M}_r^2$.

Proof. Let us denote by $\mathfrak{R}(H, Q(\mathscr{P}))$ the coordinate H-type consisting of all $\mathfrak{R}^\alpha$ with the origin $Q(\mathscr{P})$, where $H = G_c(Q(\mathscr{P}))$. According to Proposition VII it suffices to prove that the coordinate systems $\mathfrak{R}^\alpha$ satisfying the requirements $a)$, $b)$, $c)$ of Proposition 19 form a non empty open set in $\mathfrak{R}(H, Q(\mathscr{P}))$. Obviously it will be sufficient to show this property for the set of all $\mathfrak{R}^\alpha \in \mathfrak{R}(H, Q(\mathscr{P}))$ satisfying (81). But (81) is equivalent to the relation $b\beta_0 + n\beta_0 - a \neq 0$. If we denote by $\varphi : : g_c(Q(\mathscr{P})) \to \chi(\mathfrak{R}(H, Q(\mathscr{P})))$ the Lie algebra homomorphism induced by the action $G_c(Q(\mathscr{P})) \times \mathfrak{R}(H, Q(\mathscr{P})) \to \mathfrak{R}(H, Q(\mathscr{P}))$, we obtain $\varphi(x\, \partial/\partial y)(b\beta_0 + n\beta_0 - a) = = n - \beta_0 \neq 0$ in arbitrary coordinate system $\mathfrak{R}^\alpha \in \mathfrak{R}(H, Q(\mathscr{P}))$. ($a$, b, β_0, n can be understood as local differentiable functions on $\mathfrak{R}(H, Q(\mathscr{P}))$.) After Proposition V, the set of all coordinate systems in question is a H-covering set of $\mathfrak{R}(H, Q(\mathscr{P}))$, and hence follows our assertion.

From Proposition 19 and Theorem 16 we obtain the following *classification of orbits on the manifold* $\mathfrak{M}_r^2$:

Consider an invariant decomposition

$$\mathfrak{M}_r^2 = \bigcup \mathfrak{M}_r^2(i, j, k) = \bigcup \mathfrak{M}(i, j, k) \tag{82}$$

where $i = \operatorname{sgn}\left[(u_1 + 1)^2 + 4u_2u_3\right]$ (cf. (76)),

$$j = \begin{cases} 0 \ldots Q(\mathscr{P}) \in q(\mathscr{P}) \\ 1 \ldots Q(\mathscr{P}) \notin q(\mathscr{P}) \end{cases} \qquad k = \begin{cases} 0 \ldots \alpha(\mathscr{P}) \subset k(\mathscr{P}) \\ 1 \ldots \alpha(\mathscr{P}) \not\subset k(\mathscr{P}) \, . \end{cases}$$

Then

$\mathfrak{M}(1, 1, 1)$, $\mathfrak{M}(-1, 1, 1)$ *consist of* ∞^1 *orbits of dimension* 6 *each, any orbit is characterized by a division ratio of three points of the line* $q(\mathscr{P})$,

$\mathfrak{M}(0, 1, 1)$ *consists of* 2 *orbits of dimension* 6,

$\mathfrak{M}(1, 0, 1)$ *and* $\mathfrak{M}(-1, 0, 1)$ *consist of* ∞^1 *orbits of dimension* 5 *each, any orbit is characterized by a cross-ratio of* 4 *lines of the pencil with the center* $Q(\mathscr{P})$,

$\mathfrak{M}(0, 0, 1)$ *consists of* 2 *orbits of dimension* 5,

$\mathfrak{M}(1, 1, 0)$ *consists of* 2 *orbits of dimension* 6,

$\mathfrak{M}(-1, 1, 0) = \emptyset$, $\mathfrak{M}(-1, 0, 0) = \emptyset$,
$\mathfrak{M}(0, 1, 0)$ *is an orbit of dimension* 5,
$\mathfrak{M}(1, 0, 0)$ *consists of* 2 *orbits of dimension* 5,
$\mathfrak{M}(0, 0, 0)$ *is an orbit of dimension* 4.

*

Let us consider now the manifold $\mathfrak{M}_r^1$. If $\mathscr{P} \in \mathfrak{M}_r^1$, then $\xi^\infty(\mathscr{P}) \in k^\infty(\mathscr{P})$ (cf. (71), (74)), and we have $R = 0$. The couple $k^\infty(\mathscr{P})$ is real. Let us remark first that the d-line $\hat{\xi}^1$ from Proposition 18 is defined on the manifold $\mathfrak{M}_r^1 \cup \mathfrak{M}_e$, too. Only its homogeneous coordinates (79) assume, because of $R = 0$, a simpler form

$$(83) \qquad a = v_0\,, \quad b = v_0^2\,, \quad c = -1\,, \quad d = -v_0\,.$$

Proposition 20. *For $\mathscr{P} \in \mathfrak{M}_r^1 \cup \mathfrak{M}_e$ the d-line $\hat{\xi}^1$ constructed in Proposition* 18 *either does not admit any singularity, or it admits a pointwise singular plane. The last case arises if and only if*

$$(84) \qquad W = v_0 v_1 + v_0^2 v_2 - v_3 = 0\,.$$

Proof. We apply (67) and (83).

Because of reality of the couple $k^\infty(\mathscr{P})$ we have an invariant decomposition

$$(85) \qquad \mathfrak{M}_r^1 = \mathfrak{M}_r^1(1) \cup \mathfrak{M}_r^1(0)$$

according to the sign of the discriminant $(u_1 + 1)^2 + 4u_2u_3$ (see (71)). For any block $\mathscr{P} \in \mathfrak{M}_r^1(1)$, *let us limit ourselves to the coordinate systems $\mathfrak{R}^\alpha$ such that no coordinate axis passes through a point of $k^\infty(\mathscr{P})$.* (Cf. the asumption **B** 1) of Proposition VI.) For any admissible coordinate system $\mathfrak{R}^\alpha$ and a block $\mathscr{P} \in U_{26}^\alpha$ the coordinates u_2, u_3, v_0 of $\mathscr{P}$ are non-zero. We find easily that the second improper point $\eta^\infty(\mathscr{P})$ of the couple $k^\infty(\mathscr{P})$ is given by

$$(86) \qquad \eta^\infty(\mathscr{P}) \equiv v_y - wv_x = 0\,, \quad w = -\frac{u_3}{u_2 v_0}\,, \quad w \neq v_0\,.$$

Here we use admissible coordinates for $\mathscr{P}$ and the corresponding homogeneous coordinates v_x, v_y in A_∞^1. There is exactly one d-element $\eta_\tau(\mathscr{P}) \subset \mathfrak{t}$ having the improper singularity $\eta^\infty(\mathscr{P})$; it is given by

$$(87) \qquad \eta_\tau(\mathscr{P}) = \left(\frac{\partial}{\partial x} + w\frac{\partial}{\partial y}\right).$$

Let us remark that the following relations are valid on the manifold $\mathfrak{M}_r^1$:

$$(88) \qquad u_3 = -v_0 w u_2\,, \quad u_1 + 1 = -(v_0 + w)\,u_2\,,$$
$$(u_1 + 1)\,w + u_2 w^2 - u_3 = 0\,.$$

Assume in the following that $\mathscr{P} \in \mathfrak{M}_r^1(1)$.

Proposition 21. *There is exactly one* d-*line* $\hat{\eta}^1$ *in* $\mathscr{P} \cap \mathfrak{g}_e$ *such that* $[\eta_\tau(\mathscr{P}), \hat{\eta}^1] = 0$. *The homogeneous coördinates of that d-line are given by*

$$a = w\,, \quad b = w^2\,, \quad c = -1\,, \quad d = -w \tag{89}$$

and the corresponding pointwise singular line has the equation

$$r(\mathscr{P}) \equiv (w - v_0)(wx - y) + wv_1 + w^2v_2 - v_3 = 0\,. \tag{90}$$

Proof. For determining $\hat{\eta}^1$ we have the system $a + wc = 0$, $b - wa = 0$, $(u_1 + 1)\,a + u_2b + u_3c = 0$, the rank of which is always 2. From the first and second relation and using the condition $a + d = 0$ we obtain (89). The formula (90) follows from (67).

Proposition 22. *There is exactly one d-line* $\hat{\eta}^2 \subset \mathscr{P}$ *such that*

$$[\xi_\tau(\mathscr{P}), \hat{\eta}^2] \subseteq \xi_\tau(\mathscr{P})\,, \quad [\eta_\tau(\mathscr{P}), \hat{\eta}^2] \subseteq \eta_\tau(\mathscr{P})\,.$$

This d-line has homogeneous coordinates

$$a = 1 + u_1 + 2u_2u_3\,, \quad b = (1 - u_1)\,u_3\,, \quad c = (1 - u_1)\,u_2\,, \tag{91}$$

$$d = u_1 + u_1^2 + 2u_2u_3\,.$$

Proof. For determining $\hat{\eta}^2$ we have the system $(a + cv_0)\,v_0 = b + dv_0$, $(a + cw)\,w = b + dw$, $d = u_1a + u_2b + u_3c$. It is obvious that the values (91) satisfy these equations. It remains to show that our functions are not all equal to zero and that the system above is of rank 3. First of all, from the relations $a = b = c = d = 0$ in (91) would follow $u_1 = 1$, $1 + u_2u_3 = 0$, whence $(1 + u_1)^2 + 4u_2u_3 = u_2^2(w - v_0)^2 = 0$ – a contradiction. Now let us write the system in the usual form, where the right sides are all zeros. If both the determinants formed by the coefficients of the unknowns a, b, d and by the coefficients of the unknowns a, b, c respectively were zero, we should easily obtain $u_1 = 1$, $u_1(u_1 + 1) + 2u_2u_3 = 0$ and hence $1 + u_2u_3 = 0$ – a contradiction. Thus the system is of rank 3, q.e.d. We can see easily that the relations $[\xi_\tau(\mathscr{P}), \hat{\eta}^2] = 0$, $[\eta_\tau(\mathscr{P}), \hat{\eta}^2] = 0$ can not hold simultaneously on $\mathfrak{M}_r^1(1)$; otherwise the coordinates a, b, c, d of η^2 satisfy the relations $a + cv_0 = 0$, $a + cw = 0$, $b + dv_0 = 0$, $b + dw = 0$, and hence they are all equal to zero. Let us consider an invariant decomposition

$$\mathfrak{M}_r^1(1) = \mathfrak{M}_p \cup \mathfrak{M}_v \cup \mathfrak{M}_w \tag{92}$$

where

$$\mathscr{P} \in \mathfrak{M}_p \Leftrightarrow [\xi_\tau(\mathscr{P}), \hat{\eta}^2] = \xi_\tau(\mathscr{P})\,, \quad [\eta_\tau(\mathscr{P}), \hat{\eta}^2] = \eta_\tau(\mathscr{P})\,,$$

$$\mathscr{P} \in \mathfrak{M}_v \Leftrightarrow [\xi_\tau(\mathscr{P}), \hat{\eta}^2] = 0\,,$$

$$\mathscr{P} \in \mathfrak{M}_w \Leftrightarrow [\eta_\tau(\mathscr{P}), \hat{\eta}^2] = 0\,.$$

It is obvious that for $\mathscr{P} \in \mathfrak{M}_r^1(1)$ we have

(93) $$\mathscr{P} \in \mathfrak{M}_v \Leftrightarrow 1 + u_2 v_0 = 0 \,, \quad \mathscr{P} \in \mathfrak{M}_w \Leftrightarrow 1 + u_2 w = 0 \,.$$

Proposition 23. *Let $\mathscr{P} \in \mathfrak{M}_p \cup \mathfrak{M}_v$. Then there is a pointwise singular line $s(\mathscr{P})$ corresponding to the d-line $\hat{\eta}^2$ from Proposition 22. It is given by*

(94) $$s(\mathscr{P}) \equiv u_2(w - v_0)(1 + wu_2)[v_0 x - y] + \\ + u_2(w + v_0 + 2u_2 w v_0) v_1 + u_2(2 + (w + v_0) u_2)(w v_0 v_2 - v_3) = 0$$

Proof. We use (65), (91) and (88). Further we take into account that for $\mathscr{P} \in \mathfrak{M}_p \cup \cup \mathfrak{M}_v$ we have $1 + wu_2 \neq 0$ (cf. (93)). For any $\mathscr{P} \in \mathfrak{M}_p \cup \mathfrak{M}_v$, let us denote by $H(\mathscr{P})$ the intersection point of the lines $r(\mathscr{P})$, $s(\mathscr{P})$, defined in (90) and (94). (Those lines are not parallels because of $w \neq v_0$.) Choose a coordinate system $\mathfrak{R}^\alpha$ with the origin $H(\mathscr{P})$. We obtain

(95) $$r(\mathscr{P}) \equiv y - wx = 0 \,, \quad s(\mathscr{P}) \equiv y - v_0 x = 0$$

(96) $$w v_1 + w^2 v_2 - v_3 = 0 \,,$$
$$(w + v_0 + 2u_2 w v_0) v_1 + w v_0[2 + (w + v_0) u_2] v_2 - [2 + (w + v_0) u_2] v_3 = 0 \,.$$

Proposition 24. *Let $\mathscr{P} \in \mathfrak{M}_p$. Then the following assertions hold:*

a) There is exactly one d-element $\eta^1 \in \hat{\eta}^1$ having singularities in A^2. This d-element is generated, in any coordinate system $\mathfrak{R}^\alpha$ with the origin $H(\mathscr{P})$, by the vector

(97) $$Z_1^\alpha = wx \frac{\partial}{\partial x} + w^2 x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} - wy \frac{\partial}{\partial y} \,.$$

b) There is exactly one d-element $\eta^2 \subset \hat{\eta}^2$ having a singularity at the point $H(\mathscr{P})$. η^2 is generated, in any coordinate system $\mathfrak{R}^\alpha$ with the origin $H(\mathscr{P})$, by the vector

(98) $$Z_2^\alpha = (u_1 + 1 + 2u_2 u_3) x \frac{\partial}{\partial x} + u_3(1 - u_1) x \frac{\partial}{\partial y} + u_2(1 - u_1) y \frac{\partial}{\partial x} + \\ + (u_1 + u_1^2 + 2u_2 u_3) y \frac{\partial}{\partial y} \,.$$

c) There is exactly one d-element $\xi^1 \subset \hat{\xi}^1$ such that the d-element $[\xi^1, \eta^1]$ has a singularity at the point $H(\mathscr{P})$. This d-element is generated, in any $\mathfrak{R}^\alpha$ with the origin $H(\mathscr{P})$, by the vector

(99) $$Y_1^\alpha = v_0 x \frac{\partial}{\partial x} + v_0^2 x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} - v_0 y \frac{\partial}{\partial y} + \frac{W}{w - v_0} \left(\frac{\partial}{\partial x} + w \frac{\partial}{\partial y} \right)$$

where the function W is given by (84).

