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LEX-SUBGROUPS OF LATTICE-ORDERED GROUPS[1])

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1. Introduction. A convex l-subgroup C of an l-group G will be called a *lex-subgroup* if C is a proper lexicographic extension of a convex l-subgroup. These subgroups are extremely useful in determining the structure of G. The main reasons for this are that two lex-subgroups are either disjoint or comparable, and a maximal lex-subgroup is the double polar of a special element. In Section 3 we derive these and other useful properties of lex-subgroups and use them to determine structure theorems for l-groups. In particular, we obtain the main structure theorems in [3] and [7] as corollaries of Theorem 5.1.

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Notation. We shall use the standard notation for l-groups (see for example [5]). If $\{A_\lambda : \lambda \in \Lambda\}$ is a set of l-groups, then $\sum A_\lambda$ ($\prod A_\lambda$) will denote the small (large) cardinal sum of the A_λ. In particular, if $\Lambda = 1, \ldots, n$ is finite, then $A_1 \oplus \ldots \oplus A_n$ will denote the cardinal sum (that is, the direct sum, where $(a_1, \ldots, a_n)$ is defined to be positive if each $a_i \geqq 0$). If X and Y are subsets of an l-group G, then $[X]$ will denote the subgroup of G that is generated by X and $X \parallel Y$ will denote that X and Y are not comparable with respect to inclusion, and $X \backslash Y$ will denote the elements in X that are not in Y. If $g \in G$, then $G(g)$ will denote the principal convex l-subgroup that is generated by g. Thus

$$G(g) = \{x \in G : |x| \leqq n|g| \text{ for some } n > 0\} .$$

2. Lex-extensions and polars. In this section we collect some well known facts that will be used throughout this paper. The material on prime subgroups and lex-extensions may be found in [3] and [4], and most of the material on polars is due to ŠIK [8] and [9]. *Throughout this section let G be an l-group.*

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A convex l-subgroup C of G is said to be *prime* if the lattice of (right) cosets of C. in G is totally ordered. In particular, if $C \lhd G$, then G/C is an o-group. Moreover, the following are equivalent

(1) C is prime.

(2) If $a, b \in G^+ \setminus C$, then $a \wedge b \in G^+ \setminus C$.

(3) The convex l-subgroups of G that contain C form a chain.

G is a *lex-extension* of a convex l-subgroup C if

(i) C is prime, and

(ii) $g \in G^+ \setminus C$ implies $g > C$.

If $C \neq 0$, then (ii) implies (i). An element $a \in G$ is a *non-unit* if $a > 0$ and $a \wedge b = 0$ for some $0 < b \in G$. If N is the set of all non-units of G, then $[N]$ is an l-ideal of G

Theorem 2.1. *Let C be a convex l-subgroup of an l-group G. G is a lex-extension of C if and only if $C \supseteq [N]$, and all other convex l-subgroups of G are contained in $[N]$. If $0 \neq C \subset [N]$, then there exists a prime subgroup D of G such that $C \parallel D$ and hence $[N]$ is the smallest (non-zero) convex l-subgroup of G that is comparable with every convex l-subgroup of G.*

If G is a lex-extension of C, and $C \subseteq E$, where E is a convex l-subgroup of G, then G is a lex-extension of E. Finally, the following are equivalent for $C \neq 0$.

(1) *G is a lex-extension of C.*

(2) *C is comparable with all other convex l-subgroups of G.*

There are two other characterizations of $[N]$ due to LAVIS [6]. For $g \in G$ Lavis defined $g \approx 0$ if there exist $g_1, \ldots, g_n \in G$ such that

$$g \parallel g_1 \parallel g_2 \parallel \cdots \parallel g_n \parallel 0 .$$

Theorem 2.2. $[N] = [\{g \in G : g \parallel 0\}] = \{g \in G : g \approx 0 \text{ or } g = 0\}$ [2]).

We shall call $[N]$ the *lex-kernel* of G and denote it by $L(G)$. A *value* of $0 \neq g \in G$ is a convex l-subgroup of G that is maximal without containing g. Each value of g is prime, and $g > 0$ if and only if $M + g > M$ for all values M of g. If M is the only value of g, then g is said to be *special* and in this case M is also called special.

The *polar* of a subset X of G is the convex l-subgroup

$$X' = \{g \in G : |g| \wedge |x| = 0 \text{ for all } x \in X\}$$

Šik [8] has shown that the set of all polars in G is a complete Boolean algebra.

[2]) Lavis used the convex hull of $K = [\{g \in G : g \parallel 0\}]$, but for l-groups K is convex. Also it can be shown that $[N]$ is the join of all the minimal prime subgroups in the lattice of convex l-subgroups of G.

Theorem 2.3. *For a convex l-subgroup $A \neq 0$ of G the following are equivalent.*

(a) A is an o-group.
(b) If $0 < a \in A$, then $a' = A'$.
(c) A' is a prime subgroup.
(d) A' is a minimal prime subgroup.
(e) A'' is a maximal convex o-subgroup.
(f) A'' is a minimal polar.
(g) A' is a maximal polar.
(h) Each $0 \neq a \in A$ is special.

Proposition 2.4. *If A and B are convex l-subgroups of G and $0 = A \cap B = (A \oplus B)'$, then $A'' = B'$.*

Proof. Since $A \cap B = 0$, $A \subseteq B'$ and hence $A'' \subseteq B''' = B'$. $A' \cap B' \subseteq (A \oplus B)' = 0$ and hence $B' \subseteq A''$.

3. Lex-subgroups. A convex l-subgroup C of an l-group G is a *lex-subgroup* if C is a proper lex-extension of a convex l-subgroup. If, in addition, there does not exist a proper lex-extension of C in G, then C is a *maximal lex-subgroup*. A po-set S is a *root system* if for each $s \in S$, $\{x \in S : x \geqq s\}$ is totally ordered.

In the next four propositions we shall assume that A and B are lex-subgroups of G and that $A(B)$ is a proper lex-extension of $U(V)$.

3.1. *If $A \parallel B$, then $A \cap B = 0$. In particular, the set of all lex-subgroups of G form a root system with respect to inclusion.*

Proof. Select $0 < a \in A \backslash (B \cup U)$ and $0 < b \in B \backslash (A \cup V)$. Since $A \cap B$ is a convex l-subgroup of A, it is comparable with U (Theorem 2.1). If $A \cap B \subseteq U$, then $a > U \supseteq A \cap B$ and if $A \cap B \supseteq U$, then by Theorem 2.1, A is a lex-extension of $A \cap B$ and once again $a > A \cap B$. Similarly $b > A \cap B$ and hence since $a \wedge b \in A \cap B$, it follows that $a \wedge b$ is the largest element in $A \cap B$. Therefore $A \cap B = 0$.

3.2. (Clifford) $(A \oplus A')^+ = \{x \in G^+ :$ *x does not exceed every element in $A\}$. In particular, $G = A \oplus A'$, provided that A is not bounded in G.*

This is part of Lemma 6.2 in [3].

