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CONTINUA STRUCTURED BY FAMILIES OF SIMPLE CLOSED CURVES

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1. Introduction. The object of this paper is to generalize the concept of two-manifold to include certain spaces which triangulate like a compact two-manifold without boundary. Compact, locally connected, metric continua which partition into elements whose boundaries fit together like the boundaries of the two-simplexes of a triangulation of a two-manifold are considered using results obtained by ANDERSON and KEISLER [1, pp. 55–58].

If there is a sequence of such partitions, with mesh tending to zero, of such a space, M, and if successive collections of bounding simple closed curves can be mapped "nicely" onto preceding collections, then, for M homogeneous, easy characterizations, obtained by Anderson and Keisler, exist. These "nice" partitions and maps correspond, roughly, to successive subdivisions or refinements of a triangulation of a two-manifold. It is shown (Section 3) that a space in which a decreasing mesh sequence of partitions exists, but for which the maps of successive boundary collections are not given, i.e., a space for which the given partitions lack the sequential or "subdividing" nature suggested above is still a space for which a sequential structure exists if the following condition is satisfied: If $\{P_n\}_{n=1}^{\infty}$ is the sequence of partitions and $C \in P_{n+1}$ is a simple closed curve of the $(n + 1)$st, then $C \cap \bigcup_{i=1}^{n} P_i^*$ is the union of a finite number of components.

On the basis of our theorem, by its homogeneity and the Anderson-Keisler characterizations, the Menger Universal Curve is excluded from the class of such spaces.

2. Preliminary developments. The definitions of the Greek-letter collections of simple closed curves, of inverse incidence system and of $\varkappa$-inverse incidence limit are as in [1].

Since our aim is to generalize, in a sense, the concept of two-manifold to cover objects which triangulate like two manifolds, it is desirable to generalize the definitions of $\varkappa$, λ, μ, ν-collections to allow our simple closed curves to fit together like the one-skeletons of the elements of a triangulation of a two-manifold.

Definition. A finite collection of simple closed curves, G, is called a *$\varkappa'$-collection* if:

1) The intersection of any two is an arc or a point or is null,

2) The intersection of any three or more is a point or is null. If the point p is in exactly n ($n \geqq 3$) of the elements of G, then there is an ordering $C_1, \ldots, C_n$, of these elements, such that for $i = 1, \ldots, n$,

$$C_i \cap C_j = \begin{cases} \text{arc} & \text{if } \; j \equiv i \pmod{n}, \\ p & \text{if } \; j \not\equiv i \pmod{n}, \end{cases}$$

3) G^* is connected, and

4) Except for a finite point set, each point of G^* is in exactly two elements of G.

The definitions for λ', μ' and ν'-collections are analogous to those for the unprimed case. It is possible to show that a $\varkappa$-inverse incidence limit of $\varkappa'$-collections is still a $\varkappa$-inverse incidence limit of $\varkappa$-collections, but the proof is very dull. The argument amounts to showing that either may be regarded as an ordinary inverse limit of a sequence of finite collections of simple closed curves and that, for a sequence of $\varkappa'$-collections, there is an inverse limit in which both it and an appropriate sequence of $\varkappa$-collections appear. This seemingly pointless generalization is justified in the sequel, where $\varkappa'$-collections are vastly more convenient to work with. The primes on Greek letters are dropped henceforth.

3. A characterization. We shall show that if a compact, locally connected metric continuum M has a sequence of $\varkappa$-partitions with mesh tending to zero, then, even though each does not ν-refine the preceding and the connecting maps required of a $\varkappa$-inverse incidence limit are lacking, M is still a $\varkappa$-inverse incidence limit if the partitions satisfy a finiteness condition with respect to their intersections.

Our theorem is not the most desirable theorem here; however, to generalize it by removing this restriction appears to present grave technical difficulties.

Definition. The simple closed curve S *biseparates* M if $M \setminus S$ is the sum of two components. S *locally biseparates* M if for $p \in S$ and $\varepsilon > 0$, there is an open set U, containing p, and contained in the ε-sphere about p, such that S separates U into two components with an arc of S as common boundary.

Theorem. *Let M be a compact, locally connected, metric continuum with the following property: There exists a sequence of $\varkappa$-partitionings, $\{F_n\}_{n=1}^{\infty}$, of M such that:*

1. *Mesh $F_n \to 0$, and*

2. *$C \in F_{n+1}$ implies $C \cap \bigcup_{i=1}^{n} F_i^*$ has only a finite number of components. Then M is a $\varkappa$-inverse incidence limit. (The $\varkappa$-collection $\{C_1, \ldots, C_n\}$ of simple closed curves in M $\varkappa$-partitions M if each C_i biseparates and locally biseparates in M.)*

The proof of the theorem requires three lemmas to justify a basic construction and seven more to show that this construction effects a proof. Hereafter, M and the sequence $\{F_n\}$ are to be as in the statement of the theorem.

Lemma 1. *The sequence* $\{F_n\}$ *is such that for each* i *there is a* $\delta(i) > 0$ *such that if mesh* $F_j < \delta(i)$, *then no element of* F_j *contains in the closure of its interior the closure of the interior of an element of* F_k, $1 \leqq k \leqq i$.

We may suppose, without loss of generality, that the original sequence, $\{F_n\}_{n=1}^{\infty}$, is such that the closure of the interior of no element of F_j contains the closure of the interior of an element of F_i, $i < j$.

Lemma 2. *Each component of* Int $C_1 \cap \ldots \cap$ Int C_n, $C_i \in F_i$, *is bounded by the union of a finite number of simple closed curves.*

Proof. Suppose $n = 2$, then Bdry [Int $C_1 \cap$ Int C_2] is the union of a finite number of arcs from Cl [$C_2 \cap$ Int C_1] and from Cl [$C_1 \cap$ Int C_2], plus a finite number of arcs from $C_1 \cap C_2$.