Proof. ad a) According to Proposition 21 $\hat{\eta}^1 \subset \mathfrak{g}^0$ (cf. (89)) and obviously the case d) of Proposition 16 holds. Thus there is exactly one d-element $\eta^1 \subset \hat{\eta}^1$ having the line $r(\mathscr{P})$ as its pointwise singular line; in particular, having a singularity at $H(\mathscr{P})$. With respect to our choice of the coordinate origin and with respect to (12) we obtain $u = v = 0$ for η^1. From (89) we obtain (97).

ad b) We find easily that $\hat{\eta}^2 \subset \mathfrak{g}^0$ if and only if either $\mathscr{P} \in \mathfrak{M}_v$ or $\mathscr{P} \in \mathfrak{M}_w$. Thus we have $\hat{\eta}^2 \not\subset \mathfrak{g}^0$ and the case c) of Proposition 16 holds. From (12) follows $u = v = 0$ and with respect to (91) we obtain (98).

ad c) For any $\xi \subset \hat{\xi}^1$ we have $[\xi, \eta^1] \subset d_e\mathscr{P}$ because $\hat{\xi}^1 \subset \mathfrak{g}_e$, $\hat{\eta}^1 \subset \mathfrak{g}_e$. Thus the d-element $[\xi, \eta^1]$ is generated by a vector of the form (69). Since $(u_1 + 1)^2 + 4u_2u_3 = u_2^2(w - v_0)^2 \neq 0$, we have $[\xi, \eta^1] \not\subset \mathfrak{g}^0$. Thus there is at most one d-element of the form $[\xi, \eta^1]$ having a singularity at $H(\mathscr{P})$. Expressing it in the form (69) we obtain $m = n = 0$. By direct calculation we find that $[\xi, \eta^1]$ possesses that property if and only if ξ is given by (99), q.e.d.

Let $X_0 \in \xi_\tau(\mathscr{P})$, $Y_0 \in \eta_\tau(\mathscr{P})$ be arbitrary vectors. From (98) we see that $[X_0, Z_2^\alpha] = (v_0 - w)\, u_2(1 + u_2v_0)\, X_0$, $[Y_0, Z_2^\alpha] = (w - v_0)\, u_2(1 + u_2w)\, Y_0$. Hence follows

Proposition 25. *There is exactly one vector $Z_2^* \in \eta^2$ such that $[X_0, Z_2^*] = X_0$ for any $X_0 \in \xi_\tau(\mathscr{P})$, and exactly one vector $Z_2^{**} \in \eta^2$ such that $[Y_0, Z_2^{**}] = Y_0$ for any $Y_0 \in \eta_\tau(\mathscr{P})$. Moreover we have*

$$(100) \qquad Z_2^{**} = \lambda(\mathscr{P})\, Z_2^*\,, \quad \lambda(\mathscr{P}) = -\frac{1 + v_0u_2}{1 + wu_2}\,, \quad \lambda(\mathscr{P}) \neq 0, -1\,.$$

$\lambda(\mathscr{P})$ is a point invariant on the manifold $\mathfrak{M}_p$.

Let us introduce another invariant decomposition

$$(101) \qquad \mathfrak{M}_p = \mathfrak{M}_p^* \cup \mathfrak{M}_p^0$$

where $\mathfrak{M}_p^0 = \{\mathscr{P} \in \mathfrak{M}_p \mid W = 0\}$. (Cf. (84)).

Denote by $S(\mathscr{P})$ the set of all d-lines $\hat{\eta} \subset \mathscr{P} \cap \mathfrak{g}^0$ and by $\sum(\mathscr{P})$ the corresponding set of pointwise singular lines. (For the existence of a pointwise singular line it suffices that $\hat{\eta} \neq \hat{\xi}^1$.)

Proposition 26. *Let $\mathscr{P} \in \mathfrak{M}_p^*$. The set $\sum(\mathscr{P})$ is a one-parametric family of lines in A^2; its envelope is a parabola given by the equation*

$$(102) \qquad p(\mathscr{P}) \equiv (w - v_0)(1 + wu_2)(v_0x - y)^2 - 4(1 + v_0u_2)\, W(wx - y) = 0$$

in any $\mathfrak{R}^\alpha$ with the origin $H(\mathscr{P})$.

Proof. Any d-line $\hat{\eta} \in S(\mathscr{P})$, $\hat{\eta} \neq \hat{\xi}^1, \hat{\eta}^1$ can be represented, in any coordinate system $\mathfrak{R}^\alpha$ with the origin $H(\mathscr{P})$, by a vector of the form $Z^\alpha = Z_2^\alpha + f^\alpha Z_1^\alpha + g^\alpha Y_1^\alpha$, where f^α, g^α are real numbers. In fact, we find easily that the vectors Z_1^α, Z_2^α, Y_1^α

given in (97)–(99) are linearly independent over $\mathfrak{k}$. From the inclusion $Z^\alpha \in \mathfrak{g}^0$ we obtain

(103) $$f^\alpha g^\alpha = (u_2)^2 (1 + u_2 v_0)(1 + u_2 w).$$

Hence we can see that $S(\mathscr{P})$ is a one-parametric family and that the pointwise singular line corresponding to a d-line of the family is given by

(104) $$f^\alpha F_2(x, y) + (f^\alpha)^2 F_1(x, y) + (u_2)^2 (1 + u_2 v_0)(1 + u_2 w) F_3(x, y) = 0,$$

$F_2(x, y)$ denoting the left side of the equation (94), $F_1(x, y)$ the left side of (90) and $F_3(x, y) = W$. Here (96) holds with respect to our choice of coordinates. The equation of the envelope $[F_2(x, y)]^2 - 4(u_2)^2 (1 + u_2 v_0)(1 + w u_2) W F_1(x, y) = 0$ can be re-written in the form (102), q.e.d. Let us remark that the parabola $p(\mathscr{P})$ contacts the line $r(\mathscr{P})$ at the point $H(\mathscr{P})$ and the improper line A^1_∞ at the point $\xi^\infty(\mathscr{P})$.

Denote now by $\mathfrak{M}^*_{p\lambda}$ the subset of the manifold $\mathfrak{M}^*_p$ determined by the equation $\lambda(\mathscr{P}) = \lambda$, where $\lambda \neq 0, -1$ is an arbitrary real number. (See Proposition 25.)

Proposition 27. *Let $p_0 \subset A^2$ be a parabola and H_0 a point of p_0. Let $\lambda \neq 0, -1$ be a real number. Let us denote by r_0 the tangent of p_0 at H_0, and by s_0 the line joining the point H_0 with the improper point of tangency of the parabola. Further let $\mathfrak{R}^\alpha$ be a coordinate system with the origin $H(\mathscr{P})$ satisfying the relation*

(105) $$R(r_0, s_0, \vec{x}, \vec{y}) + \lambda \neq 0$$

and such that neither of the axes $\vec{x}, \vec{y}$ is parallel to r_0 or s_0. Then there is exactly one block $\mathscr{P} \in \mathfrak{M}^*_{p\lambda} \cap U^\alpha_{26}$ such that $p(\mathscr{P}) \equiv p_0$, $H(\mathscr{P}) \equiv H_0$.

Proof. From our conditions of coincidence follows, in particular, $r(\mathscr{P}) \equiv r_0$, $s(\mathscr{P}) \equiv s_0$. Put $r_0 \equiv y - w_0 x = 0$, $s_0 \equiv y - \beta_0 x = 0$. Then there is a number $m \neq 0$ such that $p_0 \equiv (y - \beta_0 x)^2 - m(w_0 x - y) = 0$. The conditions $r(\mathscr{P}) \equiv r_0$, $s(\mathscr{P}) \equiv s_0$ and (95) imply $v_0 = \beta_0$, $w = w_0$. With respect to the requirement $\lambda(\mathscr{P}) = \lambda$ we have $-(1 + u_2\beta_0)/(1 + u_2 w_0) = \lambda$, whence $(\lambda w_0 + \beta_0) u_2 + (\lambda + 1) = 0$. According to (105) $\lambda w_0 + \beta_0 \neq 0$, and thus

(106) $$u_2 = -\frac{\lambda + 1}{\lambda w_0 + \beta_0} \neq 0$$

because $\lambda + 1 \neq 0$. Using the condition $p(\mathscr{P}) \equiv p_0$ and (102) we obtain $[4(1 + u_2\beta_0) W]/[(1 + w_0 u_2)(w_0 - \beta_0)] = m$, and taking into account (106) we obtain $W = \beta_0 v_1 + (\beta_0)^2 v_2 - v_3 = [(\beta_0 - w_0) m]/4\lambda \neq 0$. We have found a linear equation for v_1, v_2, v_3. From (96) follow other linear equations

$$w_0 v_1 + (w_0)^2 v_2 - v_3 = 0,$$
$$(w_0 + \beta_0 + 2u_2 w_0 \beta_0) v_1 + w_0\beta_0[2 + (w_0 + \beta_0) u_2] v_2 -$$
$$- [2 + (w_0 + \beta_0) u_2] v_3 = 0.$$

The determinant of coefficients of the system is equal to $(\beta_0 - w_0)^3 \neq 0$, whence v_1, v_2 and v_3 are uniquely determined. The remaining coordinates u_1, u_3 will be obtained using (88) and (106). It is obvious that the block $\mathscr{P}$ just evaluated belongs to $\mathfrak{M}^*_{p\lambda}$ and it satisfies all demands of the Proposition. Especially we have $W \neq 0$ and $1 + u_2 v_0 \neq 0$, $1 + u_2 w \neq 0$ follows easily from (106) hence $\mathscr{P} \in \mathfrak{M}^*_p$, q.e.d. If we apply Proposition VII to the last one, we obtain

Theorem 17. *The equivariant objects $p(\mathscr{P})$, $H(\mathscr{P}) \in p(\mathscr{P})$ form a representing frame on each submanifold $\mathfrak{M}^*_{p\lambda}$. $\mathfrak{M}^*_p$ consists of ∞^1 orbits of dimension 5. Each orbit is determined by the value of the invariant*

$$\lambda(\mathscr{P}) = -\frac{1 + u_2 v_0}{1 + u_2 w}.$$

Similarly, let us denote by $\mathfrak{M}^0_{p\lambda} \subset \mathfrak{M}^0_p$ the submanifold determined by the relation $\lambda(\mathscr{P}) = \lambda$; $\lambda \neq 0, -1$.

Theorem 18. *The lines $r(\mathscr{P})$, $s(\mathscr{P})$ form a representing frame on each manifold $\mathfrak{M}^0_{p\lambda}$. The manifold $\mathfrak{M}^0_p$ consists of ∞^1 orbits of dimension 4. Each orbit is determined by a value of the invariant $\lambda(\mathscr{P})$.*

Proof is quite similar to that of Theorem 17. Only instead of an additional object $p(\mathscr{P})$ we have an additional relation $W = 0$. For the investigation of the manifolds $\mathfrak{M}_v$, $\mathfrak{M}_w$ we shall need the following Proposition:

Proposition 28. *Let $\mathscr{P}'$, $\mathscr{P}''$ be two blocks in Γ^1_4 such that, in a suitable coordinate system $\mathfrak{R}^\alpha$, we have $u'_i = u''_i$ for $i = 1, 2, 3$, $v'_0 = v''_0$, and $v'_i = \varrho v''_i$ for $i = 1, 2, 3$, where the triplet (v'_1, v'_2, v'_3) is non-zero (and consequently $\varrho \neq 0$). Let h be the dilatation from the origin of the coordinate system $\mathfrak{R}^\alpha$ with the modul ϱ. Then*

$$\mathscr{P}' = h \,.\, \mathscr{P}''.$$

Proof can be performed by direct computation. Now let us consider an invariant decomposition

$$\mathfrak{M}_v = \mathfrak{M}^*_v \cup \mathfrak{M}^0_v, \tag{107}$$

where $\mathfrak{M}^0_v = \{\mathscr{P} \in \mathfrak{M}_v \mid W = 0\}$.

