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CONTINUA STRUCTURED BY FAMILIES OF SIMPLE CLOSED CURVES – II

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1. Introduction. In an earlier paper, [2], the concept of two-manifold was generalized to include certain spaces which triangulate like a compact two-manifold without boundary. Compact, locally connected, metric continua which partition into elements whose boundaries fit together like the boundaries of the two-simplexes of a triangulation of a two-manifold were considered, using results obtained by ANDERSON and KEISLER, [1].

If there is a sequence of such partitions, with mesh tending to zero, of such a space, M, and if successive collections of bounding simple closed curves can be mapped "nicely" onto preceding collections, then, for M homogeneous, easy characterizations, obtained by Anderson and Keisler exist. (See [1].) These "nice" partitions and maps correspond, roughly, to successive subdivisions or refinements of a triangulation of a two-manifold. It was shown in [2] that a space in which a decreasing mesh sequence of partitions exists, but for which the maps of successive boundary collections are not given, i.e., a space for which the given partitions lack the sequential or "subdividing" nature suggested above is still a space for which a sequential structure exists if the following condition is satisfied: If $\{P_n\}_{n=1}^{\infty}$ is the sequence of partitions and $C \in P_{n+1}$ is a simple closed curve of the $(n + 1)$st, then $C \cap \bigcup_{i=1}^{n} P_i^*$ is the union of a finite number of components.

The object of this paper is to show that if a space is of the Anderson-Keisler type locally, then it is so globally. That is, if local sequential partitioning structures exist, there is one for the space as a whole.

2. Theorem. The notation and terminology employed here is that of [1], as extended by [2].

Definition. Let M be a compact, locally connected metric continuum. Let M be such that, for each point $p \in M$, there exists an open set, U, $p \in U$, such that $\overline{U}$ is homeomorphic to F union the interior of F, where F is a biseparating and locally biseparating simple closed curve of some stage of a defining sequence of $\varkappa$-collections of some inverse incidence limit. Then M is said to be an inverse incidence limit locally.

Theorem. *If M is an inverse incidence limit locally, it is one globally.*

The proof of the theorem will be borne by the following two lemmas plus an application of the theorem of [2]. The order of the lemmas would seem to be reversed; however, the basic techniques used in the proofs of each occur more naturally initially in the context of the first.

Lemma 1. *Let M be an inverse incidence limit locally and let $\{P_i\}_{i=1}^m$ be a sequence of $\varkappa$-partitionings of M such that for $C \in P_i$, $i = 1, \ldots, m$, $C \cap (\bigcup_{j=1}^m P_j^*)$ is a finite number of components. Let, given $0 < \varepsilon <$ mesh P_i, $i = 1, \ldots, m$, there exist a $\varkappa$-partitioning, P, of mesh less than $\varepsilon/3$ with the following property: There exist a finite number of open sets $\{V_i\}_{i=1}^n$ in M such that each $\overline{V}_i$ is homeomorphic to F union the interior of F where F is a biseparating and locally biseparating simple closed curve of some stage of a defining sequence of $\varkappa$-collections of some inverse incidence limit. In addition, there are to be an at most finite number of points, $\{p_i\}_{i=1}^L$, of P and sequences of open spheres, $\{S_j(p_i)\}_{j=1}^\infty$, diameter $S_j(p_i) <$ minimum of diameter $S_{j-1}(p_i)$ and $1/j$, centered at p_i, such that:*

1) $P^* \setminus (\bigcup_{i=1}^L S_j(p_i))$ is the union of arcs, $\{A_i(j)\}_{i=1}^{s_j}$, each of which is contained in the interior of some $\overline{V}_i$ and in the point set union of the curves of some stage of the sequential structure associated with that V_i, and

2) For $k > j$, each $A_i(j)$, $i = 1, \ldots, s_j$, is interior to an arc of $\{A_i(k)\}_{i=1}^{s_k}$ from the point set union of the curves of some stage of the same sequential structure.

Then, finally, there exists a $\varkappa$-partitioning P_{m+1} of M such that $C \in P_{m+1}$ implies $C \cap (\bigcup_{i=1}^m P_i^*)$ is a finite number of components and mesh $P_{m+1} < \varepsilon$.