We show now how these arcs may be expressed as union of a finite number of simple closed curves. Consider, in a three-face of a Hilbert cube, a simple closed curve identified as C_1. The configuration is completed by adding arcs in the Hilbert cube which are copies of each of the open arcs of $C_2 \cap$ Int C_1. At the points corresponding to those at which C_2 crosses C_1 from Int C_1 to Ext C_1, we tie the ends to C_1. Where the endpoints of an arc of $C_2 \cap$ Int C_1 are endpoints of arcs, possibly degenerate, shared by C_1 and C_2, we terminate them on C_1 and identify the arcs of C_1 (the copy) corresponding to these arcs of $C_1 \cap C_2$ in M. Now, given an orientation on C_1 and starting from a point of C_1 in Int C_2 (We do not really have a problem if there are no such points.), we proceed to an intersection with an arc of the copied arcs of C_2 if such an arc (and intersection) exists.

Suppose such an arc does not exist. Then either Ext C_1 in M is contained in Int C_2 or Cl (Int C_2) $\subset$ Cl (Int C_1). We exclude the first case by requiring each of the original F_i's to contain more than one element and by recalling the stipulation about the F_i's following Lemma 1. In the second case, Int $C_1 \cap$ Int C_2 is bounded by C_2.

If intersections of C_1 with the closures of the copied arcs of C_2 do exist, then we proceed, in the given orientation along C_1 to such an intersection. This point of intersection may be a point at which C_2 crosses C_1 in M or an endpoint of a common arc of C_1 and C_2. If this point is a crossing, we turn off on the arc of C_2 leading into the interior of C_1 (See Figure 1a).

If the common arc, possibly degenerate, of C_1 and C_2 is bounded at both ends by arcs leading into Int C_1, we turn off onto the first of these in the given orientation (See Figure 1b). If the arc of $C_2 \cap$ Int C_1 at which we have arrived leads into Int C_1, we turn onto it from C_1 (See Figure 1c). (To stay on C_1 past the endpoint of the common arc and into its interior would be to traverse points which are not boundary points of Int $C_1 \cap$ Int C_2.) If the arc of $C_1 \cap C_2$ at which we have arrived is bounded at this end by an arc of C_2 coming from outside C_1, we stay on the common arc of $C_1 \cap C_2$ and turn off into Int C_1 along the arc of C_2 at the other end (See Figure 1d).

At the other end of the copy of an arc of $C_2 \cap$ Int C_1, we turn onto the arc of C_1 which is interior to C_2 or common to C_1 and C_2, etc.

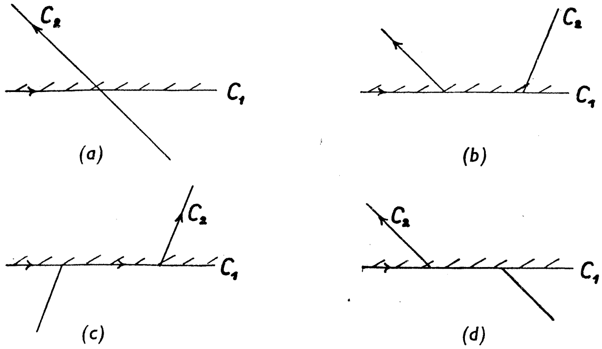

Figure 1

The set of points traversed in this way is one dimensional, has no cut points and no local cut points of order greater than two – a simple closed curve. The same procedure for other arcs of $C_1 \cap$ Int C_2, not already traversed, gives other simple closed curves, and after finitely many circuits all the boundary points (in our copy) of Int $C_1 \cap$ Int C_2 are covered in this way. The boundary of Int $C_1 \cap$ Int C_2 is a finite number of non-overlapping (in the sense that no arc of one is shared with another) simple closed curves.

The proof of the assertion for boundaries of components of Int $C_1 \cap \ldots \cap$ Int C_n, $C_i \in F_i$, $n > 2$, is an easy generalization of the argument above. For example, Int C_3 is intersected with the components of Int $C_1 \cap$ Int C_2, each of which is bounded by unions of curves – instead of just one – and so on.

Definition. Let, for $i > 1$, $B(C_1, \ldots, C_i)$ denote the boundary, the union of a finite collection of simple closed curves, of Int $C_1 \cap \ldots \cap$ Int C_i, $C_j \in F_j$. It will also be convenient to denote by $P(C_1, \ldots, C_j)$ the finite set of points (Lemma 2) common to two or more of the simple closed curves of $B(C_1, \ldots, C_i)$.

Lemma 3. *The sequence $\{F_n\}_{n=1}^{\infty}$ is such that for each i, each $B(C_1, \ldots, C_i)$ and each finite subset $Q(C_1, \ldots, C_i)$ of points of $B(C_1, \ldots, C_i)$, there is a $\delta = \delta(i; C_1, \ldots$ $\ldots, C_i; Q(C_1, \ldots, C_i)) > 0$ such that if* mesh $F_j < \delta$, *then there is, for each pair of maximal open arcs A_1 and A_2 (or for each pair of simple closed curves or pair of maximal open arc and simple closed curve) in $B(C_1, \ldots, C_i) \setminus Q(C_1, \ldots, C_i)$ connected by a component of* Int $C_1 \cap \ldots \cap$ Int C_i, *an arc of F_j^* with endpoints in A_1 and A_2 (or in the pair of simple closed curves or in the arc and in the simple closed curve) and otherwise missing $B(C_1, \ldots, C_i)$.*

(This says that for small enough mesh $\varkappa$-collections, F_j, the boundary simple closed curves of a component of Int $C_1 \cap \ldots \cap$ Int C_i are connected in the F_j-structure of M.)

Proof. Consider a component, K, of some Int $C_1 \cap \ldots \cap$ Int $C_i \neq \emptyset$, fixed i, bounded by the union, B_K, of some subcollection of the simple closed curves determining $B(C_1, \ldots, C_i)$. Consider a pair of maximal open arcs A_1 and A_2 in $B_K \setminus \setminus Q(C_1, \ldots, C_i)$, with fixed collection $Q(C_1, \ldots, C_i)$. Let C be an open arc in $K \setminus B_K$ with endpoints in each of A_1 and A_2. Then there is a $\delta(C, A_1, A_2) > 0$ such that, for an F_j of mesh $< \delta(C, A_1, A_2)$, there is a (not necessarily simple) chain, D, of closures of interiors of elements of F_j which contains Cl C such that there is an arc of Bdry $D(\subset F_j^*)$ from A_1 to A_2 in $K \setminus B_K$. $\delta(C, A_1, A_2)$ can be chosen as the δ of a sufficiently small δ-neighborhood of C in M.