Theorem 19. *$\mathfrak{M}^*_v$ is an orbit of dimension 5. $\mathfrak{M}^0_v$ is an orbit of dimension 4.*

Proof. Let r_0, s_0 be two non-parallel lines in A^2 with the intersection point H_0. Choose $\mathfrak{R}^\alpha$ with the origin at H_0 and such that neither of the axes $\vec{x}$, $\vec{y}$ coincides with r_0 or with s_0. Let $\mathscr{P}'$, $\mathscr{P}''$ be two blocks from the set $\mathfrak{M}^\alpha(r_0, s_0) = \{\mathscr{P} \in \mathfrak{M}^*_v \cap U^\alpha_{26} \mid r(\mathscr{P}) \equiv r_0, s(\mathscr{P}) \equiv s_0\}$. Let us put $r_0 \equiv y - w_0 x = 0$, $s_0 \equiv y - \beta_0 x = 0$.

From (95) follows $v_0' = v_0'' = \beta_0$, $w' = w'' = w_0$ and with regard to (91) we obtain $u_2' = u_2'' = -1/\beta_0$. From (88) we have $u_3' = u_3'' = w_0$, $u_1' = u_1'' = -w_0/\beta_0$. The relations (96) give two independent linear equations for v_1, v_2, v_3 and since $W' \neq 0$, $W'' \neq 0$, the triplets (v_1', v_2', v_3'), (v_1'', v_2'', v_3'') are non-zero and proportional to each other. According to Proposition 28 there is a dilatation h from H_0 in A^2 such that $\mathscr{P}' = h \,.\, \mathscr{P}''$. Thus the set $\mathfrak{M}(r_0, s_0) = \{\mathscr{P} \in \mathfrak{M}_v^* \mid r(\mathscr{P}) \equiv r_0,\ s(\mathscr{P}) \equiv s_0\}$ is an orbit of dimension 1 in $\mathfrak{M}_v^*$. Now all configurations $\{r_0, s_0\}$ form an orbit of dimension 4 in A^2; consequently $\mathfrak{M}_v^*$ is an orbit of dimension 5, q.e.d.

Further we can show easily that the objects $r(\mathscr{P})$, $s(\mathscr{P})$ form a representing frame on the manifold $\mathfrak{M}_v^0$ and thus $\mathfrak{M}_v^0$ is an orbit of dimension 4. *It will be very natural to join this orbit to the one-parametric system* $\{\mathfrak{M}_{p\lambda}^0\}$ *as the element corresponding to the value* $\lambda(\mathscr{P}) = 0$. (See (100).)

On the other hand we have not any reason for joining the orbit $\mathfrak{M}_v^*$ to the system $\{\mathfrak{M}_{p\lambda}^*\}$. (Their isotropy groups are of different types.)

*

Let us consider the manifold $\mathfrak{M}_w$. We start with the following problem: *Find all the d-lines* $\eta \subset \mathscr{P}$ *for which* A^2 *is a pointwise singular plane.* For the homogeneous coordinates of the wanted d-lines we obtain the system (see (65)): $b - av_0 = 0$, $d - cv_0 = 0$, $au_1 + bu_2 + cu_3 = d$, $av_1 + bv_2 + cv_3 = 0$. Now from the relations $R = 0$, $1 + u_2 w = 0$ follows $u_3 = v_0$, $u_1 + u_2 v_0 = 0$ on the manifold $\mathfrak{M}_w$. Then our system of equations can be re-written as a new one, consisting of the relation $d = au_1 + bu_2 + cu_3$ and of two equations

$$(108) \qquad b - av_0 = 0\,, \quad a(v_1 + v_0 v_2) + cv_3 = 0\,.$$

Let us consider an invariant decomposition

$$(109) \qquad \mathfrak{M}_w = \mathfrak{M}_w^1 \cup \mathfrak{M}_w^2$$

according to the rank of the system (108).

Let $\mathscr{P} \in \mathfrak{M}_w^2$, then the numbers $v_1 + v_0 v_2$, v_3 are not both equal to zero and there is exactly one d-line $\hat{\eta}^3$ meeting our demands. Its homogeneous coordinates are given by

$$(110) \quad a = -v_3\,, \quad b = -v_0 v_3\,, \quad c = v_1 + v_0 v_2\,, \quad d = -v_0(v_1 + v_0 v_2)\,.$$

Now there is exactly one d-element $\zeta_\tau(\mathscr{P}) \subset \mathfrak{t}$ such that $[\zeta_\tau(\mathscr{P}), \hat{\eta}^3] = 0$. This d-element is determined, in any $\mathfrak{R}^\alpha$, by the vector

$$Z_\tau'^\alpha = (v_1 + v_0 v_2)\frac{\partial}{\partial x} + v_3 \frac{\partial}{\partial y}\,.$$

Assume that neither of the coordinate axes of $\mathfrak{R}^\alpha$ is singular with respect to the d-element $\zeta_\tau(\mathscr{P})$; then $\zeta_\tau(\mathscr{P})$ can be represented by a vector

$$(111) \qquad Z_\tau^\alpha = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y}, \quad z = v_3/(v_1 + v_0 v_2).$$

Let us denote by $\zeta_\tau^\infty(\mathscr{P})$ the improper singularity of the d-element $\zeta_\tau(\mathscr{P})$; then we have

$$(112) \qquad \zeta_\tau^\infty(\mathscr{P}) \equiv v_y - z v_x = 0.$$

Further, consider an invariant decomposition defined as follows:

$$(113) \qquad \begin{aligned} \mathfrak{M}_w^2 &= \mathfrak{M}_{vw}^{2,*} \cup \mathfrak{M}_w^{2,v} \cup \mathfrak{M}_w^{2,w} \\ \mathfrak{M}_w^{2,*} &= \{\mathscr{P} \in \mathfrak{M}_w^2 \mid \zeta^\infty(\mathscr{P}) \notin k^\infty(\mathscr{P})\} \\ \mathfrak{M}_w^{2,v} &= \{\mathscr{P} \in \mathfrak{M}_w^2 \mid \zeta^\infty(\mathscr{P}) \equiv \xi^\infty(\mathscr{P})\} \\ \mathfrak{M}_w^{2,w} &= \{\mathscr{P} \in \mathfrak{M}_w^2 \mid \zeta^\infty(\mathscr{P}) \equiv \eta^\infty(\mathscr{P})\} \end{aligned}$$

(Cf. (112), (74), (86).) Obviously $\mathscr{P} \in \mathfrak{M}_w^{2,*}$ if and only if $z \neq w$ and $z \neq v_0$.

Proposition 29. *Let $\mathscr{P} \in \mathfrak{M}_w^{2,*}$. Then there is exactly one d-line $\hat{\eta}^4 \subset \mathscr{P}$ such that $[\eta_\tau(\mathscr{P}), \hat{\eta}^4] = 0$, $[\zeta_\tau(\mathscr{P}), \hat{\eta}^4] = \zeta_\tau(\mathscr{P})$. This d-line has homogeneous coordinates*

$$(114) \qquad a = w, \quad b = wz, \quad c = -1, \quad d = -z$$

and it admits a pointwise singular line

$$(115) \qquad r_1(\mathscr{P}) \equiv (v_0 - z)(y - wx) + wv_1 + wzv_2 - v_3 = 0.$$

Proof. First we solve the system $a + wc = 0$, $b + wd = 0$, $(a + cz)z = b + zd$, $d = u_1 a + u_2 b + u_3 c$, which is of rank 3. Then we use (65). Choose a coordinate system $\mathfrak{R}_\alpha$ with the origin lying on $r(\mathscr{P})$ (see (90)). Then we have $r(\mathscr{P}) \equiv y - wx = 0$, $wv_1 + w^2 v_2 - v_3 = 0$ and

$$(115') \qquad r_1(\mathscr{P}) \equiv (v_0 - z)(y - wx) + wv_2(z - w) = 0.$$

Now $z - w = (v_3 - v_1 w - v_0 v_2 w)/(v_1 + v_0 v_2)$ holds and with respect to $wv_1 + w^2 v_2 - v_3 = 0$ we obtain $z - w = wv_2(z - v_0)/(v_1 + v_0 v_2) \neq 0$. Hence $wv_2 \neq 0$ and $wv_2(z - w) \neq 0$. According to (115') $r_1(\mathscr{P}) \parallel r(\mathscr{P})$ but $r_1(\mathscr{P}) \not\equiv r(\mathscr{P})$.

Proposition 30. *Let be given two real parallels $r_0 \not\equiv r_{10} \subset A^2$ and two improper points $\xi_0^\infty \not\equiv \zeta_0^\infty$ different from the improper point of r_0. Let $\mathfrak{R}^\alpha$ be a coordinate system with the origin lying on r_0 and such that neither of the axes $\vec{x}, \vec{y}$ is parallel to r_0 or passes through one of the improper points $\xi_0^\infty, \zeta_0^\infty$. Then there is exactly one block $\mathscr{P} \in \mathfrak{M}_w^{2,*} \cap U_{26}^\alpha$ such that $r(\mathscr{P}) \equiv r_0$, $r_1(\mathscr{P}) \equiv r_{10}$, $\xi^\infty(\mathscr{P}) \equiv \xi_0^\infty$, $\zeta^\infty(\mathscr{P}) \equiv \equiv \zeta_0^\infty$.*

Proof. Assume that $r_0 \equiv y - w_0 x = 0$, $r_{10} \equiv y - w_0 x + \alpha = 0$, $\alpha \neq 0$; $\xi_0^\infty \equiv \equiv v_y - \beta_0 v_x = 0$, $\zeta_0^\infty \equiv v_y - z_0 v_x = 0$. With respect to (90), the condition $r(\mathscr{P}) \equiv r_0$ means that $w = w_0$, $w_0 v_1 + (w_0)^2 v_2 - v_3 = 0$. From the coincidence of improper points follows $v_0 = \beta_0$, $z = z_0$. Finally the requirement $r_1(\mathscr{P}) \equiv r_{10}$ implies, according to (115'), $w_0 v_2 (z_0 - w_0)/(\beta_0 - z_0) = \alpha$; hence we obtain $v_2 \neq 0$. According to (111), the relation $z = z_0$ can be re-written as $z_0 v_1 + z_0 \beta_0 v_2 - v_3 = 0$. Hence and from the equation $w_0 v_1 + (w_0)^2 v_2 - v_3 = 0$ we obtain the values of v_1 and v_3. Finally u_1, u_2, u_3 will be obtained from the relations $v_0 = \beta_0$, $w = w_0$, $1 + w_0 u_2 = 0$ and from (88). We can see easily that the evaluated block $\mathscr{P}$ satisfies all our demands, q.e.d.

Theorem 20. *The lines $r(\mathscr{P})$, $r_1(\mathscr{P})$ and the improper points $\xi^\infty(\mathscr{P})$, $\zeta^\infty(\mathscr{P})$ form a representing frame on the manifold $\mathfrak{M}_w^{2,*}$ with values in $A^2 \cup A_\infty^1$. $\mathfrak{M}_w^{2,*}$ is an orbit of dimension* 5.

Proof follows from Proposition 30 and from Propositions VII and IX.

Theorem 21. *The manifolds $\mathfrak{M}_w^{2,v}$, $\mathfrak{M}_w^{2,w}$ are orbits of dimension* 4.

Proof. Let us start with $\mathfrak{M}_w^{2,v}$. We can see easily that the coordinates $u_1^\alpha, u_2^\alpha, u_3^\alpha, v_0^\alpha$ of a block $\mathscr{P} \in \mathfrak{M}_w^{2,v} \cap U_{26}^\alpha$ are uniquely determined by a given position of the line $r(\mathscr{P})$ and of the improper point $\xi^\infty(\mathscr{P}) = \zeta^\infty(\mathscr{P})$. (See (88), (90), (93).) Moreover, if the origin of our coordinate system lies on $r(\mathscr{P})$, then the triplet (v_1, v_2, v_3) is non-zero and it is determined exact up to a proportionality factor by the independent relations $w_0 v_1 + (w_0)^2 v_2 - v_3 = 0$, $\beta_0 v_1 + (\beta_0)^2 v_2 - v_3 = 0$. (The last one is a consequence of $z = v_0$.) According to Proposition 28, any two blocks $\mathscr{P}, \mathscr{P}' \in \mathfrak{M}_w^{2,v}$ with the same position of $r(\mathscr{P})$ and $\xi^\infty(\mathscr{P})$ correspond to each other in a dilatation of A^2 from a centre lying on $r(\mathscr{P})$. Hence follows easily our first assertion. Let us now consider the submanifold $\mathfrak{M}_w^{2,w}$. If we prescribe $r(\mathscr{P})$, $\xi^\infty(\mathscr{P})$ and a coordinate system $\mathfrak{R}^\alpha$ with the origin lying on $r(\mathscr{P})$, then from the relations $z = w$, $z \neq v_0$ follows $v_2 = 0$. Besides that we have another equation $w_0 v_1 - v_3 = 0$, while the coordinates u_1, u_2, u_3, v_0 are uniquely determined. We can use Proposition 28 again to obtain the second assertion.