Proof. P would do as P_{m+1} if $P^* \cap (\bigcup_{i=1}^m P_i^*)$ were a finite number of components. However, suppose $C \in P$ intersects $C' \in P_i$, some i, in infinitely many components. Suppose also, as a temporary simplification, that $C \cap C'$ is contained in some open subset of M in which C' does not intersect $P^* \setminus C$. We may select a finite number of points $\{q_i\}_{i=1}^t$ of C' such that:

1) $C \cap C' \cap \{p_i\}_{i=1}^L \subset \{q_i\}_{i=1}^t$, and

2) $C \cap C' \setminus \{q_i\}$ is contained in a finite number of open arcs, $\{A_i\}_{i=1}^v$, of C', each of diameter $< \delta < \varepsilon/3$ and with endpoints in $\{q_i\}$. The number δ, guaranteed by the Lebesgue Covering Lemma, is such that each subset of M of diameter less than δ is contained in some V_i.

We shall inclose each A_i in a biseparating simple closed curve which intersects C' at the endpoints of A_i: We may cover a subset of each A_i by an open set with the following properties:

1) Each is bounded by a simple closed curve,

2) The closure of each is a "chain" in some local inverse incidence limit structure such that C' intersect the bounding simple closed curve is contained in the first and last "links" only,

3) The closure of no such open set intersects $P^* \setminus C$,

4) The bounding simple closed curve of each open set is such that each point is in an arc contained in some stage of the local inverse incidence limit structure, except, possibly, for the points of C intersect the bounding simple closed curve,

5) The closures of the open sets are pairwise disjoint,

6) The first and last "links" of each chain are contained, one in each, in the open spheres $S_j(q_i)$ and $S_j(q_i')$, $i \neq i'$, defined below, and

7) Each open set contains all of its associated A_i except for subsets of $A_i \cap S_j(q_i)$ and $A_i \cap S_j(q_i')$, q_i and q_i' its endpoints, and the bounding simple closed curve intersect A_i does not "extend as far as" q_i and q_i' in two opposite orderings of $\bar{A}_i$ from some point of A_i interior to the open set.

A "chain" in a local inverse incidence limit structure is the closure of the union of a finite number of open sets (whose closures are "links"), each bounded by a simple closed curve (obtained from the union of the bounding simple closed curves at some stage of the sequential structure) which form a simple chain in the usual sense when the intersection of two consecutive links is understood to be a common arc of their boundaries.

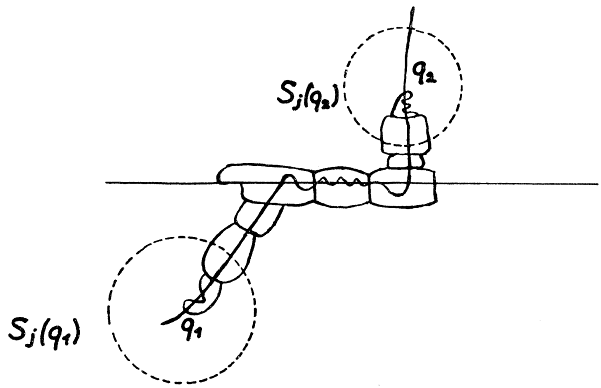


Figure 1a

The spheres $S_j(q_i)$ are understood to be those of the statement of the lemma when $q_i \in \{p_i\}$, and spheres, centered at q_i of diameter less than $1/j$ otherwise. In any case, we want j large enough that the closures of the $S_j(q_i)$'s are disjoint and that $S_j(q_i)$ does not intersect $P^* \setminus C^*$.

One such chain, containing q_1 and q_2 as q_i and q_i' is shown in Figure 1a. We shall suppose q_2 is in the sphere, $S_j(q_2)$, containing the last link of the chain for A_i

between q_1 and q_2. Let B_j be the common boundary arc shared by this last link and the next to the last in the ordering; $B_j \subset S_j(q_2)$. Because of the local sequential structure, we may construct another simple-closed-curve-bounded chain entirely in $S_j(q_2)$ such that:

1) It intersects, in the first link only, and in a subarc of B_j, the closure of the original chain minus its last link,