3.3. *If $a \in A \backslash U$, then $a' = A'$ and $a'' = A''$ is a lex-extension of A and of U, and a maximal lex-subgroup of G. If $U = 0$, then A'' is the largest convex o-subgroup of G that contains A. If $U \neq 0$, then $U' = A'$ and $U'' = A''$ is the largest lex-extension of U in G.*

Proof. If $U = 0$ and $0 \neq a \in A$, then A is an o-group and hence by Theorem 2.3, $a' = |a|' = A'$ and A'' is a maximal convex o-subgroup of G. If M is a convex o-subgroup of G and $M \supseteq A$, then $M \cap A'' \supseteq A \neq 0$ and hence by 3.1 $M \subseteq A''$. Therefore A'' is the largest convex o-subgroup of G that contains A.

Suppose that $U \neq 0$. Clearly $A' \subseteq U'$. If $0 < x \in U' \backslash A'$, then $x \wedge y > 0$ for some $0 < y \in A$ and hence $x \geqq x \wedge y \geqq u > 0$ for some $u \in U$, but this contradicts the fact that $x \in U'$. Therefore $U' = A'$. If $a \in A \backslash U$, then $a > U$ and hence $G(a) \supset U$. Thus $a' = G(a)' \subseteq U' = A'$ and since $a \in A$, $a' \supseteq A'$. Therefore $a' = A' = U'$. Now

$$G \supseteq A'' \oplus A' \supseteq A \oplus A' .$$

If $0 < g \in A'' \backslash A$, then $g \in G^+ \backslash (A \oplus A')$ and hence by 3.2 $g > A$. Thus U'' is a lex-extension of A and hence a lex-extension of U. If M is a proper lex-extension of U in G, then by the above argument $M \subseteq M'' = U''$. Therefore U'' is the largest lex-extension of U in G.

3.4. *If C is a convex l-subgroup of G and $C \supset A''$, then $C \supseteq A'' \oplus D$ for some non-zero convex l-subgroup D of G.*

Proof. Let D be the polar of A'' in C. If $D = 0$, then by 3.2 each $0 < x \in C \backslash A''$ must exceed A''. Thus C is a proper lex-extension of A'', but this contradicts the fact that A'' is a maximal lex-subgroup.

The following theorem is an immediate consequence of 3.3.

Theorem 3.5. *Let $M \neq 0$ be a convex l-subgroup of G. The lex-extensions of M in G form a chain in M''. In particular, a non-zero polar admits no proper lex-extensions, and the set of all lex-subgroups of G form a root system with respect to inclusion. If M is a lex-subgroup of G or if M admits a proper lex-extension, then M'' is a maximal lex-subgroup and the largest lex-extension of M in G.*

The following theorem is proven in [4].

Theorem 3.6. *For $g \in G$ the following are equivalent.*

(1) *$G(g)$ is a lex-subgroup.*

(2) *g is special in G.*

(3) *g is special in $G(g)$.*

3.7. *For $0 < g \in G$ the following are equivalent.*

(a) *$g \notin L(G)$ the lex-kernel of G.*

(b) *g is special and a unit.*

Proof. a) $\to$ b). If $0 < g \in G \backslash L(G)$, then $G(g)$ is a proper lex extension of $L(G)$ and hence by Theorem 3.6 g is special and clearly g is a unit.

b) $\to$ a). By Theorem 3.6 $G(g)$ is a proper lex-extension of $U = L(G(g))$ and $g \in$ $\in G(g) \backslash U$. Since g is a unit, $g' = 0$ and hence $g'' = G$. By 3.3 $G = g''$ is a lex-extension of U and hence by Theorem 2.1 $U \supseteq L(G)$. Therefore $g \notin L(G)$.

Theorem 3.8. *For a convex l-subgroup A of G the following are equivalent.*

(a) *A is a lex-subgroup.*

(b) *$G(a) \subseteq A \subseteq a''$ for some special element a of G.*

Proof. a) → b). Let $U = L(A)$ and consider $0 < a \in A \backslash U$. By 3.3

$$U \subset G(a) \subseteq A \subseteq A'' = a''$$

and a'' is a lex-extension of U. Thus $G(a)$ is a proper lex-extension of U and hence by Theorem 3.6 a is special.

b) → a). By Theorem 3.6 $G(a)$ is a lex-subgroup and hence a proper lex-extension of $V = L(G(a))$. Clearly $a \in G(a) \backslash V$ and hence by 3.3 a'' is a lex-extension of V. Therefore A is a proper lex-extension of V.

Note that if A is a maximal lex-subgroup, then $A = a''$.

Corollary I. *For a convex l-subgroup A of G the following are equivalent.*

(a) *A is a maximal lex-subgroup.*

(b) *$A = a''$ for some special element a of G.*

(c) *A is a lex-subgroup and also a polar.*

In particular if a is a special element of G, then a'' is a maximal lex-subgroup and $|a| > L(a'')$.

Proof. We have shown that (a) implies (b). If (b) holds, then by the theorem A is a lex-subgroup and clearly A is a polar. Finally since a non-zero polar admits no proper lex-extensions (Theorem 3.5) it follows that (c) implies (a).

Corollary II. *If $a_1, a_2, \ldots, a_n$ are disjoint special elements of G and no a_i'' is bounded in G, then $G = a_1'' \oplus a_2'' \oplus \ldots \oplus a_n'' \oplus D$ for some convex l-subgroup D of G.*

Proof. Since a_1'' is a lex-subgroup, we have by 3.2 that $G = a_1'' \oplus a_1'$. Consider a_i, $i \neq 1$. Since $a_i \in a_1'$, $a_i'' \subseteq a_1'$. By Theorem 3.6 a_i is special in $G(a_i) \subseteq a_1'$ and hence by Theorem 3.6 a_i is special in a_1'. Thus by induction $a_1' = a_2'' \oplus \ldots \oplus a_n'' \oplus D$, and hence $G = a_1'' \oplus \ldots \oplus a_n'' \oplus D$.

Theorem 3.9. *For an l-group G the following are equivalent.*

(a) *There exists a maximal disjoint subset $\{s_\lambda : \lambda \in \Lambda\}$ of G, and in addition each s_λ is special and no s_λ'' is bounded in G.*

(b) *There exists an l-isomorphism σ of G such that*

$$\sum A_\lambda \subseteq G\sigma \subseteq \prod A_\lambda \, (\lambda \in \Lambda)$$

where A_λ is an l-group and $A_\lambda \neq L(A_\lambda)$ for each $\lambda \in \Lambda$. In any such representation $\{\bar{A}_\lambda \sigma^{-1} : \lambda \in \Lambda\}$ is the set of all unbounded maximal lex-subgroups of G, where

$$\bar{A}_\lambda = \{(\ldots, x_\mu, \ldots) \in \prod A_\lambda : x_\mu = 0 \text{ for all } \mu \neq \lambda\}.$$