Theorem 22. *The line $r(\mathscr{P})$ and the improper point $\xi^\infty(\mathscr{P})$ form together a representing frame on the manifold $\mathfrak{M}_w^1$.* (See (109).) *The manifold $\mathfrak{M}_w^1$ is an orbit of dimension* 3.

Proof. For the coordinates of a block $\mathscr{P} \in \mathfrak{M}_w^1$ we have four characteristic relations, namely $u_3 = v_0$, $u_1 + u_2 v_0 = 0$, $v_3 = 0$, $v_1 + v_2 v_0 = 0$. We find easily that if we prescribe a position to $r(\mathscr{P})$ and $\xi^\infty(\mathscr{P})$ we obtain three additional conditions, which are independent of the former ones and determine uniquely the block $\mathscr{P}$. Some details are left to the reader.

*

Let us consider the manifold $\mathfrak{M}_r^1(0)$ determined in Γ_4^1 by the relations $R = 0$, $w = v_0$, or using (88) by the relations

$$u_3 = -u_2(v_0)^2\,, \quad u_1 + 1 = -2u_2v_0\,. \tag{116}$$

Consider an invariant decomposition

$$\mathfrak{M}_r^1(0) = \mathfrak{N}_1 \cup \mathfrak{N}_2\,, \tag{117}$$

where we put $\mathscr{P} \in \mathfrak{N}_2$ if and only if $\mathscr{P}$ contains an infinitesimal dilatation of A^2 from a point. With respect to (64) and (116) we see that $\mathscr{P} \in \mathfrak{N}_2$ if and only if $u_1 = 1$ or $1 + u_2v_0 = 0$. Thus

$$\mathfrak{N}_1 = \{\mathscr{P} \in \Gamma_4^1 \mid u_1 \neq 1,\ 1 + v_0u_2 \neq 0\,,\ (116)\ \textit{holds}\}\,. \tag{118}$$

Let us consider an invariant decomposition

$$\mathfrak{N}_1 = \mathfrak{N}_1^* \cup \mathfrak{N}_1^0\,, \tag{119}$$

where $\mathfrak{N}_1^0 = \{\mathscr{P} \in \mathfrak{N}_1 \mid W = 0\}$. (See (84).) Let $\mathscr{P} \in \mathfrak{N}_1$ be given. Denote by $S(\mathscr{P})$ the set of all d-lines $\eta \subset \mathscr{P} \cap \mathfrak{g}^0$ and by $\sum(\mathscr{P})$ the set of corresponding pointwise singular lines.

Proposition 31. *Let $\mathscr{P} \in \mathfrak{N}_1^*$. Then $\sum(\mathscr{P})$ is a one-parametric family of lines. Its envelope is the parabola $p(\mathscr{P})$ given by*

$$(1 + u_2v_0)^2\,(y - v_0x) + Ax + By + C = 0\,, \tag{120}$$

$$\begin{aligned} A &= -2(u_3 + v_0)\,G - 4H - 4u_3v_0v_2 \\ B &= 2(u_1 - u_2v_0)\,G + 4u_1v_0v_2 - 4u_2H \\ C &= G^2 - 4v_2H \\ G &= u_2v_3 - u_3v_2 + v_1\,, \\ H &= u_1v_3 - u_3v_1\,. \end{aligned} \tag{121}$$

Proof. Let us choose an admissible $\mathfrak{R}^\alpha$. To determine the envelope $p(\mathscr{P})$, let us limit ourselves to the d-lines of $\sum(\mathscr{P})$ having the homogeneous coordinate $a \neq 0$ and let us assume that $a = 1$. From the condition $\eta \subset \mathfrak{g}^0 \cap \mathscr{P}$ follows $d = u_1 + bu_2 + cu_3$, $u_1 + bu_2 + cu_3 - bc = 0$. Hence d and c can be expressed by means of the coordinate b. From (65) we obtain the equation of a general line from $\sum(\mathscr{P})$ in the form

$$\begin{aligned} b^2(x + u_2y + v_2) + b[(u_1 - u_2v_0)\,y - (u_3 + v_0)\,x + G] + \\ + (H + v_0u_3x - v_0u_1y) = 0\,. \end{aligned} \tag{122}$$

Now the equation (120) can be derived in usual manner. We find easily that $A + v_0B = 2(1 + u_2v_0)^2\,W \neq 0$ and thus $p(\mathscr{P})$ really is a parabola.

Proposition 32. *Let $p_0 \subset A^2$ be a parabola and $u_2^0 \neq 0$ a real number. Let $\mathfrak{R}^\alpha$ be a coordinate system with the origin lying on p_0 and such that neither of the axes $\vec{x}, \vec{y}$ passes through the improper point of tangency of the parabola. Then for a general choice of $\mathfrak{R}^\alpha$, there is exactly one block $\mathcal{P} \in \mathfrak{N}_1^* \cap U_{26}^\alpha$ such that $p(\mathcal{P}) \equiv \equiv p_0$, $u_2^\alpha = u_2^0$. (By a general choice of $\mathfrak{R}^\alpha$ will be meant its choice in an open subset of a coordinate $G_c(\mathbf{x})$-type, where $\mathbf{x} \in p_0$.)*

Proof. The equation of the parabola can be written in the form

$$p_0 \equiv (y - \beta_0 x)^2 + A_0 x + B_0 y = 0\,, \quad A_0 + \beta_0 B_0 \neq 0\,. \tag{123}$$

If we compare the leading terms in (120) and (123) we obtain $v_0 = \beta_0$, and from the relations $(1 + u_2 v_0)^2 A = A_0$, $(1 + u_2 v_0)^2 B = B_0$ we obtain, putting $u_2 = u_2^0$,

$$G = \frac{u_2^0 A_0 - B_0}{(1 + u_2^0 \beta_0)^4} - 2\beta_0 v_2 \tag{124}$$

$$H = \frac{(u_3^0 + \beta_0)\, B_0 + (u_1^0 - u_2^0 \beta_0)\, A_0}{2(1 + u_2^0 \beta_0)^4} + (\beta_0)^2\, v_2\,.$$

Here u_1^0, u_3^0 are the values of u_1, u_3 determined by (116). For a general choice of $\mathfrak{R}^\alpha$ we have obviously $1 + u_2^0 \beta_2 \neq 0$ and the relations (124) are sensible.

From the equation $C = 0$ and from (124) follows easily

$$v_2 = \frac{-(u_2^0 A_0 - B_0)^2}{2(1 + u_2^0 \beta_0)^5\,(A_0 + \beta_0 B_0)}\,.$$

For determining of v_1, v_3 we have the system (124). The determinant of coefficients of the unknowns v_1, v_3 is equal to $(1 + u_2^0 \beta_0)^2 \neq 0$. (Cf. (121).) This completes our proof.

Now let us find all vectors $X \in \mathfrak{g}$ such that the parabola p_0 is a singular set for X. We can see easily that the coordinates of the vectors in question satisfy the relations

$$\begin{aligned} \beta_0(a - d) - b + (\beta_0)^2\, c &= 0 \\ a + 3\beta_0 c - 2d &= 0 \\ 2v - 2\beta_0 u &= B_0 d - (A_0 + 2\beta_0 B_0)\, c \\ A_0 u + B_0 v &= 0\,. \end{aligned} \tag{125}$$

Hence a, c may be supposed arbitrary and the other coordinates depend on the former ones. (The independence of the 3-rd and 4-th equation of the system follows from $A_0 + \beta_0 B_0 \neq 0$.) Let $\mathfrak{R}^\alpha$ be a general coordinate system in the sense of Proposition 32; then the coordinate u_2^α is a local function on $U_{26}^\alpha \cap \mathfrak{N}_1^*$. To any vector $X(a, b, c, d, u, v) \in \mathfrak{g}$ satisfying (125) we have a fundamental vector field on the submanifold

$\mathfrak{N}_1^*(p_0) \equiv \{P \in \mathfrak{N}_1^* \mid p(\mathscr{P}) \equiv p_0\}$; the function u_2^α is transformed on $\mathfrak{N}_1^*(p_0) \cap U_{26}^\alpha$ according to the rule

$$\frac{\mathrm{d}u_2^\alpha}{\mathrm{d}t} = \tfrac{1}{2}(a + \beta_0 c)\, u_2^\alpha(1 + \beta_0 u_2^\alpha)\,. \tag{126}$$

We find easily that $a + \beta_0 c = 0$ if and only if the vector X belongs to a block $\mathscr{P} \in \mathfrak{N}_1^*(p_0) \cap U_{26}^\alpha$. Whenever this is the case, X belongs to *each* block $\mathscr{P} \in \mathfrak{N}_1^*(p_0)$. If we choose $X \notin \mathscr{P}$ for $p(\mathscr{P}) \equiv p_0$, we have $a + \beta_0 c \neq 0$, and (126) is a special equation by Riccati. By integration we find that the group $G(X)$ acts transitively on the variable $u_2^\alpha \neq 0$ on the submanifold $\mathfrak{N}_1^*(p_0) \cap U_{26}^\alpha$. Any two blocks $\mathscr{P}_1, \mathscr{P}_2 \in \mathfrak{N}_1^*$ can be brought into the same U_{26}^α by a suitable choice of $\mathfrak{R}^\alpha$. Thus the group $G(X)$ acts transitively on $\mathfrak{N}_1^*(p_0)$ (See Proposition 32). Hence and from Proposition 32 follows

Theorem 23. *The manifold $\mathfrak{N}_1^*$ is an orbit of dimension 5.*

Proposition 33. *Let $\mathscr{P} \in \mathfrak{N}_1^0$, then all lines of the family $\sum(\mathscr{P})$ pass through a point $M(\mathscr{P})$ given by the relations*

$$x = \frac{-u_2 v_1 + u_1 v_2}{1 + u_2 v_0}\,, \quad y = \frac{v_1 + v_0 v_2}{1 + u_2 v_0}\,. \tag{127}$$

Proof. Owing to the relation $W = 0$ (see (119)) the values (127) make all coefficients of (122) equal to zero, q.e.d. Let us denote by $m(\mathscr{P})$ the line joining the point $M(\mathscr{P})$ with the improper point $\xi^\infty(\mathscr{P})$.

Proposition 34. *Let m_0 be a line in A^2 and $M_0 \in m_0$ a point. Let $\mathfrak{R}^\alpha$ be a coordinate system with the origin M_0 and such that neither of the axes $\vec{x}, \vec{y}$ coincides with m_0. Let $u_2^0 \neq 0$ be a real number. Then for a general choice of the coordinate system $\mathfrak{R}^\alpha$, there is exactly one block $\mathscr{P} \in \mathfrak{N}_1^0 \cap U_{26}^\alpha$ such that $m(\mathscr{P}) \equiv m_0$, $M(\mathscr{P}) \equiv M_0$, $u_2^\alpha = u_2^0$.*

Proof. With respect to our choice of $\mathfrak{R}^\alpha$ we can put $m_0 \equiv y - \beta_0 x = 0$ and $M_0 = [0, 0]$. From the condition $M(\mathscr{P}) \equiv M_0$ and because of $W = 0$ we obtain, with regard to (84) and (127), $v_1 = v_2 = v_3 = 0$. The condition $m(\mathscr{P}) \equiv m_0$ implies $v_0 = \beta_0$. According to (116), from the condition $u_2 = u_2^0$ follows $u_3 = -u_2^0(\beta_0)^2$, $u_1 + 1 = -2\beta_0 u_2^0$. Finally, for a general choice of $\mathfrak{R}^\alpha$ $1 + u_2^0\beta_0 \neq 0$ holds, and thus we have $1 + u_2 v_0 \neq 0$ for the block $\mathscr{P}$ just evaluated. Consequently, $\mathscr{P} \in \mathfrak{N}_1^0$, q.e.d.

Let us consider u_2^α as a local function on the submanifold $\mathfrak{N}_1^0(m_0, M_0) \equiv \{\mathscr{P} \in \mathfrak{N}_1^0 \mid m(\mathscr{P}) \equiv m_0,\ M(\mathscr{P}) \equiv M_0\}$. If we choose $\mathfrak{R}^\alpha$ as in Proposition 34, we can show easily that any group $G(X)$ of dilatations from the line m_0 acts transitively on the coordinate $u_2^\alpha \neq 0$. Hence it follows that $G(X)$ acts transitively on $\mathfrak{N}_1^0(m_0, M_0)$. Finally we obtain

Theorem 24. $\mathfrak{N}_1^0$ *is an orbit of dimension* 4.