2) It covers all of $A_i \cap (S_j(q_2) \setminus S_{j+1}(q_2))$ which is not in the original chain minus its last link,

3) Its last link lies inside $S_{j+1}(q_2)$, and,

4) Its bounding simple closed curve intersect C' does not "extend as far as" q_2. This new chain, replacing the last link of the old, union the remaining links of the old chain, has a simple closed curve boundary and covers A_i as far as $S_{j+1}(q_2)$ at least. In the new chain, in $S_{j+1}(q_2)$, let B_{j+1} be the common boundary arc shared by the last link and the next to the last in the ordering. A repetition of this process leads to another arc, B_{j+2}, and so on, the B_{j+k}'s converging to q_2, as illustrated in Figure 1b).

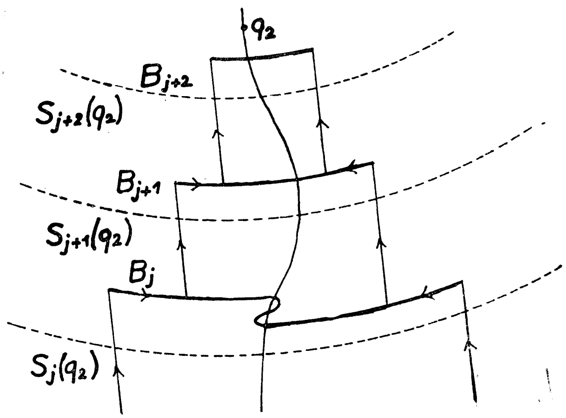


Figure 1b

The result, S, partially indicated in the figure by the arrows, is the naturally defined biseparating (due to the connected open sets or "bands" along either "side" of the simple closed curve – the result of local biseparation) and – as we shall show – locally biseparating simple closed curve intersecting C' in q_1 and q_2 only (The same process is understood to have taken place at the other end of the chain), and C in exactly two components. If either of q_1 or q_2 is an endpoint of some other A'_i, we may carry on the same procedure from the other side (on C'), in a possibly different local structure, so that each finite stage misses each finite stage in the construction of S and so that the resulting simple closed curve intersects S only in q_1 or q_2.

We show S locally biseparates at q_1 and q_2, the only points in question, as follows: Since M is a local inverse incidence limit, given any open set V containing q_1, let U be an open set containing q_1 and such that $\overline{U}$ is homeomorphic to F union the interior of F where F is a biseparating and locally biseparating simple closed curve of some stage of a defining sequence of $\varkappa$-collections of some inverse incidence limit and such that $\overline{U} \subset V$. In the local structure of U, we may construct a simple closed curve, S', just as we constructed S and such that S' biseparates M, intersects S in exactly two components and contains q_1 in its interior. The closure of $S \cap \operatorname{Int} S'$ makes a θ-curve of S' and separates its interior: $\operatorname{Int} S' \setminus S$ is $(\operatorname{Int} S \cap \operatorname{Int} S') \cup (\operatorname{Ext} S \cap \operatorname{Int} S')$. If $\operatorname{Int} S' \setminus S$ did not separate $\operatorname{Int} S \cap \operatorname{Int} S'$ from $\operatorname{Ext} S \cap \operatorname{Int} S'$, there would be an arc in $\operatorname{Int} S'$ from $\operatorname{Ext} S$ to $\operatorname{Int} S$, an impossibility. $\operatorname{Int} S' \setminus S$ is biseparated: If not, one of $\operatorname{Int} S \cap \operatorname{Int} S'$ or $\operatorname{Ext} S \cap \operatorname{Int} S'$, say the former, would have two or more components. One of these components contains an open connected subset of which $\overline{(S' \cap \operatorname{Int} S) \cup (\operatorname{Int} S' \cap S)} - q_1$ is part of the boundary. This follows from local biseparation: Each point p of the above set comes from some stage of some local inverse incidence limit and is thus part of a locally biseparating arc. The point q_1 must then be the only boundary point of a second component of $\operatorname{Int} S \cap \operatorname{Int} S'$, but it is not eligible either, since if it were a boundary point, then M is the sum of two subsets having only q_1 in common – a contradiction of local inverse incidence limit structure. Hence $\operatorname{Int} S \cap \operatorname{Int} S'$ has only one component.