Since there are only finite numbers of pairings of open arcs like A_1 and A_2 in K, and of components in Int $C_1 \cap \ldots \cap$ Int C_i, there is a δ small enough to serve all simultaneously. The argument for a pair of simple closed curves or for a pair consisting of a maximal open arc and a simple closed curve is similar.

Since there is only a finite number of non-empty intersections of interiors of elements of $\{F_n\}_{n=1}^{i}$ and since the union of the sets $P(C_1, \ldots, C_i)$ is a finite point set, there is a $\delta(i)$ sufficiently small to insure that for F_j of mesh $< \delta(i)$, there is for each component K of each Int $C_1 \cap \ldots \cap$ Int C_i, $C_k \in F_k$, an arc of F_j^* between each pair of the maximal open arcs (or simple closed curves, etc.) of Bdry $K \setminus P(C_1, \ldots, C_i)$ if Bdry $K \cap P(C_1, \ldots, C_i) \neq \emptyset$. We shall suppose, hereafter, that F_{i+1} is always of fine enough mesh to connect the boundary components of each component of each Int $C_1 \cap \ldots \cap$ Int C_i in the above manner.

We now construct a manifold associated with each collection $\{F_1, \ldots, F_n\}$: Consider a copy of $\bigcup_{k=1}^{n} F_k^*$. For each component K of each Int $C_1 \cap \ldots \cap$ Int $C_n \neq \emptyset$, $C_k \in F_k$, $1 \leqq k \leqq n$, in M, consider a two-sphere with tubes leading off and "sewn-in" along each of the boundary simple closed curves (Lemma 2) of K (in the copy of $\bigcup_{k=1}^{n} F_k^*$), the "end" of one tube for each of the curves. For a component bounded by a single simple closed curve, the corresponding manifold is just a disk. In fact, in general we shall refer to the component-of-intersection "manifold" corresponding, in the copy, to K in M even though identification of finite numbers of points of the bounding simple closed curves makes this inaccurate. It is clear that these component-of-intersection manifolds – allowed to intersect only on $\bigcup_{k=1}^{n} F_k^*$ – fill in all the simple closed curves identified with boundaries of intersections, Int $C_1 \cap \ldots \cap$ Int $C_n \neq \emptyset$ in M and that even along the arcs of the copy $\bigcup_{k=1}^{n} F_k^*$, we get a space which is locally E^2 since each "side" of an arc is used as a boundary for a "sewing" just once.

It must be mentioned here that the tubes leading to the bounding simple closed curves of a component-of-intersection manifold cannot be sewn on in a purely arbitrary fashion. We might, for example, fill in with a component-of-intersection manifold to yield a non-orientable manifold when M was an orientable manifold to start with.

To make sure the two-spheres with tubes filling in the boundary simple closed curves of a component of intersection do so "properly", we must examine an additional F_m-structure, $m > n$. Let K be a component of intersection of interiors of elements of $F_1, \ldots, F_n$, in M. Let K have as boundary the collection of simple closed curves B, with point set union B^*. By Lemma 3, F_{n+1} is of small enough mesh that the simple closed curves of B are all connected by arcs in F^*_{n+1}. Consider the manifold determined by a copy of the union of $\{F_1, \ldots, F_n, F_{n+1}\}$ in which the copies of boundaries of components of intersections of interiors of elements of F_{n+1} with interiors of elements of the other collections are filled in with two-spheres and tubes leading off to boundary simple closed curves in an arbitrary sewing. Now the copy of B bounds a "manifold" which is a two-sphere with tubes leading to the simple closed curves of B and possibly added crosscaps and handles introduced by the F_{n+1}-structure. If each of these extra features is inclosed in a biseparating simple closed curve such that the simple closed curves so obtained are pairwise disjoint, and if the closure of the interior of each such simple closed curve is identified to a point, the resulting "manifold" is a two-sphere with tubes leading off to boundary simple closed curves. Making these identifications for each of the collections $F_1, \ldots, F_n$, gives a manifold determined by $\{F_1, \ldots, F_n\}$ which is "consistent" with later structurings.

Two observations remain to be made regarding this process: First, there may be more than one way to decompose a component-of-intersection manifold in the manifold determined by $\{F_1, \ldots, F_n, F_{n+1}\}$ to get a two-sphere with tubes leading to the boundary simple closed curves. However, the "sewing" to the boundary curves are at least determined as they must be for finer future structures – orientability or non-orientability preserved, for example. Second, since we shrank out the handle-producing ones in the decomposition manifold, it is clear that it does not matter how the two-spheres with tubes filling in interiors of elements of the copy of F_{n+1} are sewn in.

If it is desired – and it will be – to construct such a copy in a particular Hilbert Cube, one might start with a copy of $\bigcup_{k=1}^{n} F_k^*$ in a three-face and then, for each component-of-intersection manifold added, retreat into a higher dimensional face to avoid unwanted intersections.

It is also natural at this point to require that each component-of-intersection manifold, in each such imbedding, have diameter no more than some fixed $\delta > 1$ times the diameter of its boundary.

Note that there is a natural comparison between M', determined by $\{F_1, \ldots, F_n\}$, and M'', determined by $\{F_1, \ldots, F_n, F_{n+1}, \ldots, F_{n+j}\}$, $j \geqq 1$. Each is $\varkappa$-partitioned by a copy of F_k, $1 \leqq k \leqq n$. Also, corresponding to each component K, in M, of each $\text{Int } C_1 \cap \ldots \cap \text{Int } C_i$, $C_k \in F_k$, $1 \leqq k \leqq i$, is a component-of-intersection manifold K_n in M' and one, K_{n+j}, in M''. By the note following Lemma 3, F_{n+1} represents a "fine enough" structuring of M that all the boundary simple closed curves of K are connected in the F_{n+1} "framework" of M and hence K_{n+1} is the two-sphere with tubes of K_n with, possibly, additional crosscaps and handles. This says that the existence of a $\varkappa$-partitioning collection, P', of simple closed curves in M' implies the existence of a $\varkappa$-partitioning homeomorphic copy, P'', in M'' of P' in M'. Further, P'' may be chosen so that the closure of the intersection of P''^* with the interor of an element of F_k in M'' is homeomorphic to the closure of the intersection of P'^* with the interior of the "same" (corresponding) element of F_k in M'.