Proof. a) → b). By Corollary I of Theorem 3.8 each s''_λ is a maximal lex-subgroup, and hence by 3.2 $G = s''_\lambda \oplus s'_\lambda$ for each $\lambda \in \Lambda$. Thus each $g \in G$ has a unique representation $g = g_\lambda + g^\lambda$, where $g_\lambda \in s''_\lambda$ and $g^\lambda \in s'_\lambda$. The mapping $g \to g_\lambda$ is an l-homomorphism of G onto s''_λ with kernel s'_λ. Define

$$g\sigma = (\ldots, g_\lambda, \ldots) \in \prod s''_\lambda .$$

Then σ is an l-homomorphism with kernel $\bigcap s'_\lambda$ and since $\{s_\lambda : \lambda \in \Lambda\}$ is a maximal disjoint subset, $\bigcap s'_\lambda = 0$. Therefore σ is an l-isomorphism of G into $\prod s''_\lambda$. Consider $0 < x \in s''_\lambda$. If $\alpha \neq \lambda$, then $s_\alpha \wedge s_\lambda = 0$ and hence $s_\alpha \in s'_\lambda$. Thus $x \wedge s_\alpha = 0$ and hence $x \in s'_\alpha$. Therefore

$$(x\sigma)_\alpha = \begin{cases} x & \text{if} \quad \alpha = \lambda \\ 0 & \text{otherwise} \end{cases}$$

and it follows that $\sum s''_\lambda \subseteq G\sigma \subseteq \prod s''_\lambda$.

b) → a). For each $\lambda \in \Lambda$ pick $0 < a_\lambda \in A_\lambda \backslash L(A_\lambda)$ and let $\bar{a}_\lambda$ be the element in $\prod A_\lambda$ with λ-th component a_λ and all other components 0, and let $s_\lambda = \bar{a}_\lambda \sigma^{-1}$. Then $\{\bar{a}_\lambda : \lambda \in \Lambda\}$ is a maximal disjoint subset of $G\sigma$ and hence $\{s_\lambda : \lambda \in \Lambda\}$ is a maximal disjoint subset of G. Moreover, $\bar{a}''_\lambda = \bar{A}_\lambda$ which is an unbounded lex-subgroup of $G\sigma$. It follows that s''_λ is unbounded in G and that $G(s_\lambda)$ is a lex-subgroup. Thus each s_λ is special.

Suppose that $\{M_\alpha : \alpha \in A\}$ is the set of all unbounded lex-subgroups of G. By 3.1 $M_\alpha \cap M_\beta = 0$ if $\alpha \neq \beta$ and hence by Theorem 2.1 in [3]

$$M = [\bigcup M_\alpha] = \sum M_\alpha .$$

By Theorem 3.9 there exists an l-isomorphism σ of M'' such that

$$\sum M_\alpha \subseteq M''\sigma \subseteq \prod M_\alpha .$$

Now $G \supseteq M'' \oplus M'$ and it would be useful to know under what conditions $G = M'' \oplus M'$; but the author has not been able to answer this question.

Theorem 3.10. *The subgroup S of an l-group G that is generated by the special elements of G is an l-ideal.*

Proof. Suppose that $0 < a \in G$ is special and consider $0 < x \in G(a)$. Then $a < a + x \in G(a)$ and hence $G(a) = G(a + x)$. Thus by Theorem 3.6 $a + x$ is special and hence $x = -a + a + x \in S$. Thus we have shown that $G(a) \subseteq S$ and it follows that

$$S = [\bigcup\{G(a) : a \text{ is special in } G\}] = \bigvee G(a)$$

and hence S is a convex l-subgroup of G. If $G(a)$ is a lex-subgroup, then so is $G(-g + a + g)$ for each $g \in G$. Therefore $S \lhd G$ and hence S is an l-ideal of G.

If $\{C_\alpha : \alpha \in A\}$ is a chain of lex-subgroups of G, then $C = \bigcup C_\alpha$ need not be a lex-subgroup or a polar.

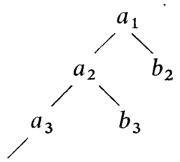
The following theorem gives an important relationship between lex-subgroups and polars (see Theorem 5.2). An l-group G is said to be *finite valued* if each $0 \neq g \in G$ has only a finite number of values or equivalently if each value of g is special (Theorem 3.8 in [4]).

Theorem 3.11. *For an l-group G the following are equivalent.*

(1) *The lattice of all filets of G satisfies the* DCC (*descending chain condition*).

(2) *G is finite valued and the root system M(G) of all maximal lex-subgroups of G satisfies the* DCC.

Proof. A *filet chain* is a set of strictly positive elements of G



such that $a_i \wedge b_i = 0$ and $a_i \geqq a_{i+1} \vee b_{i+1}$. McAlister ([7] Proposition 2.1) has shown that (1) holds if and only if each filet chain is finite.

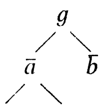
1) → 2). If $a_1'' \supset a_2'' \supset \ldots$ is a descending chain in $M(G)$, then by 3.4 $a_i'' = a_{i+1}'' \oplus$ $\oplus B_{i+1}$, where $0 \neq B_{i+1}$ is a convex l-subgroup of G. Thus by selecting $0 < b_{i+1} \in$ $\in B_{i+1}$ we get a filet chain which is necessarily finite. Thus there are only a finite number of a_i'' and hence $M(G)$ satisfies the DCC.

Suppose (by way of contradiction) that $0 < g \in G$ has an infinite number of values. Then by Theorem 3.8 in [4] at least one, say G_α, is not special. Let G^α be the convex l-subgroup of G that covers G_α and let G_β be another value of g. Pick $0 < a \in$ $\in (G^\alpha \setminus G_\alpha) \cap G_\beta$ and $0 < b \in (G^\beta \setminus G_\beta) \cap G_\alpha$. Then it follows by Theorem 3.8 in [4] that a has an infinite number of values. Without loss of generality we may assume that g exceeds a and b. Moreover

$a = a \wedge b + \bar{a}$. $\bar{a} \in G^\alpha \setminus G_\alpha$ and hence has an infinite number of values.

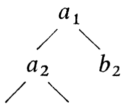
$b = a \wedge b + \bar{b}$. $\bar{b} \in G^\beta \setminus G_\beta$. $\bar{a} \wedge \bar{b} = 0$.

Thus we can construct an infinite filet chain

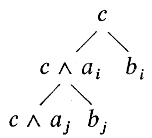


but this contradicts (1).

2) → 1). Suppose (by way of contradiction) that



is an infinite filet chain. Since each b_i is the join of disjoint special elements, we may assume that each b_i is special. Also $a_1 = c_1 + \ldots + c_n$, where the c_i are disjoint and special. Thus without loss of generality we may assume that $c = c_1$ exceeds an infinite number of the b_i. Pick $i > j$ such that $c > b_i$ and b_j. If $c \wedge a_i = c$, then $a_i \geqq \geqq c > b_i$, a contradiction. If $c \wedge a_j = c \wedge a_i$, then $c \wedge a_j \geqq b_j$, a contradiction. If $c \wedge a_i = b_j$, then $c \wedge b_j = b_j \geqq c \wedge a_j$, a contradiction. Therefore



and hence we have an infinite filet chain in which the largest element is special.

Now repeat the argument on $c \wedge a_k$, where k is the least positive integer such that $c > b_k$. In this way we get an infinite filet chain of special elements, but this contradicts the fact that $M(G)$ satisfies the DCC.[3])

4. Root systems. The proofs in this and the next section are conceptually simplified by the following abstraction of the root system $M(G)$ of all maximal lex-subgroups of an l-group G.