Although we have handled the simplest case – concerning ourselves only with $C \cap C'$ – if there had been other arcs of $\bigcup_{i=1}^{m} P_i^*$ to account for, their intersections with our newly formed biseparating simple closed curve could have been easily controlled by this procedure. We note, too, that any other arcs of curves of P which are not shared with C are as "alien" to S as was C'. That is, other curves of P may not come from the same local structures in which S was constructed. As for C', however, they may be required to intersect S in exactly two points – or not at all.

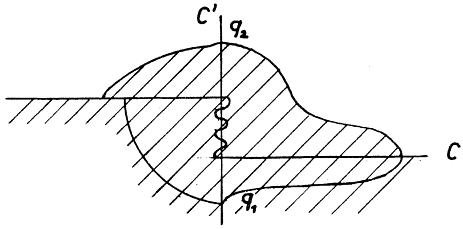


Figure 2

To determine the $\varkappa$-partitioning P_{m+1}, we add the newly formed biseparating simple closed curve S, and its interior, as a "bump" on C. See Figure 2. This means that we take as an element of P_{m+1} not C but the boundary of the interior of C plus the interior of S. This is the union of an arc from each of C and S.

Our requirement that each of the new biseparating simple closed curves, like S, intersect each $C' \in P_i$, $i = 1, \ldots, m$, and each $C \in P$ in exactly two components, if at all, says that all they do is to make θ-curves out of the simple closed curves of P and carve out chunks which may be added where we like – as to the interior of C above – without destroying the $\varkappa$-nature of the collection P. We can certainly require that no curve like S contains in its interior all of any $C \in P$.

Since our new $\varkappa$-partitioning, meeting the P_i's nicely, has "bumped" out less than $\varepsilon/3$ from the boundaries of sets of diameter less than $\varepsilon/3$, it satisfies the ε-condition of the hypothesis as well, and we may call it P_{m+1}.

Lemma 2. *If M is an inverse incidence limit locally, then given $\varepsilon > 0$, M has a $\varkappa$-partitioning of mesh less than ε which satisfies the conclusion (about P_{m+1}) of Lemma* 1.

Proof. By the hypothesis, we may consider M to be covered by a finite collection, $\{U_i\}_{i=1}^n$ of open sets such that:

1) $\overline{U}_i \subset V_i$ where V_i is an open set which is homeomorphic to the interior of a biseparating and locally biseparating simple closed curve in some stage of the sequential structure of some inverse incidence limit,

2) Mesh $U_i < \varepsilon$, and

3) Bdry $U_i = C_i$, a simple closed curve in some stage of the V_i-structure. The first and last requirements say that the U_i-inverse incidence limit structure extends slightly past C_i.

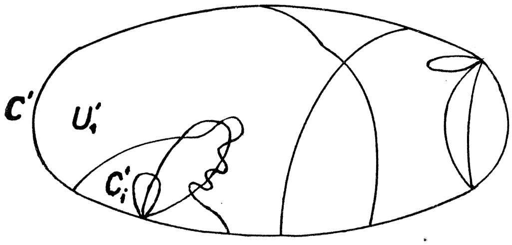


Figure 3

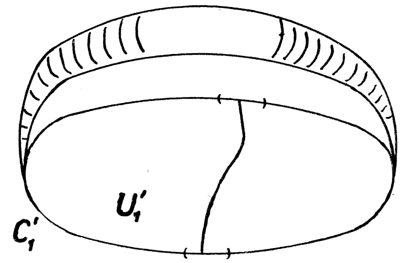


Figure 4

The intersection of the C_i's may be very messy; however, the endpoints of all of the components of $U_1 \cap (\bigcup C_i)$ lie in a finite number of disjoint intervals of C_1 whose diameters sum to less than $1/10$ the mesh of the C_i's and to less than $\delta < \varepsilon$ where δ is a number, guaranteed by the Lebesgue Covering Lemma, for which any subset of M of diameter less than δ is in some U_i. Consider a decomposition space, M_i', in which each of these intervals is identified as a point. This gives a space in which many components of $\overline{U}_1 \cap (\bigcup C_i)$ are pulled into loops (Figure 3, with C_1' and U_1' in M_1').