Consider the manifold M_1 determined by the first N of the F_i's. Since it is a two-manifold, there is a $\varkappa$-partitioning, P, which ν-refines each of the $\varkappa$-partitionings of the copies of the F_i's. It will be convenient to locate the distinguished points (those common to more than two of the curves) of P on $\bigcup_{i=1}^{n} F_i^*$ for some sufficiently large $n \geqq N$ in a manifold M_n structured by copies of each of $F_1, \ldots, F_N, \ldots, F_n$.

Let p be a distinguished point of P (if there is such a point) which is not contained in $\bigcup_{i=1}^{N} F_i^*$. A homeomorphic copy, $P(1)$, of P may then be chosen in the manifold M_2, determined by $\{F_i\}_{i=1}^{N+1}$, so that the distinguished point corresponding to p is in $F_{N+1}^* \cap \text{Int } C_1 \cap \ldots \cap \text{Int } C_N$ where C_i is the element of F_i in whose interior p lay in M_1. This may be accomplished by "sliding" the copy of p in $P(1)$ in M_2 over to the F_{N+1}-structure. Likewise, for a second distinguished point q of P in M_1, we may require the copy of q in M_3, determined by $\{F_i\}_{i=1}^{N+2}$, to be contained in F_{N+2}^*. The net effect of all of this is that from some $n = N + k$ on, we have a manifold M_n which is $\varkappa$-partitioned by a ν-refinement of each of $F_1, \ldots, F_N$ and such that all the distinguished points of P lie on $\bigcup_{i=1}^{n} F_i^*$.

Now we need to say something about the "size" of crosscaps and handles in a manifold M_k, determined by $\{F_i\}_{i=1}^{k}$, some k. Actually, all we really need to discuss is the "size" of handles and crosscaps in an M determined by a single F_i—as we shall see below.

In such an M', consider a handle and denote by U the union of the closures of the disk-interiors of a sub-collection of the elements of F_i. U may be thought of as bounded by the union, possibly empty, of a finite collection of simple closed curves, Bdry U. Let U be such that it contains in its interior a simple closed curve inscribed on the given handle which is not homotopic to a constant in M'. We say U contains the handle if each such simple closed curve is still not homotopic to a constant in the decomposition manifold obtained from U by identifying each of the boundary simple closed curves, if any, to a point. This says, that, in some sense, U provides a "base" for the handle.

We define the diameter of the handle in M' to be the minimum of the diameters, in the metric of M, of the collections of elements of F_i determining sets U which contain it. It is a measure, in terms of the structuring of M' by F_i, of the size of the handle.

Similarly, for a given crosscap, define its diameter to be the minimum of the diameters of the collections of elements of F_i determining, with their disk-interiors, the sets which contain it.

We shall wish in the sequel to be able to identify as the "same", handles and crosscaps in different manifolds determined by different collections $\{F_i\}_{i=1}^{n}$. To do this we make the following – promised – basic construction: Consider the manifolds M_n, determined by $\{F_j\}_{j=1}^n$, imbedded successively in the same Hilbert cube. Let the imbeddings be such that each M_n is contained in a finite-dimensional face of the Hilbert cube and such that the imbedded manifolds intersect in exactly the copies of the F_k's. This second condition may require retreating to higher-dimensional faces with each such successive imbedding to prevent component-of-intersection manifolds from intersecting except along boundaries. If the diameters of component-of-intersection manifolds are kept bounded by a common factor of δ times the diameters of their boundaries, as has been our practice, the limit set of the sequence is clearly M. We shall presume, henceforth, that such an imbedding has been made for a sequence $\{M_n\}_{n=1}^{\infty}$.

Now handles and crosscaps on some M_n, produced by the F_j-structures, can be identified with corresponding handles and crosscaps on an M_m, $m > n$, in terms of the common points of $\bigcup_{j=1}^n F_j^*$. Of course, the corresponding handles or crosscaps in the more finely structured M_m may themselves be studded with handles and crosscaps.

We may always choose an $n > k$ such that the M'' determined by $\{F_j\}_{j=1}^n$ reproduces at least those crosscaps and handles of M', determined by $\{F_j\}_{j=1}^k$, which are obtained by sewing together the manifolds bounded by elements of F_k. Each such crosscap or handle may, as noted above, have additional crosscaps and handles sewn on it by the component-of-intersection structurings of the other F_j's. Without loss of generality, we shall henceforth assume that the basic sequence $\{F_i\}_{i=1}^{\infty}$ of the theorem has the property that the manifold M'', obtained by filling in with disks the interiors of elements of a copy of F_{i+1} is, except for possible additional crosscaps and handles, a homeomorph of M', the corresponding manifold obtained for F_i. This is what we have indicated as possible above, relative to the given sequence of imbeddings in the Hilbert cube and with each of the crosscaps and handles in which we were interested possibly carrying further crosscaps and handles produced by the other F_j-structurings. In fact, since F_{i+1} is already of small enough mesh to connect boundary components of a component of intersection, Int $C_1 \cap \ldots \cap$ Int C_i, $C_j \in F_j$, and thus to reproduce, with possible additional features, the component-of-intersection manifolds chosen for M_i, determined by $\{F_j\}_{j=1}^i$, we may more generally require that M_{i+1}, obtained from $\{F_j\}_{j=1}^{i+1}$, is, except for possible additional crosscaps and handles, a homeomorph of M_i.