Let S be a root system that satisfies the DCC and consider $s \in S$. Each chain in S for which s is an upper bound is a well ordered set and hence has an ordinal number for its "length". We define the *length* of s to be the least upper bound of the lengths of the chains strictly below s. In particular, the minimal elements of S have length 0. The *α-th level* of S consists of the elements of length α together with those elements b of length $\beta < \alpha$ such that b is maximal in S or b is covered by an element of length $> \alpha$.

4.1. *If $a \neq b$ belong to the α-th level of S, then $a \parallel b$.*

Proof. If $a > b$, then b has length $< \alpha$ and is not maximal in S. Thus b is covered by an element c of length $> \alpha$ and hence $a \geqq c > b$, but this means that a has length $> \alpha$, a contradiction.

4.2. *Each o-permutation π of S permutes the elements in the α-th level.*

Proof. a has length α if and only if $a\pi$ has length α. a is maximal in S if and only if $a\pi$ is maximal in S. b covers c if and only if $b\pi$ covers $c\pi$.

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[3]) Byrd [2] shows that for any l-group G the lattice of all filets is isomorphic to the lattice of all principal polars.

4.3. *If $\alpha \leqq \beta < \gamma$, a has length α and a is in the γ-th level, then a is in the β-th level.*

Proof. If a is not covered, then a is maximal in S and hence belongs to the β-th level. Clearly a belongs to the α-th level. If $\alpha < \beta$ and b covers a, then since a is in the γ-th level it follows that b has length $> \gamma$ and hence a is in the β-th level.

4.4. *If b covers a and b has length $\beta + 1$, then a is in the β-th level.*

Proof. If a has length $< \beta$, then since a is covered by an element of length $> \beta$, a is in the β-th level.

4.5. *If a has length $\alpha + 1$, then a covers an element of length α.*

Proof. There exists a chain below a of length $> \alpha$ and hence one of length $\alpha + 1$. Let b be the maximal element in this chain. Then a covers b and b has length α.

Suppose that $\{a_\lambda : \lambda \in \Lambda\}$ is a maximal disjoint subset of G and that each a_λ is special. For each $\lambda \in \Lambda$ let $A_\lambda = a''_\lambda$. If $\alpha \neq \beta$, then $A_\alpha \cap A_\beta = 0$ and hence

$$A = [\bigcup A_\lambda] = \sum A_\lambda .$$

Let

$$T = \{C \in M(G) : C \supseteq A_\lambda \text{ for some } \lambda \in \Lambda\} .$$

Then T is a root system and we shall first show that each $C \in T$ is determined by the A_λ that it contains.

4.6. *If $\Delta \subseteq \Lambda$, then $(\sum A_\delta)'' = (\sum A_\lambda)'$, where $\delta \in \Delta$ and $\lambda \in \Lambda \backslash \Delta$, and each $C \in T$ is of this form. In particular, if $D \in T$ and $D \supset C$, then there exists $A_\lambda \parallel C$ such that $D \supset A_\lambda$.*

Proof. $A = \sum A_\delta \oplus \sum A_\lambda$ and if $0 < x \in A'$, then $x \wedge a_\lambda = 0$ for all λ and hence $x = 0$. Thus by Proposition 2,4, $(\sum A_\delta)'' = (\sum A_\lambda)'$. If $C \in T$, then $C \supseteq A_\gamma$ for some $\gamma \in \Lambda$, and if $\lambda \in \Lambda$, then $A_\lambda \cap C = 0$ or $A_\lambda \subseteq C$. For otherwise by 3.1 $A_\lambda \supset A_\gamma$ which is impossible. Thus there exists a subset Δ of Λ such that $C \supseteq \sum A_\delta \, (\delta \in \Delta)$ and $C' \supseteq \sum A_\lambda \, (\lambda \in \Lambda \backslash \Delta)$ and hence $(\sum A_\delta)'' \subseteq C \subseteq (\sum A_\lambda)'$.

Now let

$S = \{C : C$ is the join of a chain in T and C has no proper lex extension in $G\}$. Note that $T \subseteq S$. Moreover $C \in S$ is a lex-subgroup if and only if $C \in T$. For if $\{X_\beta : \beta \in B\}$ is a chain from T with no maximal element, and $\bigcup X_\beta$ is a lex-subgroup, then $\bigcup X_\beta = a''$ for some special element, but then $a \in X_\beta$ for some β and hence $a'' \subseteq X_\beta$, a contradiction.

4.7. *If $C = \bigcup C_\gamma$ and $D = \bigcup D_\delta$ belong to S and $C \parallel D$ then $C \cap D = 0$. In particular S is a root system.*

Proof. If $0 = C_\gamma \cap D_\delta$ for all γ and δ, then

$$C \cap D = C \cap (\bigcup D_\delta) = \bigcup(C \cap D_\delta) = \bigcup((\bigcup C_\gamma) \cap D_\delta) = \bigcup(C_\gamma \cap D_\delta) = 0 .$$

If $C_\gamma \cap D_\delta \neq 0$ for some γ and δ, then by 3.1 we may assume that $C_\gamma \supseteq D_\delta$. Thus since the elements of T that contain D_δ form a chain it follows that C and D are comparable a contradiction.

4.8. *If $C, D \in S$ and C covers D, then $C \in T$.*

Proof. If $C \notin T$, then $C = \bigcup C_\gamma$ where $\{C_\gamma : \gamma \in \Gamma\}$ is a chain in T and each $C_\gamma \subset C$. If each $C_\gamma \subseteq D$ then $C \subseteq D$ and if $C_\gamma \cap D = 0$ for all γ, then $C \cap D = 0$. Thus there exists a C_γ such that $C \supset C_\gamma \supset D$, a contradiction.

4.9. *If T satisfies the* DCC, *then so does S.*

Proof. Suppose that $M_1 \supset M_2 \supset \ldots$, where the $M_i \in S$. $M_1 = \bigcup C_\gamma$ is the join of a chain from T. If $C_\gamma \cap M_2 = 0$ for all γ, then $M_1 \cap M_2 = 0$ and if $C_\gamma \subseteq M_2$ for all γ, then $M_1 \subseteq M_2$. Therefore at least one C_γ properly contains M_2 and hence we have

$$M_1 \supseteq K_1 \supset M_2 \supseteq K_2 \supset M_3 \supseteq \ldots$$

where the K_i belong to T, and hence there can only be a finite number of the M_i.

Remark. We can derive 4.7, 4.8 and 4.9 in terms of abstract root systems, but the formulation becomes somewhat messy.

Now suppose that T and hence S satisfies the DCC and let $\{A^\alpha_\lambda : \lambda \in \Lambda_\alpha\}$ be the α-th level of S. In particular $\Lambda_0 = \Lambda$. If $\lambda_1, \lambda_2, \in \Lambda_\alpha$, then by 4.1 $A^\alpha_{\lambda_1} \parallel A^\alpha_{\lambda_2}$ and hence by 4.7 $A^\alpha_{\lambda_1} \cap A^\alpha_{\lambda_2} = 0$. Therefore

4.10. $A^\alpha = [\bigcup A^\alpha_\lambda] = \sum A^\alpha_\lambda$.