Identify as points all the intervals of $\overline{U}_1 \cap (\bigcup C_i)$ of diameter less than δ

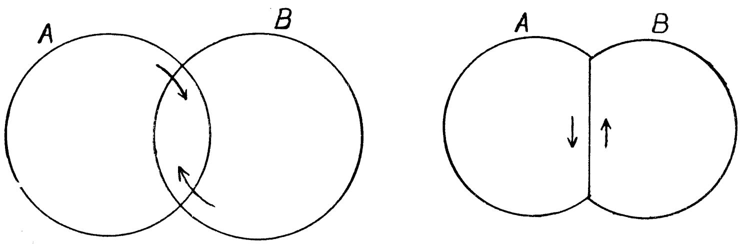
yielding these loops with the possible finite exception of those whose identification would give rise to the situation of Figure 4. That is, in which $\overline{U}_1'$ cannot be separated, in general, by a spanning arc connecting a pair of points, one in each of two arbitrary open sets of C_1' minus the decomposition points. Since there is only a finite number of sets in $\overline{U}_1$ big in this sense, leaving a finite number of loops is sufficient.

In pulling in these loops to points for some C_i', we may as suggested in Figure 3, create an infinity of loops in some C_j' at a decomposition point. The same process is applied again, to those of diameter less than δ, and since there are only n C_i's eventually we run out of troubles.

In U_2', what is left of U_2, we may go through the above procedure again with respect to what is left, C_2', of C_2. C_2' is not necessarily a simple closed curve anymore. It is, however, the union of intervals of which a finite number of subintervals may be identified as points to make the intersection of C_2' "nice" with respect to each of $C_3, \ldots, C_n$. It is already finite (in number of interval-components) with respect to C_1' from the construction of M_1'. Call the resulting decomposition space of M_1', M_2'. Continue in this way through M_n', denoted by M' for short.

The final form of $\bigcup C_i$, in M', is as a union of a finite number of simple closed curves (not uniquely determined) and $M \setminus (\bigcup C_i')^*$, C_i' the final form of C_i, is a collection of components each bounded by a finite number of simple curves and points in some particular, and hereafter fixed, description of $\bigcup C_i'$.

We wish to bound each such component of $M \setminus (\bigcup C_i')^*$ by a single simple closed



(a) (b)

Figure 5

curve and so we describe how the boundary simple closed curves might intersect. Our preliminary goal is to get the simple closed curves bounding each given component – if a component has some bounding simple closed curves – to intersect only in finite point sets or to be disjoint. Suppose, for a particular component of $M \setminus (\bigcup C_i')^*$, that two bounding simple closed curves, A and B, intersect so that an arc of one spans the other. See Figure 5a.

Since each of these simple closed curves locally biseparate, except at a finite number of points, there is a connected open set, or "band", along the inside of each which is in the component. At a point of intersection of A and B, then, a subset of the arc of A spanning B is entirely inside the component and hence is not a subset of the boundary. Neither then, except for isolated points, is the rest of the arc. Likewise for the spanning arc of B. Hence, we may drop from consideration, as part of the boundary, all but finite point sets of two of the spanning arcs (All of the arcs of $A \setminus A \cap B$ and $B \setminus A \cap B$ are actually spanning arcs) and substitute for the simple closed curves a single simple closed curve and possibly a finite number of points. It is clear which arcs are to be deleted – except for finite point sets: Those which, near their ends, are contained in the interior "bands" of the simple closed curves they span.

A similar principle governs in case two simple closed curves intersect along an arc; the common arc is not a part of the boundary except possibly for a finite number of points. For more frequent intersections of the simple closed curves bounding a component, the arguments clearly generalize, and we may say that each component is bounded by the union of a finite set of simple closed curves which intersect, pairwise, in finite point sets without spanning arcs or else not at all.

Now, if we could modify each component so that its boundary is a collection of disjoint simple closed curves, and a finite number of points, and then connect these curves with "slits" in the component we would have broken up M' into a finite number of components each bounded by at most one simple closed curve and a finite number of points.

Suppose two boundary simple closed curves of a component intersect at a point p as in Figure 6a. We shall give a construction whereby one of these, C, is spanned by an arc, A, which with the arc containing p, of the two into which A divides C, forms a simple closed curve, C', which biseparates and locally biseparates, except possibly at p, in M'.