Let us now return to our earlier discussion, in which we assume a $\varkappa$-partitioning P of M_n, determined by $\{F_i\}_{i=1}^n$, which ν-refines each of the first N of the F_i's and such that the distinguished points of P all lie in $\bigcup_{i=1}^n F_i^*$. Before the next lemma, we need to say what we mean by the statement that the arc $A \subset P^*$, connecting distinguished points of P, *separates in the manifold* M_k *to within* ε. Suppose we have a not-necessarily-partitioning copy of P in M_k, determined by $\{F_i\}_{i=1}^k$, $k \geqq n$, such that all of A, except for ε-small sets containing its endpoints, is contained in $\bigcup_{i=N+1}^k F_i^*$. This will be the situation in the sequel. Suppose, further, that the copy of P would be a $\varkappa$-partitioning of M_k, were it not for the possible existence of crosscaps and handles in the interiors of elements of F_N whose diameter-determining collections of elements, in the collections F_j, $j > N$, are of diameter $\leqq \varepsilon$ and intersect $A \cap \bigcup_{i=N+1}^{k} F_i^*$ in their interiors. What this amounts to is that P would be in M_k a $\varkappa$-partitioning except, possibly, for ε-small crosscap and handle "leaks" in the neighborhood of A. With this definition and M_n and P as above, we have:

Lemma 4. *Let A be an arc of P^* intersecting the distinguished points of P only in its endpoints, p and q. Let A be contained in* Cl (Int C), $C \in F_n$. *For each $\varepsilon > 0$ there is a $k > n$ such that in the interior of C in M_k, determined by $\{F_i\}_{i=1}^{k}$, there is an arc, $A(\varepsilon)$, between p and q which separates to within ε (A copy of P may be inscribed in M_k which agrees with P in M_n on $P^* \cap \bigcup_{i=1}^{n} F_i^*$) and which is contained in F_k^*, except for two mutually separated sets of diameters $< \varepsilon$, each containing one of p and q.*

Proof. Let k be large enough that δ times the mesh of F_k is $< \varepsilon$, (δ the bound on the size of the component-of-intersection manifolds), $(\delta/2)\, k < \varepsilon$, and that p and q in M_k are in the closures of the interiors of disjoint elements of F_k in M. Further, let k be large enough that p and q do not lie in the closures of the interiors of any elements of F_k which are in minimal diameter collections containing handles or crosscaps of diameter $\geqq \varepsilon$. Our imbedding of the manifolds in the Hilbert cube, agreeing on the F_j^*'s, again permit us to identify those handles and crosscaps for which diameter-defining collections, when they appear, will always be as big as ε in each manifold. Still further, let k be large enough that there exists, for each crosscap and handle of diameter $\geqq \varepsilon$, a minimal diameter collection containing it, with closure missing C. Consider M_k minus the interiors of those elements of F_k which are elements of a minimal diameter collection determining the F_k-diameter of each crosscap or handle of diameter $\geqq \varepsilon$, one such collection for each crosscap or handle. (This may require us to choose an F_k of still smaller mesh.) Subtract also those arcs of F_k^* which in this new space have had the interiors of elements on "both sides" removed – but only if they were removed as interiors of the same crosscap or handle collection. We are, in effect, leaving the "outer" boundary of each such collection intact.

We claim that what is left of M_k is connected: Each of the sets – let us denote them by $U_1, \ldots, U_m$, some m – which has been removed is the union of the interiors of elements of F_k plus common boundary arcs. Hence, each of these sets, and each component of $U_j \cap \bigcup_{i \neq j} U_i$, $j = 1, \ldots, m$, is an open set bounded by the union of a finite number of simple closed curves in F_k^*. Removing these non-overlapping open sets (the components) does not disconnect the manifold M_k. It is now possible to pass an arc in the remainder of the F_k^*-structure interior to C from near ($< \varepsilon$) p to near q. Any crosscaps or handles preventing separation by such an arc will necessarily be of diameter $< \varepsilon$ (less than δ times mesh F_k, in the usual sense, for component-of-intersection manifolds in Int C). To complete our arc $A(\varepsilon)$, and to get it to terminate at p

and q, it will, in general, be necessary to leave the F_k^*-structure — but only within ε of each of p and q — and finish the arc in what is left of M_k. The remaining arcs of a copy of P may now be inscribed with the result that the arc $A(\varepsilon)$ of P separates to within ε in M_k.

Lemma 5. *Let A be an arc of P connecting distinguished endpoints p and $q \in \bigcup_{i=1}^n F_i^*$ in $\mathrm{Cl}\,(\mathrm{Int}\, C)$, $C \in F_n$ in M_n. Let A contain no other distinguished points of P. Then there is a sequence of arcs, $\{A(i)\}_{i=1}^{\infty}$, each $A(i)$ of which separates to within $1/2^i$ in M_n, determined by $\{F_j\}_{j=1}^{n_i}$, $n_i \geqq n_{i-1} \geqq \ldots \geqq n$; each is contained, except for two disjoint sets of diameter $< 1/2^i$, each containing one of p and q, in $F_{n_i}^*$, and each is such that the limiting set of the sequence is an arc, A, between p and q.*

Proof. By Lemma 4, each $A(i)$ exists; the problem here is to show that they may be chosen so that the limit set is also an arc.

Since $A(i)$ separates to within $1/2^i$ in M_{n_i} and $A(i+1)$ to within $1/2^{i+1}$ in $M_{n_{i+1}}$, we may choose n_{i+1} sufficiently larger than n_i, and mesh $F_{n_{i+1}}$ sufficiently smaller than mesh F_{n_i}, that $A(i+1) \cap F_{n_{i+1}}$ need be "perturbed" by no more than $\delta/2^i$, the bound on the diameters of crosscaps and handles it must skirt but $A(i) \cap F_{n_i}^*$ need not, from the position of $F(i) \cap F_{n_i}^*$ in the Hilbert cube in which all are imbedded. Further, even the corresponding subsets (at each end) of $A(i+1) \setminus F_{n_{i+1}}^*$ and of $A(i) \setminus F_{n_i}^*$ in $M_{n_{i+1}}$ need not be more than $\delta/2^i$ apart. In short, $A(i+1)$ and $A(i)$ need be "crooked" with respect to each other on sets of diameters no greater than $\delta/2^i$.