Let be given $\mathscr{P} \in \mathfrak{N}_2$. With respect to the relation $1 + v_0 u_2 = 1 + w u_2 = 0$ we can proceed likewise as we did above discussing the manifold $\mathfrak{M}_w$. Then we obtain the relations (108) once again and we can consider an invariant decomposition

$$\mathfrak{N}_2 = \mathfrak{N}_2(1) \cup \mathfrak{N}_2(2) \tag{128}$$

according to the rank of the system (108). For $\mathscr{P} \in \mathfrak{N}_2(2)$ we obtain the equivariant object $\zeta^\infty(\mathscr{P})$ as in (112). Here we have $\zeta^\infty(\mathscr{P}) \equiv \xi^\infty(\mathscr{P})$ if and only if $W = 0$. Therefore, let us consider another invariant decomposition

$$\mathfrak{N}_2(2) = \mathfrak{N}_2^*(2) \cup \mathfrak{N}_2^0(2) \tag{129}$$

where $\mathfrak{N}_2^0(2) = \{\mathscr{P} \in \mathfrak{N}_2(2) \mid z = v_0 \Leftrightarrow W = 0\}$.

Let us start with $\mathfrak{N}_2^*(2)$. We show easily that the object $r_1(\mathscr{P})$ (see (115)) is defined on the manifold $\mathfrak{N}_2(2)$. If we prescribe the position of the line $r_1(\mathscr{P})$ and of the improper point $\zeta^\infty(\mathscr{P})$ and if the origin of a coordinate system $\mathfrak{R}^\alpha$ lies on $r_1(\mathscr{P})$, then the block $\mathscr{P} \in \mathfrak{N}_2^*(2) \cap U_{26}^\alpha$ is uniquely determined exact up to a proportionality factor for the triplet (v_1, v_2, v_3). This last triplet is always non-zero since $W \neq 0$. According to Proposition 28, any group of dilatations of the plane A^2 from a point of the line $r_1(\mathscr{P})$ acts transitively on the submanifold $\{\mathscr{P} \in \mathfrak{N}_2^*(2) \mid r_1(\mathscr{P}) \equiv r_{10}, \zeta^\infty(\mathscr{P}) \equiv \zeta_0^\infty\}$. Hence it follows easily that $\mathfrak{N}_2^*(2)$ is an orbit of dimension 4.

For $\mathscr{P} \in \mathfrak{N}_2^0(2)$ let us consider a d-line $\hat{\eta}^5 \subset \mathscr{P}$ consisting of all those infinitesimal dilatations of the plane A^2 from a centre that are elements of $\mathscr{P}$. (See (117).) The corresponding pointwise singular line is

$$h(\mathscr{P}) \equiv y - v_0 x + v_1 = 0\,. \tag{130}$$

In fact, the d-line $\hat{\eta}^5$ has homogeneous coordinates $a = d = 1$, $b = c = 0$. Now, if we prescribe the position of $h(\mathscr{P})$ and choose the origin of a coordinate system $\mathscr{P}^\alpha$ on the line $h(\mathscr{P})$, then the coordinates u_1, u_2, u_3, v_0 are uniquely determined and for v_1, v_2, v_3 we have two relations $v_1 = 0$, $W = 0$ determining the triplet (v_1, v_2, v_3) exact up to a proportionality factor. That triplet is non-zero since at least one of the terms v_3 and $v_1 + v_0 v_2$ is non-zero. According to Proposition 28 the set $\{\mathscr{P} \in \mathfrak{N}_2^0(2) \mid h(\mathscr{P}) \equiv h_0\}$ is an orbit of dimension 1 and consequently $\mathfrak{N}_2^0(2)$ is an orbit of dimension 3.

Finally, we have the same equivariant object $h(\mathscr{P})$ on the manifold $\mathfrak{N}_2(1)$, and besides that five independent relations among the coordinates:

$$u_1 + 1 = -2v_0 u_2\,, \quad u_3 = -(v_0)^2\, u_2\,, \quad 1 + u_2 v_0 = 0\,, \quad v_3 = 0\,, \quad v_1 + v_0 v_2 = 0\,.$$

We find easily that $h(\mathscr{P})$ is a representing frame on $\mathfrak{N}_2(1)$ and thus $\mathfrak{N}_2(1)$ is an orbit of dimension 2. Let us summarize our results:

Theorem 25. *The manifold* $\mathfrak{N}_2^*(2)$ *is an orbit of dimension* 4. *The manifold* $\mathfrak{N}_2^0(2)$ *is an orbit of dimension* 3. *The manifold* $\mathfrak{N}_2(1)$ *is an orbit of dimension* 2.

Consider the manifold $\mathfrak{M}_e$ determined by the relations $u_1 + 1 = u_2 = u_3 = 0$. (Cf. (68).) Let us have an invariant decomposition

$$\mathfrak{M}_e = \mathfrak{M}_e^* \cup \mathfrak{M}_e^0 \tag{131}$$

where $\mathfrak{M}_e^0 = \{\mathscr{P} \in \mathfrak{M}_e \mid W = 0\}$. Our method will be the same as that we have applied to the manifold $\mathfrak{N}_1$. Namely, we construct the parabola $p(\mathscr{P})$ in case of $\mathfrak{M}_e^*$ and the point $M(\mathscr{P})$ together with the line $m(\mathscr{P})$ in case of $\mathfrak{M}_e^0$. (See the equations (120) and (127).) In comparison with the manifold $\mathfrak{N}_1$ we have one relation more, namely $u_2 = 0$.

Proposition 32 holds literally for the manifold $\mathfrak{M}_e^*$ and Proposition 34 holds literally for the manifold $\mathfrak{M}_e^0$ with the only difference that we omit the requirement $u_2 = u_2^0$. Hence we obtain

Theorem 26. *The parabola* $p(\mathscr{P})$ *is a representing frame on the manifold* $\mathfrak{M}_e^*$. *The point* $M(\mathscr{P})$ *and the line* $m(\mathscr{P})$ *passing through* $M(\mathscr{P})$ *form together a representing frame on the manifold* $\mathfrak{M}_e^0$. $\mathfrak{M}_e^*$ *is an orbit of dimension* 4 *and* $\mathfrak{M}_e^0$ *is an orbit of dimension* 3.

3. THE MANIFOLD Γ_4^2

Let us choose a fixed point $\boldsymbol{p} \in A^2$ and an admissible coordinate system $\mathfrak{R}^\alpha$ with the origin $\boldsymbol{p}$. Consider the subalgebras $\mathfrak{t} = (\partial/\partial x, \partial/\partial y)$ $\mathbf{n} = (\partial/\partial x, \partial/\partial y, x(\partial/\partial x) + y(\partial/\partial y))$, $\mathfrak{g}_c(\boldsymbol{p}) = (x(\partial/\partial x), x\partial/\partial y, y(\partial/\partial x), y(\partial/\partial y))$, $\mathfrak{g}_{ec}(\boldsymbol{p}) = (x(\partial/\partial x) - y(\partial/\partial y), x(\partial/\partial y), y(\partial/\partial x))$ and the corresponding connected Lie subgroups $T, N, G_c(\boldsymbol{p}), G_{ec}(\boldsymbol{p})$ of $G = GA^+(2)$. Obviously we have a direct decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{g}_{ec}(\boldsymbol{p})$, where we consider $\mathfrak{g}$ as a vector space and $\mathfrak{n}$, $\mathfrak{g}_{ec}(\boldsymbol{p})$ as its vector subspaces. Let us denote by $p : \mathfrak{g} \to \mathfrak{n}$, $q : \mathfrak{g} \to \mathfrak{g}_{ec}(\boldsymbol{p})$ the corresponding projections. For $Y \in \mathfrak{n}$, $X \in \mathfrak{g}$ we obviously have $[Y, X] \in \mathfrak{t}$. Thus for any $\mathscr{P} \in \Gamma_4^2$ holds $[\mathfrak{n}, \mathscr{P}] \subset \mathscr{P}$ since $\mathfrak{t} \subset \mathscr{P}$. If we denote by $\Phi_* : \mathfrak{g} \to \chi(\Gamma_4^2)$ the Lie algebra homomorphism induced by the action $\varphi : G \times \Gamma_4^2 \to \Gamma_4^2$, then for each $Y \in \mathfrak{n}$ holds $\Phi_*(Y) = 0$ identically on Γ_4^2. If $X \in \mathfrak{g}$, then $\Phi_*(X) = \Phi_*(pX) + \Phi_*(qX) = \Phi_*(qX)$. Hence it follows easily that the action φ induces the same diffeomorphism group on the manifold Γ_4^2 as the action of the subgroup $G_{ec}(\boldsymbol{p}) \subset G$. Our classification problem then reduces to the following one: classify all orbits of the manifold Γ_4^2 with respect to the adjoint action of the isotropy troup $G_{ec}(\boldsymbol{p})$! The set of all coordinate systems $\mathfrak{R}^\alpha$ with the origin $\boldsymbol{p}$ is divided into more coordinate H_e-types, where $H_e = G_{ec}(\boldsymbol{p})$. We shall choose one of them, for instance $\mathfrak{R}(H_e)$, and call the coordinate systems belonging to $\mathfrak{R}(H_e)$ *admissible*.

Let us consider an invariant decomposition

$$\Gamma_4^2 = \mathfrak{M}_1 \cup \mathfrak{M}_2 \tag{132}$$

where $\mathscr{P} \in \mathfrak{M}_1$ or $\mathscr{P} \in \mathfrak{M}_2$ according to $\dim(\mathscr{P} \cap \mathfrak{g}_{ec}(\boldsymbol{p})) = 1$ or 2, respectively. Let us remark that $\dim(\mathscr{P} \cap \mathfrak{g}_{ec}(\boldsymbol{p})) = \dim(\mathscr{P} \cap \mathfrak{g}_e) - 2$ and thus the decomposition (132) does not depend on the choice of the origin $\boldsymbol{p}$.

Theorem 27. *Denote $H_e = G_{ec}(\boldsymbol{p})$. For any admissible coordinate system $\mathfrak{R}^\alpha$*

a) $\mathfrak{M}_1 \cap U_{45}^\alpha$ is an H_e-covering set of the manifold $\mathfrak{M}_1$,
b) $\mathfrak{M}_2 \cap U_{36}^\alpha$ is an H_e-covering set of the manifold $\mathfrak{M}_2$.

Proof. Let $\mathfrak{F}_2$ be the Stiefel manifold of all couples $\{X_1, X_2\}$ of linearly independent vectors of $\mathfrak{h} = \mathfrak{g}_c(\boldsymbol{p})$. Then $\mathfrak{F}_2$ can be made into a fibre bundle with the base Γ_4^2 and projection $p : \mathfrak{F}_2 \to \Gamma_4^2$ given by the rule $p\{X_1, X_2\} = (\partial/\partial x, \partial/\partial y, X_1, X_2) = (\mathrm{t}, X_1, X_2)$. Put $\tilde{\mathfrak{M}}_1 = p^{-1}(\mathfrak{M}_1)$, $\tilde{\mathfrak{M}}_2 = p^{-1}(\mathfrak{M}_2)$ (see (132),

$$\tilde{U}_{36}^\alpha = p^{-1}(U_{36}^\alpha \cap \Gamma_4^2)\,, \quad \tilde{U}_{45}^\alpha = p^{-1}(U_{45}^\alpha \cap \Gamma_4^2)\,.$$

With respect to Proposition IV it suffices to prove the following

Proposition 35. *For any admissible coordinate system $\mathfrak{R}^\alpha$*

a) $\tilde{\mathfrak{M}}_1 \cap \tilde{U}_{45}^\alpha$ is an H_e-covering set of $\tilde{\mathfrak{M}}_1$
b) $\tilde{\mathfrak{M}}_2 \cap \tilde{U}_{36}^\alpha$ is an H_e-covering set of $\tilde{\mathfrak{M}}_2$.