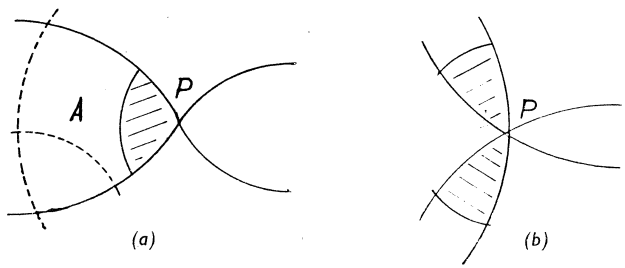


Figure 6

In other words, cutting out the interior of this simple closed curve as a new component of $M' \setminus (\bigcup C_i' \bigcup C')^*$ means the boundary simple closed curves of the original component minus the closure of the newly formed one no longer meet at p. Moreover, this may all be done in a small enough sphere about p to miss all the other common points of C and the rest of the boundary of the original component as well as any decomposition points other than p. If three or more boundary simple closed curves should meet at p, as in Figure 6b, we may repeat the process cutting out small components of all but one of these.

Let $d > 0$ be such that $S_d(p)$ contains no decomposition or "big" points of M' except possibly p. Now suppose p were interior to an arc B of some C'_i. Then, since the local sequential V_i-structure extends into U, the component of $M' \setminus (\bigcup C'_i)^*$ in question, in $(S_d(p) \cap U) \cup B$ we may draw the desired spanning, biseparating and locally biseparating arc A in the local V_i-structure. In general, however, p will be an endpoint of arcs of each of some C'_i and C'_j. The local structures which extend out past each of C'_i and C'_j overlap in an open subset, U', of U so that in $U' \cap S_d(p)$ we have two local structures, one from V_i and one from V_j.

In some stage of one of the local structures, say that of V_i, we may draw an arc A' in $U' \cap S_d(p)$ from the arc B_i of C'_i which separates U, minus a small open sphere containing the endpoint, q, of A', into two or more subcomponents. See Figure 7.

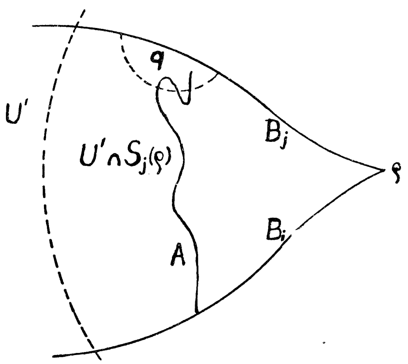


Figure 7

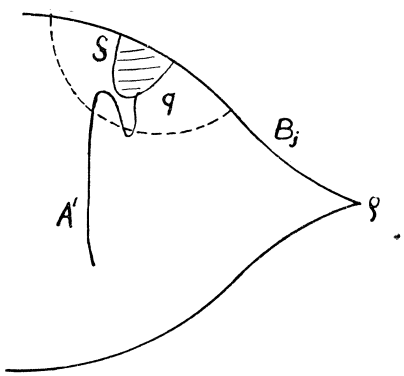


Figure 8

In the small open sphere we may construct, using the local V_j-structure, a biseparating simple closed curve, S, containing q and intersecting the arc B_j of C'_j in an arc and A' only in q. See Figure 8. The procedure for constructing S is that of Lemma 1.

A' and S together decompose U into three components, one of which is interior to S. If we throw the interior of S to either one of the others, then A' plus an arc, A'', of S from q to B_j biseparates (and locally biseparates) in U. Hence, the simple closed curve formed by A', A'' and parts of B_i and B_j is a biseparating and locally biseparating, except possibly at p, simple closed curve in M'. This procedure, the first of two we shall use, enables us to cut away from points at which boundary simple closed curves of components of $M' \setminus (\bigcup C'_i)^*$ might intersect. Denoting by $\bigcup C''_i$ the union of the C''_i's and any added curves like the one formed from A', A'' and arcs of B_i and B_j, then $M' \setminus (\bigcup C''_i)^*$ is the sum of components bounded by disjoint simple closed curves and possibly finite point sets.