This allows us to assert the existence of a homeomorphism $h_i : A(i) \to A(i+1)$ such that $h_i(p) = p$ and $h_i(q) = q$ and the distance in the Hilbert cube (or in M) between $x \in A(i) \cap F_{n_i}^*$ and $h_i(x)$ is $< \delta/2^i$. We are requiring, as we may, that $h_i(A(i) \cap \cap F_{n_i}^*) \subset A(i+1) \cap F_{n_{i+1}}^*$. We wish now to show that the family, $\{A(i)\}_{i=1}^{\infty}$, of arcs is equicontinuous[1]) and hence that the limiting set, A, is also an arc.

In the Hilbert cube, the limit set, A, of the $A(i)$'s is a continuum containing p and q. We may require of the h_i's that, in fact, for all $x \in A(i)$, the distance in the Hilbert cube between x and $h_i(x)$ is less than $\delta/2^i$. Now, given $\varepsilon > 0$, first let i be large enough that $2\delta \sum_{j=i}^{\infty} 1/2^j < \varepsilon/2$ and then let $0 < \gamma' < \varepsilon/2$ be small enough that if xy is an interval of $A(i)$ of diameter less than γ', each of the intervals $h_{i-1}^{-1}(xy)$ and $h_k^{-1} h_{k+1}^{-1} \ldots \ldots h_{i-1}^{-1}(xy)$, $1 \leqq k \leqq i-2$, is of diameter less than ε. This last takes care of the first i homeomorphs of xy and the diameters of the rest are less than $\gamma' + 2\delta \sum_{j=i}^{\infty} 1/2^j < < \varepsilon$. Thus, if we choose our γ' to be the γ of the definition of equicontinuity, we have shown $\{A(i)\}_{i=1}^{\infty}$ to be equicontinuous and then, as is well-known, A is an arc.

[1]) A collection G of arcs is equicontinuous if, for every $\varepsilon > 0$, there is a $\gamma > 0$ such that if x and y are any two points of an arc $g \in G$ at a distance apart less than γ, then the diameter of the interval xy of g is less than ε.

Lemma 6. *There exists in M a copy of P in which each of the arcs between distinguished points (and containing none in its interior) is the limit of a sequence of arcs of the $A(i)$-type described in Lemma 5.*

Proof. We have already seen in Lemma 5 how to obtain in M one such of the arcs connecting distinguished points of P. Let us suppose that, given an ordering of the finite number of such arcs composing P, we have constructed, in M, the limit arcs, $\{A_j\}_{j=1}^k$, of the first k of them and wish to construct the $(k+1)$st.

Even though they may share endpoints, there is an open set O_i, $i = 1, \ldots, k$, containing the interior of each A_i, $i = 1, \ldots, k$, in M such that the O_i's are pairwise disjoint, and such, in fact, that their closures intersect only at the distinguished endpoints of the arcs they contain. We may require the O_i's to contain only points sufficiently near their respective A_i's that $M - \bigcup_{j=1}^k O_j$ is connected. In addition, we require the O_i's to be small enough in M about their respective arcs that for $n \geqq$ $\geqq$ some N there is enough of the copy of $F_n^* \setminus \bigcup_{j=1}^k O_j$ left to provide in M_n, determined by $\{F_i\}_{i=1}^n$, homeomorphs of each of the crosscaps and handles in each of the elements of F_N in M_N. In other words, the O_i's are to be "unobtrusive" enough to permit, for some N, the F_N-structure except, possibly, near ($\leqq$ mesh F_N) its distinguished points.

Now in a sequence of arcs, $\{A_{k+1}(i)\}_{i=1}^{\infty}$, $1/2^i$-separating approximations for A_{k+1} in the M_n's, from some i on, corresponding to some sufficiently richly "veined" F_{n_i}-structure, the $A_{k+1}(i)$'s can be chosen, except for $1/2^i$-small sets containing their endpoints, in $\bigcup_{j=1}^{n_i} F_j^* \setminus \bigcup_{j=1}^k O_j$ in M_{n_i}. Hence, the limiting set, A_{k+1}, will intersect $\bigcup_{j=1}^k A_j$, if at all, only in its endpoints – from which the conclusion of the lemma follows.

Lemma 7. *Each simple closed curve C of the $\varkappa$-collection P, constructed in Lemma 6, separates (and biseparates) in M.*

Proof. Before we can claim C separates in M, we must indicate the subset claimed to be the interior of C. We proceed to a definition of the "interior" of C: Our construction, one at a time, of the limit arcs A_j, $j = 1, \ldots, L$, which determine P in M was undertaken in Lemma 6 so that approximations to the $(k+1)$st arc avoided, for large enough subscripts, those parts of F_i-structures contained in certain open sets (of M), $\{O_j\}_{j=1}^k$, containing the interiors of the first k limit arcs. This says that for some sufficiently large i_0, all approximations $A_j(i)$, $i > i_0$, are disjoint except, possibly in small open sets containing their distinguished endpoints. In each of these sets, in each of the M_{n_i}'s, we can alter, without affecting the limit arcs, the approximating arcs $A_j(i)$ – which needn't be carried in $\bigcup_{j=1}^n F_j^*$ here anyway – so that the $A_j(i)$'s intersect only at the distinguished points of P. The result is a $\varkappa$-collection of simple closed curves with union homeomorphic to P^*, each element of which separates to within $1/2^i$ in M_{n_i} – with the natural extension for simple closed curves of our definition of $1/2^i$-separation for arcs. Let us denote by $C(i)$ the simple closed curve corresponding to C in the copy of P in M_{n_i}, $i > i_0$. Each $C(i)$ has a naturally defined "interior", i.e., those points of M_{n_i} which would be interior to $C(i)$ were it not for the possible existence of crosscap and handle "leaks" of diameter $< 1/2^i$. To put it another way, each of the crosscaps and handles of diameter $< 1/2^i$ in M_{n_i} is contained *in* a set made up of the interiors of elements of F_{n_i} and of arcs of $F^*_{n_i}$ adjoined on both sides by these interiors. If each of these connected sets is decomposed to a point, then in the decomposition space, what is left of $C(i)$, not necessarily a simple closed curve anymore, separates. Those points of M_{n_i} which were in the interior of C in the copy of P in M_{n_i} and which are separated from the rest of M_{n_i}, decomposed, by $C(i)$, decomposed, we call the interior of $C(i)$.