Proof of the Proposition: let $X_1, X_2 \in \mathfrak{h}$ and let $\mathfrak{R}^\alpha$ be an admissible coordinate system. Then we can write

$$X_1 = a_1^\alpha x^\alpha \frac{\partial}{\partial x^\alpha} + \ldots + d_1^\alpha y^\alpha \frac{\partial}{\partial y^\alpha}\,, \quad X_2 = a_2^\alpha x^\alpha \frac{\partial}{\partial x^\alpha} + \ldots + d_2^\alpha y^\alpha \frac{\partial}{\partial y^\alpha}\,.$$

If moreover X_1 and X_2 are linearly independent, we have $\{X_1, X_2\} \in \mathfrak{F}_2$, and $a_1^\alpha, \ldots, d_1^\alpha, a_2^\alpha, \ldots, d_2^\alpha$ are local coordinates of the couple $\{X_1, X_2\}$ on the manifold $\mathfrak{F}_2$. Let us consider the Pluecker's coordinates p_{ij}^α of the couple $\{X_1, X_2\}$, which are differentiable functions on $\mathfrak{F}_2$. Let $\varphi : H_e \to \chi(\mathfrak{F}_2)$ be the Lie algebra homomorphism induced by the action $H_e \times \mathfrak{F}_2 \to \mathfrak{F}_2$. From (2) we obtain the following table for the infinitesimal transformations of p_{ij}^α:

	$\varphi\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)$	$\varphi\left(x\frac{\partial}{\partial y}\right)$	$\varphi\left(y\frac{\partial}{\partial x}\right)$
p_{12}	$-p_{12}$	$p_{23} - p_{14}$	0
p_{13}	p_{13}	0	$p_{13} + p_{14}$
p_{14}	0	$p_{13} - p_{34}$	$p_{24} - p_{12}$
p_{23}	0	$p_{13} + p_{34}$	$p_{24} + p_{12}$
p_{24}	$-p_{24}$	$p_{14} + p_{24}$	0
p_{34}	p_{34}	0	$-p_{10} + p_{23}$

(For the sake of simplicity we omit the index α.) Let $E(f)$ be the symbol introduced in (62) where f is a local function on the manifold $\mathfrak{F}_2$. Put $E_i(f) = E(f) \cap \widetilde{\mathfrak{M}}_i$ for $i = 1, 2$. We find easily that $\{X_1, X_2\} \in \widetilde{\mathfrak{M}}_2$ if and only if we have simultaneously $a_1^\alpha + d_1^\alpha = 0$, $a_2^\alpha + d_2^\alpha = 0$, or in Pluecker's coordinates,

$$p_{14}^\alpha = 0\,, \quad p_{13}^\alpha - p_{34}^\alpha = 0\,, \quad p_{24}^\alpha - p_{12}^\alpha = 0\,. \tag{133}$$

Let us remind that $U_{36}^\alpha = \{\mathscr{P} \in \Gamma_4 \mid \mathscr{P} \cap E_{36}^\alpha = 0\}$, $E_{36}^\alpha = (x^\alpha(\partial/\partial x^\alpha),\ y^\alpha(\partial/\partial y^\alpha))$. Hence $\widetilde{U}_{36}^\alpha = E(p_{23}^\alpha)$ and similarly $\widetilde{U}_{45}^\alpha = E(p_{14}^\alpha)$. From (133) follows $\widetilde{\mathfrak{M}}_1 = E_1(p_{14}^\alpha) \cup E_1(p_{13}^\alpha - p_{34}^\alpha) \cup E_1(p_{24}^\alpha - p_{12}^\alpha)$, and since the Pluecker's coordinates of a couple $\{X_1, X_2\} \in \mathfrak{F}_2$ cannot be all equal to zero, we obtain from (133) $\widetilde{\mathfrak{M}}_2 = E_2(p_{13}^\alpha + p_{34}^\alpha) \cup E_2(p_{24}^\alpha + p_{12}^\alpha) \cup E_2(p_{23}^\alpha)$. From our table and from Proposition V we find easily that $E_1(p_{14}^\alpha)$ is an H_e-covering set of $\widetilde{\mathfrak{M}}_1$ and $E_2(p_{23}^\alpha)$ is an H_e-covering set of $\widetilde{\mathfrak{M}}_2$, q.e.d.

Theorem 28. *The manifold $\mathfrak{M}_2$ consists of two orbits of dimension* 2 *and of one orbit of dimension* 1.

Proof. Let $\mathfrak{R}^\alpha$ be an admissible coordinate system (see our convention about coordinates). Then each block $\mathscr{P} \in \Gamma_4^2 \cap U_{36}^\alpha$ is determined by two equations of the form

$$a = u_1 b + u_2 c\,, \quad d = v_1 b + v_2 c\,, \tag{134}$$

where a, b, c, d denote the last four coordinates of a vector $X \in \mathfrak{g}$ in the coordinate system $\mathfrak{S}_\alpha$. (We shall omit the index α if there is no risk of confusion.) The manifold Γ_4^2 is of dimension 4. Thus if we restrict the coordinate system $\mathfrak{S}_{36}^\alpha : U_{36}^\alpha \to \mathfrak{R}^8$ to the intersection $U_{36}^\alpha \cap \Gamma_4^2$, we obtain a map of the form $\mathscr{P} \to (u_1, u_2, 0, 0, v_1, v_2, 0, 0)$, i.e., a map $\mathfrak{S}_{36}'^\alpha : \Gamma_4^2 \cap U_{36}^\alpha \to \mathbf{R}^4$ which is a local coordinate system on the manifold Γ_4^2 induced by the coordinate system $\mathfrak{R}^\alpha$. According to Theorem 27 the coordinates $\mathfrak{S}_{36}'^\alpha$ are H_e-covering on $\mathfrak{M}_2$. Now for $\mathscr{P} \in \mathfrak{M}_2 \cap U_{36}^\alpha$ holds $\mathscr{P} \subset \mathfrak{g}_e$ and one has additional relations

$$u_1 + v_1 = 0\,, \quad u_2 + v_2 = 0\,. \tag{135}$$

Thus by restricting the chart $\mathfrak{S}_{36}'^\alpha$ to the set $\mathfrak{M}_2 \cap U_{36}^\alpha$, we obtain a local chart $\mathfrak{S}_{36}^{\alpha''}$ on $\mathfrak{M}_2$. $\mathfrak{M}_2$ is of dimension 2 and u_1^α, u_2^α can be taken for local coordinates expressing $\mathscr{P} \in \mathfrak{M}_2 \cap U_{36}^\alpha$.

Suppose $\mathscr{P} \in \mathfrak{M}_2$ and let us look for all complex d-elements ${}^c\xi \subset C\mathscr{P}$ such that $ad - bc = 0$. Using Theorem 27 we can find an $\mathfrak{R}^\alpha$ such that $\mathscr{P} \in U_{36}^\alpha$. In local coordinates, we have to determine all groups (a, b, c, d) of complex numbers satisfying the relations (134) and $ad - bc = 0$. Hence we deduce a condition

$$(u_1)^2\, b^2 + (1 + 2u_1u_2)\, bc + (u_2)^2\, c^2 = 0\,. \tag{136}$$

If we add another requirement ${}^c\xi \subset C\mathfrak{g}_c(\mathbf{p})$, i.e., $u = v = 0$, we obtain exactly two

d-elements. They can be real different or imaginary conjugate or real and coincident. The relation $ad - bc = 0$ is invariant on Cg and thus the couple of complex d-elements just constructed is a well defined object. There is a couple of pointwise singular lines corresponding to our couple of d-elements; it is given by the equation

$$\varkappa(\mathscr{P}) \equiv u_2x^2 + xy - u_1y^2 = 0\,. \tag{137}$$

Neither of the equations (136), (137) can vanish identically. We have obtained an equivariant object on the whole $\mathfrak{M}_2$.

Proposition 36. *Let $\varkappa_0$ be a couple of lines in CA^2, which are real and different, or imaginary conjugate, or real and coincident, and have a double point at $\mathbf{p} \in A^2$. Let $\mathfrak{R}^\alpha$ be an admissible coordinate system such that the following conditions are satisfied:*

a) If the lines of the couple $\varkappa_0$ are mutually different, then they are not harmonically separated by the axes $\vec{x}, \vec{y}$. [1]

b) If the couple $\varkappa_0$ is a double line, then the axes $\vec{x}, \vec{y}$ are different from that line.

Under these conditions there is exactly one block $\mathscr{P} \in \mathfrak{M}_2 \cap U^\alpha_{36}$ such that $\varkappa(\mathscr{P}) \equiv \varkappa_0$.

Proof. Let us write the equation of $\varkappa_0$ in the coordinates $\mathfrak{R}^\alpha$ in the form $d_1x^2 + d_2xy + d_3y^2 = 0$. With respect to our assumptions a), b) we always have $d_2 \neq 0$. The rest of the proof is trivial.

Proof of Theorem 28. The coordinate systems $\mathfrak{R}^\alpha$ that are admissible in the sense of Proposition 36 form an open subset of the H_e-type of all admissible coordinate systems. According to Proposition VII $\varkappa(\mathscr{P})$ is a representing frame on $\mathfrak{M}_2$ with respect to the group $H_e = G_{ec}(\mathbf{p})$. Now it remains to discuss the domain of values of the frame.

*

Let us consider some $\mathscr{P} \in \mathfrak{M}_1$. According to Theorem 27 there is an admissible coordinate system $\mathfrak{R}^\alpha$ such that $\mathscr{P} \in U^\alpha_{45}$. Then $\mathscr{P}$ is given by two equations of the form

$$b = u_1a + u_2d\,, \quad c = v_1a + v_2d\,. \tag{138}$$

Here u_1, u_2, v_1, v_2 are to be considered as coordinates of the block $\mathscr{P} \in U^\alpha_{45}$. We can also obtain these local coordinates by restricting the chart $\mathfrak{S}^\alpha_{45} : U^\alpha_{45} \to \mathbf{R}^8$ to the intersection $U^\alpha_{45} \cap \mathfrak{M}_1$. Now the d-element $\xi = \mathscr{P} \cap \mathfrak{h}_e$ admits a singularity at $\mathbf{p}$

[1]) We say that a line couple $\{p, q\}$ separates harmonically another line couple $\{r, s\}$ if $R(p, r, q, s) = -1$.

and a couple $k(\mathscr{P})$ of singular lines given by the equation (3). In local coordinates we obtain from (3), (139) and from the relation $a + d = 0$

$$(139) \qquad k(\mathscr{P}) \equiv (u_1 - u_2)\, x^2 - 2xy - (v_1 - v_2)\, y^2 = 0\,.$$

Denote by $C\mathfrak{g}_0$ the set of all complex vectors from $C\mathfrak{g}$ satisfying the invariant relation $ad - bc = 0$. Let us consider an invariant decomposition

$$(140) \qquad \mathfrak{M}_1 = \mathfrak{M}_1^* \cup \mathfrak{M}_1^0$$

where $\mathfrak{M}_1^0 = \{\mathscr{P} \in \mathfrak{M}_1 \mid C\mathscr{P} \subset C\mathfrak{g}^0\}$. If $\mathscr{P} \in \mathfrak{M}_1 \cap U_{45}^{\alpha}$, then a complex d-element ${}^c\xi$ belongs to $C\mathscr{P} \cap C\mathfrak{g}^0$ if and only if the homogeneous coordinates of ${}^c\xi$ fulfil the, relation

$$(141) \qquad u_1 v_1 a^2 + (u_1 v_2 + u_2 v_1 - 1)\, ad + u_2 v_2 d^2 = 0\,.$$

This is an immediate consequence of (138). It is obvious that $\mathscr{P} \in \mathfrak{M}_1^0$ if and only if the equation (141) vanishes identically. The d-element $\mathscr{P} \cap \mathfrak{h}_e$ belongs to $\mathfrak{g}^0$, and the object $k(\mathscr{P})$ is a double line in this case.

Let us consider the submanifold $\mathfrak{M}_1^*$. In this case the equation (141) completed by the condition ${}^c\xi \subset C\mathfrak{g}(\boldsymbol{p})$ determines exactly two complex d-elements. These d-elements are real and different, or imaginary conjugate, or real and coincident, and they have a double point at $\boldsymbol{p}$. The corresponding couple of pointwise singular lines will be denoted by $\varkappa(\mathscr{P})$ again; in local coordinates we have

$$(142) \qquad \varkappa(\mathscr{P}) \equiv u_1 x^2 - (1 + u_1 v_2 - u_2 v_1)\, xy + v_2 y^2 = 0\,.$$

(The last formula follows from (4), (138), (141).)

Let us introduce the following notation: $\operatorname{sgn} k(\mathscr{P}) = 1, -1, 0$, according to the sign of the discriminant of (139). The symbol $\operatorname{sgn} \varkappa(\mathscr{P})$ will have the analogous signification. We consider an invariant dexomposition as follows:

$$(143) \qquad \mathfrak{M}_1^* = \bigcup_{i,j} \mathfrak{M}_1^*(i, j)\,, \quad i, j = 1, -1, 0\,,$$

where $\mathfrak{M}_1^*(i, j) = \{\mathscr{P} \in \mathfrak{M}_1^* \mid \operatorname{sgn} k(\mathscr{P}) = i,\ \operatorname{sgn} \varkappa(\mathscr{P}) = j\}$.

Proposition 37. $\mathfrak{M}_1^*(0, -1) = \emptyset$.

Proof. If $\operatorname{sgn} k(\mathscr{P}) = 0$, then according to Theorem 1, **h**), the real d-element $\mathscr{P} \cap \mathfrak{h}_e$ satisfies the relation $ad - bc = 0$ and consequently, it is one of the d-elements defining the object $\varkappa(\mathscr{P})$. Thus one of the lines of the object $\varkappa(\mathscr{P})$ coincides with the double line $k(\mathscr{P})$ and hence both lines of $\varkappa(\mathscr{P})$ are real, $\operatorname{sgn} \varkappa(\mathscr{P}) = 1$ or 0, q.e.d.