Next we try to connect the simple closed curves of the boundary of a component U, of $M' \setminus (\bigcup C_i'')^*$, in such a way that U is modified into a component with a single simple closed curve boundary (and still possibly finitely many isolated boundary points). This will be the second of our general construction procedures. Since the local V_i-structures are still "big" in M' with respect to a component U, we may construct disjoint chains (in the sense of Lemma 1), each in $\overline{U}$ and in one of the V_i-structures, which nearly "string" the boundary simple closed curves together. See Figure 9.

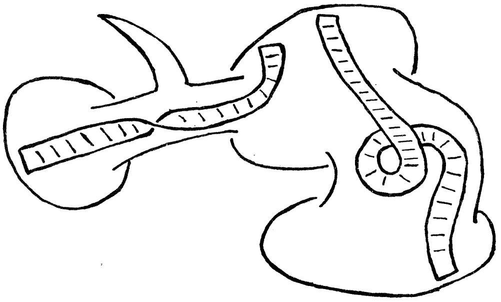


Figure 9

Now the structure of the chain whose end link comes near a given boundary simple closed curve may not be one which determined any of the arcs of this simple closed curve. Thus, it is necessary to construct connecting simple closed curves as indicated in Figure 10. It is understood that the end link of a chain associated with a given simple closed curve of the boundary of U comes close enough to lie in the local inverse incidence limit structure of an arc of that simple closed curve.

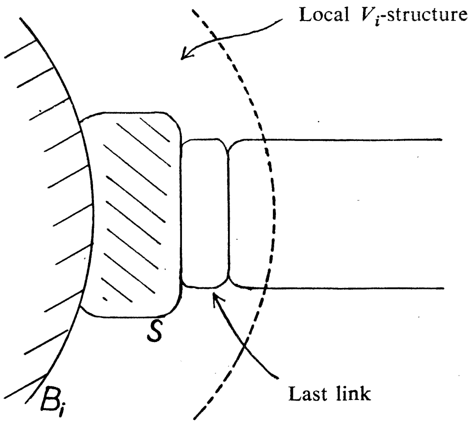


Figure 10

Suppose, for a given simple closed curve and chain, that an end link lies in the V_i-structure determining an arc of a simple closed curve of the boundary of U. Then, in this local structure, we may construct a chain which intersects the given chain (constructed, in general, in some other structure) in only the end link at the other end.

In a procedure exactly analogous to that by which we constructed the biseparating and locally biseparating simple closed curves of the preceding lemma, we construct a connecting simple closed curve, S, which biseparates and locally biseparates and intersects the boundary simple closed curve in an arc of B_i and the chain in an arc of its boundary. Figure 11, with its similarity to Figure 1 indicates the method of constructing S.

Here, the arc A to be shared by S and the boundary of the chain is to be interior to the last link at each stage of the approximation and the arcs A_i, of which A_1 and A_2

are shown, are to converge to A (rather than to a point as in Lemma 1). Again, S is partially indicated by arrows.

Now if we subtract from U the chains plus their boundaries and the connecting simple closed curves plus their interiors – as new components plus their boundaries – we will have left of U a component, of diameter less than ε, bounded by a single simple closed curve and possibly a finite number of points. Some components may still be bounded by a single simple closed curve and possibly a finite number of points.