We define the interior of C in M to be the limiting set of the sequence of interiors of $C(i)$, $i = 1, \ldots$ It is easy to see that C separates the interior of C, so defined, from the rest of M: Suppose p is in the interior of C and q is a point of M in neither C nor interior C. Then, if C does not separate p from q in M, there is an arc A in M, missing C, with endpoints p and q. A is contained in a chain, not necessarily simple, of closures of interiors of elements of F_i, each i, in M. For large enough $i >$ some i_0, and small enough mesh F_i, these chains will also miss C. Each such chain of F_k-elements, $k > i_0$, then contains an arc A_k in F^*_k in M (and in M_{n_j}, $n_j \geqq k$) from the element of F_k whose closure contains p to the element whose closure contains q. Consider what must happen for a fixed $k > i_0$ with A_k missing C. Since M_{n_j}, some $n_j > k$, contains A_k and since M_{n_j} contains only crosscaps and handles of diameter $> 1/2^i$, for some i, the $1/2^i$-separating simple closed curves, $C(i)$, must intersect A_k from some i on. This implies the limiting set C intersects A_k as well, a contradiction.

It is clear from the increasingly rich "veining" or "webbing" of the interiors of the $C(i)$'s, as i increases, that interior C is connected, implying that C not only separates but biseparates as well. Another way of seeing this is to observe that the interior of C, where "interior" now has the usual meaning, is the limiting set of the sequence of connected sets $\{\text{interior } C(i)\}_{i=1}^{\infty}$ and is thus connected. We shall prove biseparation in still another way in the sequel.

The next, and perhaps most natural step would seem to be to show each element of P locally biseparates. This, however, will be a very simple consequence of showing that M is the inverse limit, in the ordinary sense, of a sequence of $\varkappa$-collections each of which ν-refines the preceding (except, possibly, for local biseparation by each of the elements).

Lemma 8. *There exists a $\varkappa$-collection P' in* **M** *which ν-refines (except, possibly, for the local biseparation required in the definition of ν-refining) each of F_{N+1} and the $\varkappa$-collection P of the preceding lemmas.*

Proof. Let P_0 be a $\varkappa$-partitioning of M_m, determined by $\{F_i\}_{i=1}^{m}$, some $m > N +$ $+ 1$, which is homeomorphic to P in M and has all its distinguished points located at the corresponding points of P in M (They are also in M_m).

Now P, in M, and F_{N+1} may not, as collections of simple closed curves in M, have the finite-number-of-components-of-intersection property of the theorem and the sequence $\{F_i\}_{i=1}$. We may, however, choose P_0 in M_m to be such that each simple closed curve of P_0 intersects each of the curves of F_{N+1} in only a finite number of components. Let P_0' be a ν-refinement of each of P_0 and F_{N+1}. Let $n \geqq m$ be chosen large enough so that the distinguished points of P_0' which are not in P_0^* are in $\bigcup_{i=1}^{n} F_i^*$. This is a convenience we have justified before.

The strategy for the remainder of the proof will be to alter P_0' slightly so that the remaining distinguished points of P_0' (in P_0^*) are points of P in M. Then, if each of the original arcs between distinguished points of P in M can be rederived as unions of arcs between original distinguished points and the newly added distinguished points of the altered P_0', we shall be able to fit in the remaining necessary limiting set arcs for P', a homeomorph of P_0' — just as we constructed P in Lemma 6.

In M_n, each arc of P_0, with distinguished points of P_0 at each end and none in the interior, will, as an arc of $P_0'^*$, contain no more than some k distinguished points of P_0' in its interior. Our problem is to find k "accessible" points on each such arc, A, of P^* in M (distinguished endpoints and no interior distinguished points) which are available for use as distinguished points of P_0'. Suppose, for example, that a "tail" of the closure of the interior, in M, of an element of F_{N+1} spirals down around a point p, possibly distinguished, of P^* as "vortex". Clearly, p, now also a point of F_{N+1}^* need not be used as a distinguished point of the ν-subcollections of P_0' ν-refining the elements of F_{N+1} in M_m which contain it. It must also not be used as a distinguished point of the ν-subcollections refining the elements of P in which p lies, and we must show that other points of P^* are available for such use.

Each arc A, as above, of P^* is either contained in F_{N+1}^* or contains a segment, A^0, disjoint from F_{N+1}^*. Choose $n' > n$ large enough that, in M, each such arc A of P^*, or arc A^0 if it exists, contains at least k different points of F_n^*. Now, back in $M_{n'}$, determined by $\{F_i\}_{i=1}^{n'}$, we consider a copy of P_0' with distinguished points of P_0' in $\bigcup_{i=1}^{n} F_i^*$ located as before. We modify P_0' by "sliding" the arcs of P_0 containing the remaining distinguished points of P_0' (in P_0) over to the F_n^*-structure at a finite number of points to make the remaining distinguished points coincide with points of $P^* \cap F_{n'}^*$ in M. (Our choice of n' guaranteed the existence of enough such available points.) The arcs of P_0' which terminated at these "transported" points may be made to "trail along", preserving P_0' as a partition. Another way of describing the process above is to say that some or all of the arcs of P_0, not contained in $\bigcup_{i=1}^{N} F_i^*$ (The arcs of $\bigcup_{i=1}^{N} F_i^* \cap P_0^*$ are fixed), be required to contain finite numbers of points of $F_{n'}^* \setminus F_{N+1}^*$ in $M_{n'}$. These points are then to be used as the remaining (not already fixed in $\bigcup_{i=1}^{n} F_i^*$) distinguished points of P_0'.