Let $\mathscr{P} \in \mathfrak{M}_1^*$. Choose an admissible $\mathfrak{R}^{\alpha}$ such that

a) $\mathscr{P} \in U_{45}^{\alpha}$; *b*) if $\operatorname{sgn} \varkappa(\mathscr{P}) \neq 0$, then the axes $\vec{x}, \vec{y}$ do not separate harmonically the lines of $\varkappa(\mathscr{P})$; *c*) if $\operatorname{sgn} \varkappa(\mathscr{P}) = 0$, then neither of the axes $\vec{x}, \vec{y}$ coincides with $\varkappa(\mathscr{P})$.

A choice like this is possible according to Theorem 27. With respect to *b*) and *c*) the equations (139) and (142) can be written in the form

$$(144) \qquad d_1x^2 - 2xy - d_2y^2 = 0\,, \quad \tilde{d}_1x^2 - xy + \tilde{d}_2y^2 = 0\,,$$

where

$$(145) \qquad u_1 = u_2 + d_1\,, \quad v_1 = v_2 + d_2\,.$$

In view of (142), (144) the triplets $(\tilde{d}_1, -1, \tilde{d}_2)$, $(u_1, -(1 + d_1v_1 - d_2u_1), v_1 - d_2)$ are proportional to each other, which leads to the system

$$(146) \qquad (1 + d_2\tilde{d}_1)\,u_1 - d_1\tilde{d}_1v_1 = \tilde{d}_1\,, \quad d_2\tilde{d}_2u_1 + (1 - d_1\tilde{d}_2)\,v_1 = d_2 + \tilde{d}_2$$

The determinant of the system (146) is

$$(147) \qquad D = 1 + d_2\tilde{d}_1 - d_1\tilde{d}_2\,.$$

If we denote by Δ the resultant of the left sides of (144), we find easily the following expression:

$$(148) \qquad \Delta = (1 + d_1d_2)(1 - 4\tilde{d}_1\tilde{d}_2) - D^2$$

Proposition 38. *Let $k_0, \varkappa_0$ be two line couples of CA^2 given in an admissible coordinate system $\mathfrak{R}^\alpha$ by the equations (144). Then*

a) If k_0 and $\varkappa_0$ taken together consist of four mutually different lines, we have $D = 0$ if and only if the couples in view separate harmonically each other.

b) If both k_0 and $\varkappa_0$ are formed by non-parallel lines but not all lines of the configuration are mutually different, we have $D \neq 0$.

Proof. The assertion *a*) can be verified by direct computation, the assertion *b*) follows from the signification of the resultant and from the formula (148).

Proposition 39. *Let $\mathscr{P} \in \mathfrak{M}_1^*$,* sgn $k(\mathscr{P}) \neq 0$, sgn $\varkappa(\mathscr{P}) \neq 0$. *If the couples $k(\mathscr{P})$, $\varkappa(\mathscr{P})$ have not a common line, then they do not separate harmonically each other. If* sgn $k(\mathscr{P}) \neq 0$, sgn $\varkappa(\mathscr{P}) = 0$, *then the double line $\varkappa(\mathscr{P})$ does not coincide with any line of the couple $k(\mathscr{P})$.*

Proof. Let $\mathfrak{R}^\alpha$ be an admissible coordinate system such that the objects $k(\mathscr{P})$, $\varkappa(\mathscr{P})$ are given by equations of the form (144). If $D = 0$, then from the solvability of the system {(145), (146)} follows $\tilde{d}_1(1 + d_1d_2) = 0$, $\tilde{d}_2(1 + d_1d_2) = 0$. Since $D = 0$ it is impossible that $\tilde{d}_1 = \tilde{d}_2 = 0$ and hence $d_1d_2 = -1$ and sgn $k(\mathscr{P}) = 0$. Thus whenever sgn $k(\mathscr{P}) \neq 0$ we have $D \neq 0$. From part *a*) of Proposition 38 follows our first assertion and from the formula (148) we obtain the rest of the proof.

Proposition 40. *Let $k_0, \varkappa_0$ be two couples of mutually non-parallel lines (real or imaginary conjugate) crossing at $\mathfrak{p}$, and such that they do not separate harmonically* *each other. Let $\mathfrak{N}^\alpha$ be an admissible coordinate system such that the axes $\vec{x}, \vec{y}$ do not separate harmonically any of the couples $k_0, \varkappa_0$. Then there is exactly one block $\mathscr{P} \in \mathfrak{M}_1^* \cap U_{45}^\alpha$ such that $k(\mathscr{P}) \equiv k_0$, $\varkappa(\mathscr{P}) \equiv \varkappa_0$.*

Proof. With respect to our choice of $\mathfrak{N}^\alpha$ the equations of the couples $k_0, \varkappa_0$ can be written in the form (144). To determine the coordinates of the block $\mathscr{P}$ consider the system composed of (145) and (146), where $D \neq 0$ with regard to both parts of Proposition 39. Thus the system in view has a single solution $\mathscr{P} \in \mathfrak{M}_1$. Now it is impossible that $\mathscr{P} \in \mathfrak{M}_1^0$. In fact, in this case the coordinates u_1, u_2, v_1, v_2 would make zero all coefficients of (141) and from (145) we should obtain $1 + d_1 d_2 = 0$, i.e., sgn $k_0 = 0$ — a contradiction. Hence $\mathscr{P} \in \mathfrak{M}_1^*$, q.e.d.

Theorem 29. *The equivariant objects $k(\mathscr{P})$, $\varkappa(\mathscr{P})$ form a representing frame on the union $\bigcup_{i,j=\pm 1} \mathfrak{M}_1^*(i, j)$. Each of the manifolds $\mathfrak{M}_1^*(1, -1)$, $\mathfrak{M}_1^*(-1, 1)$ consists of ∞^1 orbits of dimension 3. The manifold $\mathfrak{M}_1^*(1, 1)$ consists of ∞^1 orbits of dimension 3, of two special orbits of dimension 3, and of an orbit of dimension 2. The manifold $\mathfrak{M}_1^*(-1, -1)$ consists of ∞^1 orbits of dimension 3 and of an orbit of dimension 2.*

Proof. The first assertion follows from Propositions VII, 39 and 40. Then we have to investigate, separately for each component, the domain of values of the representing frame. Particularly, we have to consider all possibilities of coincidence of the couples $k(\mathscr{P})$, $\varkappa(\mathscr{P})$. Let us remark that the general part of each $\mathfrak{M}_1^*(i, j)$ is a one-parametrical orbit family, and any orbit of that family is determined by a cross ratio of 4 mutually different lines.

Proposition 41. *Let k_0 be a couple of non-parallel lines (real or imaginary conjugate) with an intersection point at $\mathfrak{p}$, and let $\varkappa_0 \not\subset k_0$ be another real line passing through $\mathfrak{p}$. Let $\mathfrak{N}^\alpha$ be an admissible coordinate system such that its axes do not separate harmonically the couple k_0 and such that neither of them coincides with the line $\varkappa_0$. Then there is exactly one block $\mathscr{P} \in \mathfrak{M}_1^* \cap U_{45}^\alpha$ such that $k(\mathscr{P}) \equiv k_0$, $\varkappa(\mathscr{P}) \equiv \varkappa_0$.*

Proof. The couple k_0 and the double line $(\varkappa_0)^2$ are described by equations of the form (144). Since $\Delta \neq 0$ and $1 - 4\tilde{d}_1\tilde{d}_2 = 0$, it follows from (148) that $D \neq 0$ and the system composed of (145) and (146) has only one solution. We can show as in the preceding Proposition that the block $\mathscr{P} \in \mathfrak{M}_1$ justobtained belongs to $\mathfrak{M}_1^*$ and that the coincidence requirements are satisfied in a geometrical sense.

From the last assertion of Proposition 39 we obtain

Theorem 30. *The couple $k(\mathscr{P})$ and the double line $\varkappa(\mathscr{P})$ form together a representing frame on the manifold $\mathfrak{M}_1^*(1, 0) \cup \mathfrak{M}_1^*(-1, 0)$. Each of the manifolds $\mathfrak{M}_1^*(1, 0)$, $\mathfrak{M}_1^*(-1, 0)$ is an orbit of dimension 3.*

In case that sgn $k(\mathscr{P}) = 0$ the objects $k(\mathscr{P})$, $\varkappa(\mathscr{P})$ do not form a representing frame and we have to find another equivariant object. Let us start with the case sgn $\varkappa(\mathscr{P}) = 1$. As we have shown in the proof of Proposition 37, one of the d-elements determining the object $\varkappa(\mathscr{P})$ coincides with the direction element $\mathscr{P} \cap \mathfrak{h}_e$. The other one does not belong to $\mathfrak{h}_e$, it is real and it can be "provided" by two real singular lines $\alpha(\mathscr{P})$, $\beta(\mathscr{P})$. The equations of $\alpha(\mathscr{P})$ and $\beta(\mathscr{P})$ are of the form (4) or (5), respectively. Here $\alpha(\mathscr{P})$ is pointwise singular and belongs to the couple $\varkappa(\mathscr{P})$ whereas $\beta(\mathscr{P})$ is a new equivariant object. Let us remark that $\alpha(\mathscr{P})$ and $\beta(\mathscr{P})$ are always non-parallel and they pass through the point $\boldsymbol{p}$. $\varkappa(\mathscr{P})$ consists of the lines $\alpha(\mathscr{P})$ and $k(\mathscr{P})$. Because one of the d-elements determined by the relations (141) and $u = v = 0$ is $\mathscr{P} \cap \mathfrak{h}_e$, we deduce that one solution of (141) is $a + d = 0$. Hence the other solution is of the form

$$(149) \qquad u_1v_1a + u_2v_2d = 0\,, \quad u_1v_1 \neq u_2v_2\,, \quad (u_1 - u_2)(v_1 - v_2) = -1\,.$$

Here the last relation expresses the condition sgn $k(\mathscr{P}) = 0$, or else the inclusion $k(\mathscr{P}) \subset \varkappa(\mathscr{P})$. Using (4) and (5) we derive easily

$$(150) \qquad \alpha(\mathscr{P}) \equiv u_1x - v_2(u_1 - u_2)\,y = 0\,, \quad \beta(\mathscr{P}) \equiv u_2x - v_1(u_1 - u_2)\,y = 0$$

in any admissible $\mathfrak{R}^\alpha$; moreover we can see that

$$(151) \qquad k(\mathscr{P}) \equiv (u_1 - u_2)\,x - y = 0\,.$$

Now we have $k(\mathscr{P}) \neq \alpha(\mathscr{P})$, $k(\mathscr{P}) \neq \beta(\mathscr{P})$; otherwise from the relation $(u_1 - u_2) \,.\, (v_1 - v_2) = -1$ would follow $u_1v_1 = u_2v_2$ — a contradiction.

Proposition 42. *Let k_0, α_0, β_0 be three mutually different real lines in A^2 with common intersection point $\boldsymbol{p}$. Let $\mathfrak{R}^\alpha$ be an admissible coordinate system such that neither of the axes $\vec{x}, \vec{y}$ coincides with any of the lines k_0, α_0, β_0. Then there is exactly one block $\mathscr{P} \in \mathfrak{M}_1^*(0, 1) \cap U_{45}^\alpha$ such that $k(\mathscr{P}) \equiv k_0$, $\alpha(\mathscr{P}) \equiv \alpha_0$, $\beta(\mathscr{P}) \equiv \beta_0$.*

Proof is a routine and it is left to the reader. Let us only remark that the relation $u_1v_1 \neq u_2v_2$ must be verified for the block $\mathscr{P}$ formally evaluated. Hence we obtain

Theorem 31. *The objects $k(\mathscr{P})$, $\alpha(\mathscr{P})$, $\beta(\mathscr{P})$ form a representing frame on the manifold $\mathfrak{M}_1^*(0, 1)$. The manifold $\mathfrak{M}_1^*(0, 1)$ consists of two orbits of dimension 3.*

The manifold $\mathfrak{M}_1^*(0, -1)$ is an empty set according to Proposition 38; it remains to investigate the submanifolds $\mathfrak{M}_1^*(0, 0)$, $\mathfrak{M}_1^0$. If $\mathscr{P} \in \mathfrak{M}_1^*(0, 0)$ then $k(\mathscr{P}) \equiv \varkappa(\mathscr{P})$, and in arbitrary admissible $\mathfrak{R}^\alpha$ such that $\mathscr{P} \in U_{45}^\alpha$ we have invariant relations

$$(152) \qquad (u_1 - u_2)(v_2 - v_1) = 1$$

$$(153) \qquad u_1v_1 = u_2v_2\,.$$

Here both sides of (153) are non-zero. For $\mathscr{P} \in \mathfrak{M}_1^0$ the formulae (152), (153) hold, too, with the only difference that $u_1 v_1 = u_2 v_2 = 0$. From the preceding relations and from (138) follows

$$(154) \qquad ad - bc = -u_1 v_1 (a + d)^2$$

for any vector $X \in \mathscr{P}$, $\mathscr{P} \in \mathfrak{M}_1^*(0, 0) \cup \mathfrak{M}_1^0$.