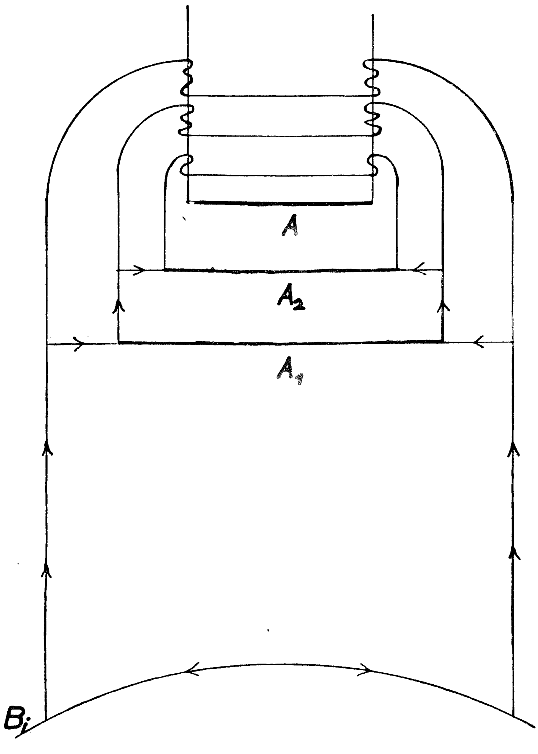


Figure 11

Some components may still be bounded by finite point sets (from the set of decomposition points) only. In this sense we have "almost" partitioned M' by a finite collection, $\{C_i''\}$, of simple closed curves. Any arc, of one of these curves, not containing a decomposition point is shared by only two of these simple closed curves – this much from local biseparation. However, a decomposition point may be a boundary point of several components of M' such as the modified U above.

Our problem, then, is to get back in M a collection of simple closed curves such that M minus the point set union of these simple closed curves is a finite number of components, each bounded by a finite number of simple closed curves. This means we must get the decomposition points out of our boundaries in M' if we are going to "blow back up" to M and continue to use the $\{C_i''\}$ as parts of boundaries: Any decomposition point is contained in a set of which the diameter of its "original" in undecomposed M is less than $\delta < \varepsilon$. Further, we take such a set to be bounded, in M', by a finite number of biseparating and locally biseparating disjoint simple closed curves, since M' is still locally an inverse incidence limit except at the "big" points. Keeping the "originals" of these sets of diameter less than δ says the bounding simple closed curves may all be chosen such that each point, except possibly for finite sets of points (a requirement explained below) lies in some finite stage of some local inverse incidence limit structure. This is allowed since each of these sets is inside some local structure from the definition of δ at the beginning of the lemma.

If a given decomposition point is not in any of the C_i'''s, we can take this set small enough that its bounding simple closed curves miss $\bigcup C_i''^*$. If a decomposition point is common to one or more of the C_i'''s, we require the simple closed curves bounding its containing set to intersect each C_i'' in only finitely many components – using the techniques of Lemma 1 which were developed for precisely this sort of situation. This explains our description above of these new simple closed curves. Call this finite collection of simple closed curves which bound sets, of diameter less than $\delta < \varepsilon$, containing decomposition points $\{T_j''\}$, and require that the T_j'''s intersect one another in only finitely many components. Then $M' \setminus (\bigcup C_i'' \bigcup T_j'')^*$ is a finite number of components each bounded by finite unions of simple closed curves (We went through the proof of this in an earlier phase of the proof of the lemma.), and decomposition points are no longer points of boundaries.

Now back in M, the T_j'''s and C_i'''s, since they do not intersect decomposition points, exist and $M \setminus (\bigcup C_i'' \bigcup T_j'')^*$ is the union of a finite number of components each bounded by a finite number of simple closed curves. Using the first construction process developed in the lemma, we may create new simple closed curves to get, finally, a finite collection, $\{S_r\}$, such that $M \setminus \bigcup S_r^*$ is a finite number of components, each bounded by a finite number of disjoint simple closed curves; each component is of diameter less than ε, and the locally biseparating S_r's intersect pairwise in only finitely many components. Using the second of our construction techniques, we may link up, working inside the closure of each component, its bounding simple closed curves to get a partitioning of M by a finite collection of biseparating simple closed curves which intersect one another nicely. All the conditions for a $\varkappa$-partitioning are now met except that two elements may intersect in more than a single arc or point. Subdividing by θ-curves, using the first of the above construction techniques, lets us finally create a $\varkappa$-partitioning of M. Since our link-ups are in sets of diameter less than ε, the mesh of our partition is less than ε and it satisfies the hypothesis of Lemma 1 in having only a finite number of points which are not part of some stage of some local inverse incidence limit structure.

Proof of theorem. From Lemmas 1 and 2, it now follows that if M is locally an inverse incidence limit, then it has a sequence, $\{P_n\}_{n=1}^{\infty}$ of $\varkappa$-partitions, mesh tending to zero, such that if $C \in P_{n+1}$, then $C \cap \bigcup_{i=1}^{n} P_i^*$ is finite. Hence, the theorem of [2] applies, and M is an inverse incidence limit globally.

References

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