One more comment needs to be made regarding our latest version of P_0'. Some of the points of P_0, which may also have been points of P, may have had to be abandoned as distinguished points of the refinement, P_0', of both F_{n+1} and P_0 — for example, the "vortex" point p above if it were in F_0^*. Such points, trailing their attendent arcs, get carried into "safe" open arcs, like the arc A^0 above, and we may find we have "stretched" or "squeezed" the original simple closed curves of P_0' (in M_m) in our new manifold $M_{n'}$ to produce a not-necessarily-$\varkappa$-partition not-necessarily ν-refining each of P_0 and F_{N+1}. The trouble is that simple closed curves of P_0' may now be pinched together or they may intersect one another in more than two components – contrary to the requirements for a $\varkappa$-partition. It is possible, however, to re-establish from P_0' a $\varkappa$-partitioning ν-refinement of each of F_{N+1} and the copy (in $M_{n'}$) of P_0, determined by P_0', by subdiving interiors of elements of P_0' with spanning separating arcs finitely often. We shall call, for reasons of notational simplicity, this new collection of simple closed curves P_0' again. It is important to note also that this readjustment requires the addition of no more distinguished points in P_0^*. Thus, we shall presume that n and n' (possibly rechosen) are large enough that all the remaining distinguished points (in P_0^*) are in $F_{n'}^* \setminus F_{N+1}^*$. Although this amounts to choosing two integers greater than or equal to each of the original n and n', we shall for subsequent simplicity keep the same notation for the newly selected integers.

We are now in a position to apply Lemma 6, with $N + 1$ replacing N, and P_0' (or P' in M) replacing P in the statement. In fact, much of the construction of P' in M ($P'^* \cap P^*$) is already completed. If the arcs of P_0' with distinguished points as endpoints and containing no distinguished points in their interiors are enumerated, $\{A_i\}_{i=1}^k$, then since the distinguished points are all in $\bigcup_{i=1}^{n'} F_i^*$, we may proceed as in the proof of Lemma 6 – with the following convention. Whenever the arc A_k of P_0' is a subarc of an arc of P_0, the limiting set arc A_k of the sequence $\{A_k(i)\}_{i=1}^{\infty}$ has already been produced for us as a subarc of an arc of P^* in M. The result of the construction is a homeomorphic copy, P', of P_0', each element of which biseparates in M, and which ν-refines (except, possibly, for local biseparation) each of P and F_{N+1}.

Lemma 9. *M can be represented as an inverse limit space,* $\lim (\{F_i\}_{i=1}^{\infty}, \{f_i\}_{i=1}^{\infty})$, *where each P_i is a $\varkappa$-collection of biseparating simple closed curves ν-refining (except, possibly, for local biseparation) P_{i-1} and $f_i : P_{i+1} \to P_i$ is any natural map taking the interior ν-subcollections of P_{i+1} onto their boundary simple closed curves in P_i, i.e., the identity on $P_{i+1}^* \cap P_i^*$.*

Proof. We note that the mesh of P above is $\leqq$ mesh $F_N \leqq 1/2^N$ and mesh P' above is $\leqq$ mesh $F_{N+1} \leqq 1/2^{N+1}$. Lemma 8 is the inductive step in the construction of a sequence, $\{P_i\}_{i=1}^{\infty}$, of the $\varkappa$-collections of biseparating (Lemma 7) simple closed curves of mesh $< 1/2^i$ each of which ν-refines (except, possibly, for local biseparation) the preceding. Since the meshes of the P_i's tend to zero, the interiors of the simple closed curves of the P_i's and the interiors of simple closed curve bounded open sets with boundaries in the P_i's form a basis of open sets for M, and the representation of M as an ordinary inverse limit of the P_i-sequence is immediate.

Finally:

Lemma 10. *Each of the simple closed curves of each of the P_i's above locally biseparates.*

Proof. Let p be a point of a simple closed curve, C, of P_i, some i. Then p is interior to an arc of each of two simple closed curves formed, possibly, by the union of two or more simple closed curves of some P_j, $j > i$, such that each simple closed curve separates M, and their union is of suitably small diameter and their union is bounded by a single simple closed curve which contains p in a spanning separating arc of C.

Now, finally, we can remove the nagging parenthetical restriction regarding ν-refinement by the P_i's above.

Note. Since M is connected and each $C \in P_i$, separates and locally biseparates, it biseparates M. This is another proof of earlier observation. Since C locally biseparates, there is a connected open set in its interior, of which it is a boundary component, and also a similar connected open set in its exterior. These connected open "bands" on either "side", since M is connected, provide places for arcs to link pairs of points in the exterior and in the interior of C-biseparation.

Proof of Theorem. Since we now have in M a sequence of $\varkappa$-collections, $\{P_i\}_{i=1}^{\infty}$, each of which $\varkappa$-partitions M (which requires biseparation and local biseparation) and each of which ν-refines the preceding, with mesh tending to zero with i, we have shown M to be what we called a $\varkappa$-inverse incidence limit – the conclusion of the theorem.

4. Conclusions. Since the converse of the theorem is obviously true for $\varkappa$-inverse incidence limits, we have obtained a characterization of such spaces. Neither of these is, perhaps, surprising. It is however, surprising that the Universal Curve should not have a "nice" (in the sense of the theorem) sequential, or $\varkappa$-inverse incidence limit, structure.

The Universal Curve has a neighborhood basis in which the boundary of each element is a simple closed curve which biseparates and biseparates locally. If, however, a given Universal Curve had a sequence $\{P_n\}_{n=1}^{\infty}$, of $\varkappa$-partitionings with mesh tending to zero, such that for $C \in P_{n+1}$, $C \cap \bigcup_{i=1}^{n} P_i^*$ was a finite number of components, and such that the elements of P_j, $j = 1, \ldots$, biseparated and biseparated locally, then it would be a $\varkappa$-inverse incidence limit. Hence, by its homogeneity and the Anderson-Keisler theorems it would be a P or T-sphere and thus two-dimensional. In short, for a given Universal Curve, one or both of two things must happen: First, there is no decreasing mesh sequence of $\varkappa$-partitions, nice with respect to one another. Second, if there is such a sequence, there is a non-zero lower bound on the mesh of the partitions. The first possibility seems unlikely, but the natural generalization of our theorem, which would imply the second, is beyond the author.

While this is a negative sort of characteristic to ascribe to the Universal Curve, it does suggest how higher dimensional universal spaces ought not be to constructed. Further, since the techniques we have used depend on simple considerations of manifold theory, generalizations of our definitions and results to higher dimensional cases, with collections of bounding two-spheres, for example, are naturally suggested.

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