Propositon 43. *The ratio $\lambda(\mathscr{P}) = (ad - bc) : (a + d)^2$ is the same for all vectors $X(u, v, a, b, c, d)$, $X \in \mathscr{P} \cap (\mathfrak{g} \dot{-} \mathfrak{g}_e)$, and all admissible $\mathfrak{R}^\alpha$.*

Proof. From (154), (153) we have

$$(155) \qquad \lambda(\mathscr{P}) = -u_1 v_1 = -u_2 v_2 .$$

Thus if we choose a fixed admissible $\mathfrak{R}^\alpha$ such that $\mathscr{P} \in U_{45}^\alpha$, the assertion holds with respect to that coordinate system. On the other hand the set $\mathscr{P} \cap (\mathfrak{g} \dot{-} \mathfrak{g}_e)$ does not depend on the coordinates and thus $\lambda(\mathscr{P})$ depends only on $\mathscr{P} \in \mathfrak{M}_1^*(0, 0) \cup \mathfrak{M}_1^0$, q.e.d.

Thus $\lambda(\mathscr{P})$ is a well-defined function on the manifold $\mathfrak{M}_1^*(0, 0) \cup \mathfrak{M}_1^0$. Moreover $\lambda(\mathscr{P})$ is a point invariant under the group H_e. In fact, the functions $ad - bc$, $a + d$ are point invariants on the manifold $\mathfrak{g}$ under H_e and for $h \in H_e$ we have $h . \mathscr{P} \cap (\mathfrak{g} \dot{-} \mathfrak{g}_e) = h[\mathscr{P} \cap (\mathfrak{g} \dot{-} \mathfrak{g}_e)]$. The value of $\lambda(\mathscr{P})$ cannot be arbitrary. In fact, from (152), (153), (155) we derive easily $(u_1 + u_2)^2 = (u_1 - u_2)^2 (1 - 4\lambda(\mathscr{P}))$; hence $\lambda(\mathscr{P}) \leqq \frac{1}{4}$.

Proposition 44. *Let k_0 be real line passing through the origin $\mathfrak{p}$ and let $\lambda \leqq \frac{1}{4}$ be a real number. Assume that $\mathfrak{R}^\alpha$ is an admissible coordinate system such that neither of the axes $\vec{x}, \vec{y}$ coincides with k_0. For $\lambda < \frac{1}{4}$ there are exactly two blocks $\mathscr{P} \in [\mathfrak{M}_1^*(0, 0) \cup \mathfrak{M}_1^0] \cap U_{45}^\alpha$ and for $\lambda = \frac{1}{4}$ there is exactly one block $\mathscr{P} \in [\mathfrak{M}_1^*(0, 0) \cup \mathfrak{M}_1^0] \cap U_{45}^\alpha$ such that $k(\mathscr{P}) \equiv k_0$, $\lambda(\mathscr{P}) = \lambda$.*

Proof. The equation of the double line $(k_0)^2$ in the coordinate system $\mathfrak{R}^\alpha$ is of the form $\mu x^2 - 2xy + y^2/\mu = 0$. The condition $k(\mathscr{P}) \equiv k_0$ yields two relations $u_1 - u_2 = \mu$, $v_2 - v_1 = 1/\mu$ and from the condition $\lambda(\mathscr{P}) = \lambda$ we obtain $u_1 v_1 = -\lambda$, $u_2 v_2 = -\lambda$. With respect to the inequality $\lambda \leqq \frac{1}{4}$ we obtain two real solutions given by

$$(156) \qquad u_1 = \tfrac{1}{2}\mu(1 \pm \sqrt{(1 - 4\lambda)}) , \quad u_2 = \tfrac{1}{2}\mu(-1 \pm \sqrt{(1 - 4\lambda)}) .$$

Proposition 45. *The function* sgn $\{|u_1^\alpha| - |u_2^\alpha|\}$ *is independent of the choice of an admissible coordinate system $\mathfrak{R}^\alpha$ such that $\mathscr{P} \in U_{45}^\alpha$.*

Proof. We can see from (156) that sgn $\{|u_1^\alpha| - |u_2^\alpha|\} = 0$ for $\lambda = \frac{1}{4}$. Let $\mathfrak{N} \subset \mathfrak{M}_1^*(0, 0) \cup \mathfrak{M}_1^0$ be an open submanifold of $\mathfrak{M}_1^*(0, 0) \cup \mathfrak{M}_1^0$ determined by the inequality $\lambda(\mathscr{P}) < \frac{1}{4}$. Assume that $\mathscr{P} \in \mathfrak{N}$. The admissible $\mathfrak{R}^\alpha$ such that $\mathscr{P} \in U_{45}^\alpha$ are just those satisfying $\vec{x} \nparallel k(\mathscr{P})$, $\vec{y} \nparallel k(\mathscr{P})$. The set of all those coordinate systems is divided into four connected components with regard to the topology of the group H_e. The continuous function sgn $\{|u_1^\alpha| - |u_2^\alpha|\} \neq 0$ is constant on each component an we find easily that its value is preserved by the transformation $x' = y$, $y' = -x$ belonging to H_e. But the last transformation mediates a passage among the components, q.e.d.

Now we find easily from (156) that the manifold $\mathfrak{N}$ splits into two invariant submanifolds $\mathfrak{N}^+$, $\mathfrak{N}^-$ determined by the relations sgn $\{|u_1^\alpha| - |u_2^\alpha|\} = \pm 1$. For $\lambda(\mathscr{P}) < \frac{1}{4}$ the two blocks $\mathscr{P} \in \mathfrak{M}$ given in Proposition 44 belong to different orbits. Finally we obtain

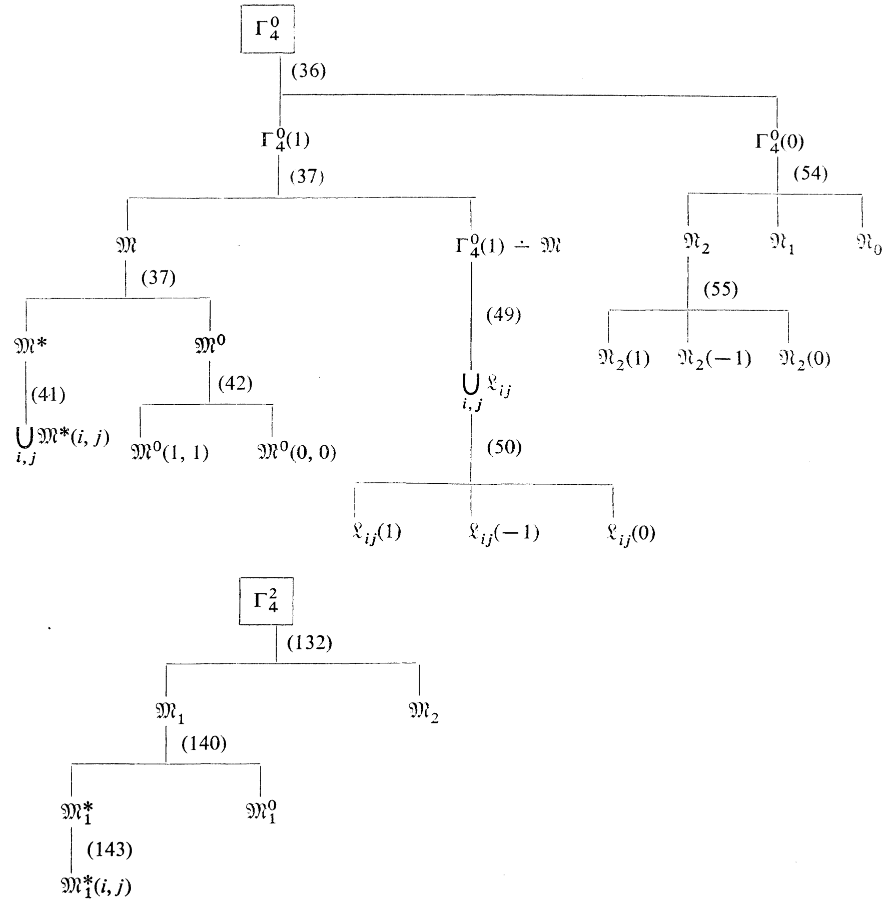
Theorem 32. *The manifold $\mathfrak{M}_1^*(0, 0) \cup \mathfrak{M}_1^0$ consists of two one-parametric systems of orbits of dimension* 1 *and of another orbit of dimension* 1. *Each orbit is completely determined by the numbers* sgn $\{|u_1| - |u_2|\}$, $\lambda(\mathscr{P})$, *and* sgn $\{|u_1| - |u_2|\} \neq 0$ *implies that it belongs to one of both the orbit systems generating $\mathfrak{N}^+$ and $\mathfrak{N}^-$. A block $\mathscr{P}$ belongs to the special orbit if, and only* sgn *if* $\{|u_1| - |u_2|\} = 0$, $\lambda(\mathscr{P}) = \frac{1}{4}$. *Otherwise we have $\lambda(\mathscr{P}) < \frac{1}{4}$, the invariant $\lambda(\mathscr{P})$ being given by* (155).

To conclude, we shall give a summary of results of this Chapter.

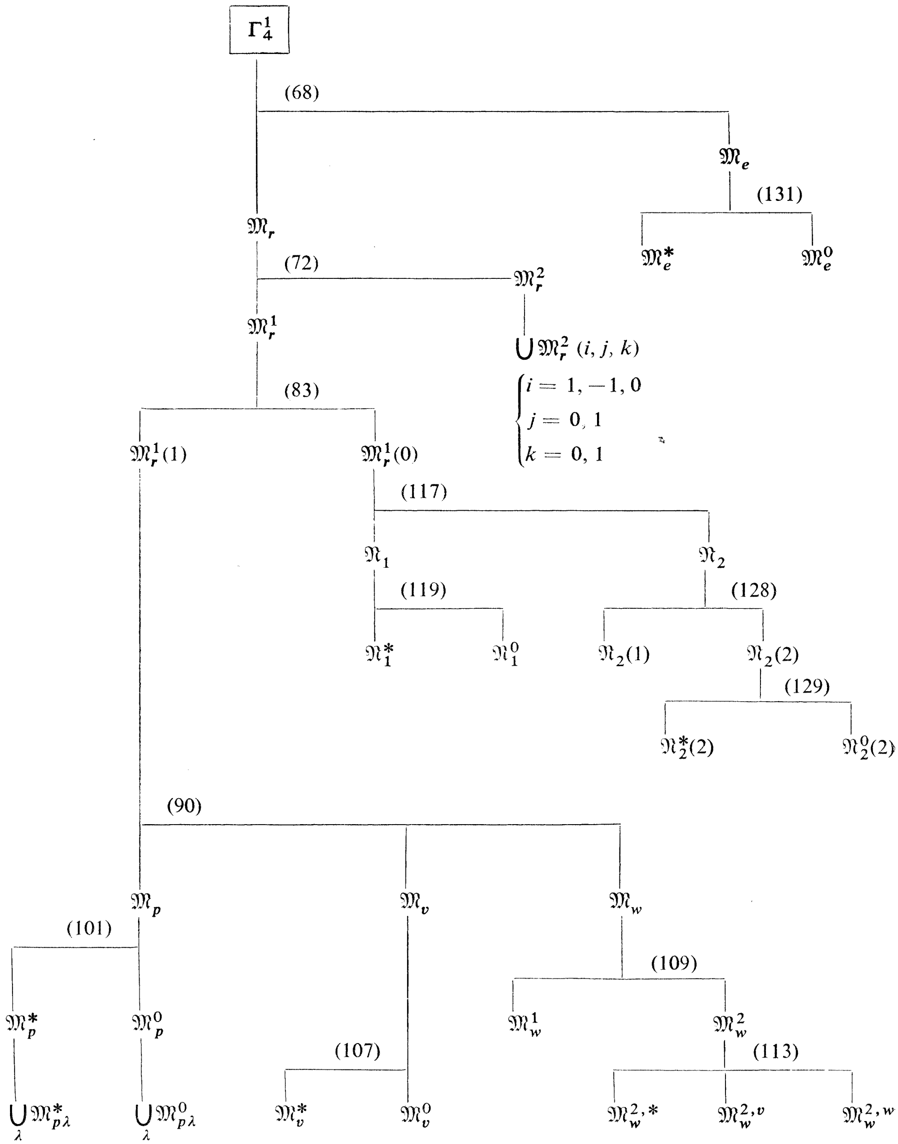
A) *A table of orbit types of the manifold* Γ_4.

Orbits \ Manifold	Γ_0^4	Γ_4^1	Γ_4^2
dim 6	$4 \times \infty^2$ $8 \times \infty^1$ 8 special	$2 \times \infty^1$ 4 special	
dim 5	$1 \times \infty^1$ 3 special	$3 \times \infty^1$ 8 special	
dim 4	2 special	$1 \times \infty^1$ 6 special	
dim 3		3 special	$4 \times \infty^1$ 6 special
dim 2	1 special (subalgebras)	1 special	4 special
dim 1			$2 \times \infty^1$ 2 special

B) *Invariant decompositions of the manifolds* Γ_4^0, Γ_4^2.



C) *An invariant decomposition of the manifold* Γ_4^1.



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