4.11. *If $A \lhd G$, then $A^\alpha \lhd G$.*

Proof. Since $A = \sum A_\lambda$ is the indecomposable representation of A it follows that each inner automorphism π of G induces a permutation on $\{A_\lambda : \lambda \in \Lambda\}$. Thus π induces a permutation on T and hence on S. By 4.2 π induces a permutation on the α-th level of S and hence $A^\alpha \pi = A^\alpha$. Therefore $A^\alpha \lhd G$.

5. Lex-sums of L-groups. An l-group G is a *lex-sum of l-groups* $\{A_\lambda : \lambda \in \Lambda\}$ if for some ordinal σ there exists a chain of convex l-subgroups

$$A^0 \subseteq A^1 \subseteq \ldots \subseteq A^\alpha \subseteq \ldots \subseteq G$$

one for each ordinal $\alpha < \sigma$, such that $G = \bigcup A^\alpha$ and $A^\alpha = \sum A^\alpha_\lambda$ $(\lambda \in \Lambda_\alpha)$, where each A^α_λ admits no proper lex-extensions and the following are satisfied.

(A) $A_0 = A$ and $A_\lambda^0 = A_\lambda$ for each $\lambda \in A$.

(B) $A_\lambda^{\alpha+1} = A_\beta^\alpha$ for some $\beta \in A_\beta$ or $A_\lambda^{\alpha+1}$ is a proper lex-extension of a small cardinal sum of two or more of the components of A^α and at least one of these components of A^α is not contained in any A^u with $u < \alpha$.

(C) If α is a limit ordinal, then there exists a cofinal sequence B in $\{\mu : \mu < \alpha\}$ and for each $\beta \in B$ a component $A_{\gamma_\beta}^\beta$ of A^β such that A_λ^α is a proper lex-extension of $\sum A_{\gamma_\beta}^\beta (\beta \in B)$ or the $A_{\gamma_\beta}^\beta$ form a chain and A_λ^α is a lex-extension of the join of this chain.

If, in addition, each A^α is an l-ideal, then we say that the lex-sum is *normal*. If $\sigma \leqq \omega$, then (C) is vacuous, and in this case we call the result an *ω-lex-sum*. An ω-lex-sum is *restricted* if the cardinal sum referred to in (B) is finite.

Remark. The concept of a restricted ω-lex-sum was introduced in [3]. The above generalization is essentially the same as McAlister's definition of a τ-lexico-sum in [7]. It differs only in (C) as follows: if α is a limit ordinal and A_λ^α is a proper lex-extension of $\sum A_{\gamma_\beta}^\beta$, then by McAlisters definition A_λ^α appears first as a component of $A^{\alpha+1}$. Also in [3] and [7] only normal lex-sums were considered.

The following is our main structure theorem, all other theorems in this section are corollaries of this one.

Theorem 5.1. *Suppose that $\{a_\lambda : \lambda \in A\}$ is a maximal disjoint subset of an l-group G and that each a_λ is special. Then G is a lex-sum of the groups $A_\lambda = a_\lambda''$ if and only if*

(a) *$T = \{C \in M(G) : C \supseteq A_\lambda$ for some $\lambda \in A\}$ satisfies the* DCC, *and*

(b) *for each $g \in G^+$ there exists an $a \in A = \sum A_\lambda$ such that $g + a$ is finite valued. If this is the case, then G is a normal lex-sum of the A_λ if and only if $A \lhd G$. Moreover, $A \lhd G$ if G is representable (as a subdirect sum of o-groups) or A is the basis subgroup of G or $|A| = n$ is finite and G does not contain $n + 1$ disjoint special elements.*

Proof. The verification that (a) and (b) are necessary conditions for G to be a lex-sum of the A_λ is straightforward and will be left to the reader. Suppose that (a) and (b) are satisfied, then we have all the material in Section 4 at our disposal.

In particular, we let $\{A_\lambda^\alpha : \lambda \in A_\alpha\}$ be the α-th level of S. Then by 4.10 $A^\alpha = [\bigcup A_\lambda^\alpha] = \sum A_\lambda^\alpha$ and $A = A^0 = \sum A_\lambda$. Thus (A) is satisfied.

(1) $G = \bigcup A^\alpha$.

For clearly $\bigcup A^\alpha \supseteq A$ and if $g \in G^+ \backslash A$, then $g + a$ is finite valued for some $a \in A$ and hence $|g + a| = g_1 + \ldots + g_n$, where the g_i are special and disjoint. Thus $g_i \in g_i'' \subseteq \bigcup A^\alpha$ and hence $|g + a| \in \bigcup A^\alpha$, but since $\bigcup A^\alpha$ is a convex l-subgroup it follows that $g \in \bigcup A^\alpha$.

(2) If $C \in S$, then $C = (\sum A_\delta)'' = (\sum A_\lambda)' = C''$, where $\delta \in \Delta$ and $\lambda \in A \backslash \Delta$. By 4.6 we may assume that $C \in S \backslash T$. Also by 4.6 $(\sum A_\delta)'' = (\sum A_\lambda)'$ for any subset Δ of A. Now $C = \bigcup C_\alpha$, where $\{C_\alpha : \alpha \in a\}$ is a chain in T. Let $\Delta = \{\delta \in A : A_\delta \subseteq C_\alpha$ for

some $\alpha \in a\}$. Then $\sum A_\delta \subseteq \bigcup C_\alpha = C$ and hence $(\sum A_\delta)'' \subseteq C''$. If $\lambda \in \Lambda \backslash \Delta$, then $A_\lambda \cap C_\alpha = 0$ and hence $A_\lambda \subseteq C'_\alpha$ for all α and so $\sum A_\lambda \subseteq \bigcap C'_\alpha = C'$. Therefore

$$\sum A_\delta \subseteq C \subseteq (\sum A_\delta)'' = (\sum A_\lambda)' = C''.$$

Suppose (by way of contradiction) that $0 < g \in C'' \backslash C$. Then $g + a$ is finite valued, where $a = a_1 + a_2$, $a_1 \in \sum A_\delta$ and $a_2 \in \sum A_\lambda$. In particular, $g \wedge |a_2| = |a_1| \wedge |a_2| = 0$ and so $|g + a_1| \wedge |a_2| = 0$. Thus if M is a value of $g + a_1$, then $a_2 \in M$ and so M is a value of $g + a$. Therefore $g + a_1$ is finite valued and belongs to $C'' \backslash C$ and hence it follows that there exists $0 < s \in C'' \backslash C$, where s is special.

If $s \in C_\alpha \oplus C'_\alpha$ for some α, then since s is special it must belong to C'_α. If $C_\beta \subseteq C_\alpha$, then $s \in C'_\alpha \subseteq C'_\beta$ and if $C_\beta \supseteq C_\alpha$ and $s \notin C'_\beta$, then $s \notin C_\beta \oplus C'_\beta$ and hence $s > C_\beta \supseteq C_\alpha$ which is impossible. Therefore $s \in \bigcap C'_\alpha = C'$ and so $s \in C' \cap C'' = 0$, a contradiction.

Therefore $s \notin C_\alpha \oplus C'_\alpha$ for all α, and hence $s > C_\alpha$ for all α. We shall show that in this case s'' is a proper lex-extension of C, but this contradicts the fact that $C \in S$. Thus to complete the proof of (2) it suffices to show that if $0 < x \in s'' \backslash C$, then $x > C$. As above $x + a$ is finite valued for $a \in \sum A_\delta \subseteq C$. Thus $x + a = x_1 + \ldots + x_n$, where each x_i is special and hence comparable to zero. If $x + a \leqq 0$, then $0 < x \leqq -a \in C$ and so $x \in C$, a contradiction. Similarly at least one of the positive x_i is not in C and so we may assume that $0 < x_n \in s'' \backslash C$ and hence $x_n > C$. Thus $x_n - a > C$ and $x_n - a$ is special with the same value as x_n. Therefore $x = |x_1| + \ldots + |x_{n-1}| + |x_n - a| > C$ and so (2) is established.

Now suppose that $C = A^\alpha_\lambda$ is in the α-th level of S. We must show that (C) (B) are satisfied according as α is a limit ordinal or not. If C has length $\beta < \alpha$, then by 4.3 C belongs to the γ-th level for all $\beta \leqq \gamma < \alpha$ and so (B) and (C) are satisfied. Thus we may assume that C has length α. By (2) $C = (\sum A_\delta)''$. If Δ consists of a single element δ, then $C = A''_\delta = A_\delta$ and so C has length 0. Thus we may assume that Δ contains at least two elements. For each $\delta \in \Delta$ let D_δ be the join of the chain of elements in T that contain A_δ and are properly contained in C.

Case I. $D_\delta = C$ for some $\delta \in \Delta$. Then C is the join of a chain $\{A^\beta_{\gamma_\beta} : \beta \in B\}$ of T each of which is properly contained in C and hence belongs to a lower level. Suppose (by way of contradiction) that for all $\beta \in B$, $\beta \leqq \delta < \alpha$. Since C has length α there exists a chain $\{C_i : i \in I\}$ of length $> \delta$ and such that each $C_i \subset C$. If $C_i \cap A^\beta_{\gamma_\beta} = 0$ for all i and all β, then

$$(\bigcup C_i) \cap C = (\bigcup C_i) \cap (\bigcup A^\beta_{\gamma_\beta}) = \bigcup (C_i \cap A^\beta_{\gamma_\beta}) = 0$$

a contradiction. It follows that there exists C_i of length $> \delta$ such that $C_i \cap A^\beta_{\gamma_\beta} \neq 0$ for some β. Thus C_i and $A^\beta_{\gamma_\beta}$ are comparable. If $C_i \subset A^\beta_{\gamma_\beta}$, then $A^\beta_{\gamma_\beta}$ has length $> \delta$. If $A^\beta_{\gamma_\beta} \subseteq C_i$, then since T is a root system and C is the join of the chain of the $A^\beta_{\gamma_\beta}$ it follows that $A^\beta_{\gamma_\beta} \supseteq C_i$ for some $s \in B$, which is again impossible. Therefore B is cofinal with $\{\mu : \mu < \alpha\}$ and so (C) is satisfied.

Case II. $D_\delta \neq C$ for all δ. Then since Δ contains more than one element $D = \sum D_\delta \subseteq L(C)$. Suppose (by way of contradiction) that $0 < g \in L(C) \backslash D$. Then $g + a$ is finite valued for some $a = a_1 + a_2$, where $a_1 \in \sum A_\delta$ and $a_2 \in \sum A_\lambda$. As above it follows that $g + a_1$ is finite valued and belongs to $L(C) \backslash D$. Thus there exists a special element $0 < q \in L(C) \backslash D$. If $q'' \subset C$ then $q \in D$ and if $q'' = C$ then $q > L(C)$ both of which are impossible. Therefore C is a proper lex-extension of $D = L(C)$.

If $\alpha = \beta + 1$, then since C covers each D_δ, the D_δ must by 4.4 have length β and hence each D_δ belongs to the β-th level. Thus (B) is satisfied.

If α is a limit ordinal, then since each chain under C must contain one of the A_δ and C has length α it follows that α is the least upper bound of the lengths of the D_δ. Thus (C) is satisfied.

Therefore G is a lex-sum of the A_λ and by 4.11 G is a normal lex-sum if and only if $A \lhd G$. All that remains to be shown is that $A \lhd G$ under any of the given hypothesis. If G is representable, then Šik [9] has shown that each polar is normal. Thus each A_λ is normal and hence $A \lhd G$. The basis subgroup of an l-group is normal (see the discussion of basic elements and the basis subgroup given below).

Suppose $|\Lambda| = n$ is finite and that G does not contain $n + 1$ disjoint special elements. If Q is a subset of G and $g \in G$, then let $Q^g = -g + Q + g$. If $A_i^g \cap A_j = 0$ for $j = 1, \ldots, n$, then $a_i^g, a_1, \ldots, a_n$ are disjoint, but this contradicts the fact that $a_1, \ldots, a_n$ is a maximal disjoint set. Thus $A_i^g \cap A_j \neq 0$ for some j and hence by 3.1

$$A_i^g \subset A_j \quad \text{or} \quad A_j^{-g} \subset A_i \quad \text{or} \quad A_i^g = A_j .$$

Suppose (by way of contradiction) that $A_i^g \subset A_j$. Then $A_k^g \subset A_j$ or $A_k^g \cap A_j = 0$ for all k, and by 3.4 $A_j \supset A_i^g \oplus Q$, where $0 \neq Q$ is a convex l-subgroup of G. Pick $0 < q \in Q$. If no other A_k^g is contained in A_j, then $q, a_1^g, \ldots, a_n^g$ are disjoint and so $q^{-g}, a_1, \ldots, a_n$ are disjoint, a contradiction. Therefore

$$A_j \supset A_i^g \oplus A_k^g .$$

But then $a_i^g, a_k^g, a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n$ are disjoint and special, a contradiction. Thus it follows that $A_i^g = A_j$ and hence $A \lhd G$. This completes the proof of Theorem 5.1.

An element $s \in G$ is *basic* if $s > 0$ and $\{x \in G : 0 \leqq x \leqq s\}$ is totally ordered. This is equivalent to the fact that $G(s)$ is an o-group [3], and hence to the fact that s'' is a maximal convex o-subgroup (Theorem 2.3). A subset $S = \{a_\lambda : \lambda \in \Lambda\}$ is a *basis* for G if S is a maximal disjoint subset and each a_λ is basic. In this case $A = \sum a_\lambda''$ is the *basis subgroup* of G, and since $\{a_\lambda'' : \lambda \in \Lambda\}$ is the set of all maximal convex o-subgroups of G, $A \lhd G$.

The equivalence of (a) and (c) in the next theorem has been proven by McAlister [7].

Theorem 5.2. *For an l-group G the following are equivalent.*

(a) *G is a normal lex-sum of o-groups* $\{A_\lambda : \lambda \in \Lambda\}$.

(b) *G is finite valued and* $M(G)$ *satisfies the* DCC.

(c) *The lattice of filets of G satisfies the* DCC.

If this is the case, then $A = \sum A_\lambda$ *is the basis subgroup of G.*

Proof. By Theorem 3.11 (b) and (c) are equivalent. a) → b). Pick $0 < a_\lambda \in A_\lambda$. Then clearly $a_\lambda'' = A_\lambda$, $A = \sum A_\lambda$ is finite valued and $M(G) = \{C \in M(G) : C \supseteq A_\lambda$ for some $\lambda \in \Lambda\}$. Thus by Theorem 5.1 (b) is satisfied.

b) → a). If $0 < g \in G$, then $g = g_1 \vee \ldots \vee g_n$, where the g_i are disjoint and special. If g_1 is not basic, then $g_1 \geqq g_{11} \vee g_{12}$, $g_{11} \wedge g_{12} = 0$ and g_{11}, g_{12} are special. If g_{11} is not basic, then find g_{111}, g_{112} etc. Thus we get a descending chain $g_1'' \supset g_{11}'' \supset \ldots$ in $M(G)$ which is necessarily finite. Therefore g exceeds a basic element and hence by Theorem 5.1 in [3] G has a basis $\{a_\lambda : \lambda \in \Lambda\}$. Thus it follows by Theorem 5.1 that G is a lex-sum of the o-groups $A_\lambda = a_\lambda''$ and since the basis subgroup $A = \sum A_\lambda \lhd G$, G is a normal lex-sum of the A_λ. Thus a lex-sum of o-groups is necessarily normal.

The following is an unpublished theorem of NORMAN REILLY.

Corollary. *For an l-group G the following are equivalent.*

(i) *G is finite valued and each element in* $M(G)$ *has finite length.*

(ii) *G is a normal* ω*-lex-sum of o-groups.*

There is a natural relationship between Theorems 5.1 and 5.2.

Theorem 5.3. *Suppose that G is a normal lex-sum of maximal lex-subgroups* $\{A_\lambda = a_\lambda'' : \lambda \in \Lambda\}$. *Then* $N = \sum L(A_\lambda)$ *is an l-ideal of G and* G/N *is a normal lex-sum of the o-groups* $(N + A_\lambda)/N$.

Proof. Since $A = \sum A_\lambda \lhd G$ and this is the irreducible representation of A, it follows that an inner automorphism of G must induce a permutation of the A_λ and hence a permutation of the $L(A_\lambda)$. Thus $N \lhd G$ and hence N is an l-ideal. By Theorem 5.1 $T = \{C \in M(G) : C \supseteq A_\lambda$ for some $\lambda \in \Lambda\}$ satisfies the DCC and each $X \in G/N$ is finite valued. Also

$$\frac{N + A_\lambda}{N} \cong \frac{A_\lambda}{N \cap A_\lambda} = \frac{A_\lambda}{L(A_\lambda)}$$

and hence $(N + A_\lambda)/N$ is an o-group and $\sum(N + A_\lambda)/N$ is the basis subgroup of G/N. Thus by Theorem 5.2 G/N is a normal lex sum of the o-group $(N + A_\lambda)/N$.

Theorem 5.4. *Suppose that* $\{a_\lambda : \lambda \in \Lambda\}$ *is a maximal disjoint subset of an l-group G and that each* a_λ *is special. If each* $0 < g \in G$ *is disjoint from all but* *a finite number of the a_λ, then G is a restricted ω-lex-sum of the groups $A_\lambda = a''_\lambda$, and a normal lex-sum of the A_λ if and only if $A = \sum A_\lambda \lhd G$.*

Conversely, suppose that G is a restricted ω-lex-sum of a set $\{B_\lambda : \lambda \in \Lambda\}$ of maximal lex-subgroups and pick $0 < b_\lambda \in B_\lambda \backslash L(B_\lambda)$ for each $\lambda \in \Lambda$. Then $\{b_\lambda; \lambda \in \Lambda\}$ is a maximal disjoint subset of G, each b_λ is special and each $0 < g \in G$ is disjoint from all but a finite number of the b_λ.

Proof. The verification of the converse is straightforward and will be left to the reader. Let $T = \{C \in M(G) : C \supseteq A_\lambda$ for some $\lambda \in \Lambda\}$ and consider $C = (\sum A_\delta)'' \in T$. If Δ is infinite and $c \in C \backslash L(C)$, then $c > L(C) \supseteq \sum A_\delta$ and hence $c \wedge a_\delta > 0$ for all $\delta \in \Delta$, a contradiction. Therefore Δ is finite and hence it follows from 4.6 that C has finite length in T. In particular, T satisfies the DCC. Moreover, if G is a lex-sum of the A_λ, then it is necessarily a restricted ω-lex-sum.

In order to complete the proof of the theorem it suffices by Theorem 5.1 to show that for each $0 < g \in G$ there exists an $a \in A$ such that $g + a$ is finite valued. Now $g \wedge a_{\lambda_i} > 0$ for $i = 1, \ldots, n$ and $g \wedge a_\lambda = 0$ for all other $\lambda \in \Lambda$. Let M be a value of $g + a = g + a_{\lambda_1} + \ldots + a_{\lambda_n}$. If $a_{\lambda_i} \notin M$, then $M \subseteq N$ the value of a_{λ_i} and if $M \subset N$, then $a_{\lambda_i} < g + a \in N$, a contradiction. Thus if $a_{\lambda_i} \notin M$, then M is the value of a_{λ_i}. Suppose that M is not a value of a_{λ_i} for any i, then $a_{\lambda_1}, \ldots, a_{\lambda_n} \in M$. Suppose (by way of contradiction) that $M \not\supseteq a'_{\lambda_i}$ for $i = 1, \ldots, n$ and pick $0 < x_i$ in $a'_{\lambda_i} \backslash M$ for $i = 1, \ldots, n$. Then $x = g \wedge x_1 \wedge \ldots \wedge x_n \notin M$ but $x \in \bigcap a'_\lambda = 0\ (\lambda \in \Lambda)$ a contradiction. Thus $M \supseteq a'_{\lambda_i}$ for some i and hence $M \supseteq G(a_{\lambda_i}) \oplus a'_{\lambda_i} = X$. But by Theorem 3.6 in [4] X is a prime subgroup of G and hence there exists at most one value of $g + a$ that contains it. Therefore $g + a$ has at most n values.

Corollary I. *Let $\{a_\lambda : \lambda \in \Lambda\}$ be a set of disjoint special elements of an l-group H and let $G = \{a_\lambda : \lambda \in \Lambda\}''$. If each $0 < g \in G$ is disjoint from all but a finite number of a_λ, then G is a lex-sum of the maximal lex-subgroups a''_λ.*

Corollary II. *If $0 < g \in G$ has only a finite number of values, then $G(g)'' = g''$ is a lex-sum of a finite number of maximal lex-subgroups.*

Proof. $g = g_1 + \ldots + g_n$, where the g_i are disjoint and special and clearly

$$G(g)'' = (G(g_1) \oplus \ldots \oplus G(g_n))'' = \{g_1, \ldots, g_n\}''.$$

The result now follows from Corollary I.

If $a_1, \ldots, a_n$ is a finite maximal disjoint subset of G and each a_i is special, then by Theorem 5.4 G is a lex-sum of the groups $A_i = a''_i$. Byrd [2] has shown that the set S of all the conjugates of the A_i is finite. Thus G is a normal lex-sum of the minimal elements in S. Thus by Theorem 5.3 there exists an l-ideal N of G such that $a_i \notin N$ for $i = 1, \ldots, n$ and G/N is a lex sum of a finite number of o-groups. Whether or not this can be generalized to an infinite set $\{a_\lambda : \lambda \in \Lambda\}$ that satisfies the hypotheses of Theorem 5.4 is not known.

6. L-Groups with a finite basis. We shall first consider l-groups that satisfy (F) each $0 < g \in G$ exceeds at most a finite number of disjoint elements or equivalently each bounded disjoint subset of G is finite. In [3] it is shown that if G satisfies (F), then G has a basis. Moreover, G satisfies (F) if and only if each $G(g)$ has a finite basis. It is easy to show that a representable l-group G satisfies (F) if and only if G is a subdirect sum of a small cardinal sum of o-groups (see for example [1]). The following is one of the main theorems in [3].

Theorem 6.1. *An l-group G is an ω-lex-sum of o-groups if and only if it satisfies* (F).

Proof. Suppose that G satisfies (F) and let $\{a_\lambda : \lambda \in \Lambda\}$ be a basis for G. Then $\{a_\lambda : \lambda \in \Lambda\}$ satisfies the hypotheses of Theorem 5.4 and hence G is an ω-lex-sum of the o-groups a''_λ. The converse also follows from Theorem 5.4.

Corollary. (Finite Basis Theorem) *An l-group G is a lex-sum of a finite number of o-groups if and only if it has a finite basis.*

Let Γ be an index set for the set of all pairs (G^γ, G_γ) of convex l-subgroups of G such that G_γ is a value of some $g \in G$ and G^γ covers G_γ. Define $\alpha < \beta$ in Γ if $G^\alpha \subseteq G_\beta$ or equivalently $G_\alpha \subset G_\beta$. Then Γ is a root system. The groups G_γ are called regular. From [3] and the theory in this paper it follows that the following statements about an l-group G are equivalent.

(1) *G has a finite basis.*
(2) *Each disjoint subset of G is finite.*
(3) *Γ contains only a finite number of maximal chains ("roots").*
(4) *Each proper convex l-subgroup of G has a finite basis.*
(5) *G is a lex-sum of a finite number of o-groups.*
(6) *Each convex l-subgroup C of G has an irreducible representation*

$$C = C_1 \oplus \ldots \oplus C_n \ (n \text{ finite}).$$

(7) *G is finite valued and $M(G)$ is finite.*
(8) *The lattice of filets of G is finite.*

Corollary. *For an l-group G the following are equivalent.*

(a) *G has only a finite number of convex l-subgroups.*
(b) *Γ is finite.*
(c) *G is a lex-sum of a finite number of o-groups and each o-group used in this construction has only a finite number of convex subgroups.*

Proof. Since each convex l-subgroup of G is the intersection of regular subgroups it follows that (a) and (b) are equivalent.

a) and b) → c). Clearly G has a finite basis, and hence G is a lex-sum of a finite number of o-groups. Let A_i^r be a group in the r-th level with $N = L(A_i^r)$. Then since there exists a one to one correspondence between the convex subgroups of A_i^r/N and the convex l-subgroups of G that lie between A_i^r and N, A_i^r/N has only a finite number of convex subgroups.

c) → a). If C is a lex-subgroup of G, then $A_i^r \supseteq C \supseteq L(A_i^r)$ for some r and i. Now for a given r and i there exist only a finite number of such subgroups C and hence it follows that there exists only a finite number of lex-subgroups. But each convex l-subgroup of G is the cardinal sum of a finite number of lex-subgroups, and hence (a) is satisfied.

This last result can be generalized. The *rank* of an o-group H is the order type of its chain of convex subgroups. In particular, H has inversely well ordered rank means that each ascending chain of convex subgroups is finite.

Lemma 6.2. *For an o-group H the following are equivalent.*

(a) *H has inversely well ordered rank.*
(b) $\Gamma = \Gamma(H)$ *is inversely well ordered.*
(c) *Each convex subgroup is principal (that is, has the form* $H(a)$*).*

Proof. Clearly (a) implies (b).

b) → c). If $0 < x \in C$ a convex subgroup, then there exists a regular subgroup $K \subset C$. Let M be the largest such subgroup and consider $0 < a \in C \setminus M$. If $0 < c \in C \setminus H(a)$, then there exists a regular subgroup N such that $M \subset H(a) \subseteq N \subset C$, a contradiction. Therefore $C = H(a)$.

c) → a). If $\mathscr{C}$ is a set of convex subgroups of H, then $S = \bigcup_{C \in \mathscr{C}} C = H(a)$ for some $a \in H$. But then $a \in C \in \mathscr{C}$ and hence $H(a) \subseteq C \subseteq S = H(a)$. Thus C is the largest element in $\mathscr{C}$.

Theorem 6.2. *For an l-group G the following are equivalent.*

(1) *Each convex l-subgroup of G is finitely generated.*
(2) *Each convex l-subgroup of G is principal.*
(3) Γ *has only a finite number of roots and satisfies the* ACC.
(4) *G has a finite basis and each of the o-groups used in lex-sum construction of G has inversely well ordered rank.*

Proof. 1) → 2). If $g_1, \ldots, g_n$ generate the convex l-subgroup C of G, then $g = |g_1| + \ldots + |g_n| \in C$ and hence $G(g) \subseteq C$, but each $|g_i| \in G(g)$ and hence $g_1, \ldots, g_n \in G(g)$. Therefore $G(g) = C$.

2) → 3). If $a_1, a_2, \ldots$ is an infinite disjoint set, then $G(a_1) \oplus G(a_2) \oplus \ldots$ is not principal. Thus each disjoint subset of G is finite, and hence Γ has only a finite number of roots. To complete the proof of this implication it suffices to show that a chain of regular subgroups that contains a given minimal prime subgroup M is inversely well

ordered. Let $\mathscr{C}$ be a set of regular subgroups that contain M. Then exactly as in the above proof of c) → a) it follows that $\mathscr{C}$ contains a largest element.

3) → 4). Clearly G has a finite basis. Consider A_i^r with lex kernel N. We must show that the regular subgroups of A_i^r containing N are inversely well ordered. But if M is a prime subgroup of G that does not contain A_i^r, then $M \cap A_i^r$ is a prime subgroup of A_i^r and this mapping σ is one to one onto (see the proof of Theorem 3.5 in [4]). The set $\mathscr{S}$ of regular subgroups of G that contain $N\sigma^{-1}$ but not A_i^r are mapped by σ onto the set of regular subgroups of A_i^r that contain N. Since $N\sigma^{-1}$ is prime in G it follows that $\mathscr{S}$ is a chain in Γ and hence it is inversely well ordered. Therefore the regular subgroups of A_i^r containing N are inversely well ordered.

4) → 1). If C is a lex-subgroup of G, then $A_i^r \supseteq C \supset N = L(A_i^r)$ for some r and i and A_i^r/N has inversely well ordered rank. Thus by Lemma 6.2 C/N is generated by a single element $N + c$, where $0 < c \in C$. If $0 < x \in C$, then $N + x < N + mc$ for some $m > 0$ and hence $x < mc$. Therefore $C \subseteq G(c)$ and clearly $C \supseteq G(c)$. Thus each lex-subgroup of G is principal. But it is easy to check that each non-zero convex l-subgroup of G is a cardinal sum of a finite number of lex-subgroups. Therefore each convex l-subgroup of G is finitely generated.

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