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ON THE POLAR DECOMPOSITION OF SCALAR-TYPE OPERATORS

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In [3] it is proved that for an arbitrary scalar-type operator S we have $S = A + iB$, where A and B are commuting scalar-type operators with a real spectrum and i is the imaginary unit. This decomposition is analogous to the expressing a complex number in the form $\lambda = \alpha + i\beta$, where α and β are real numbers.

The present paper deals with the so-called polar decomposition $S = RU$ of the scalar-type operator S, where R and U are commuting scalar-type operators, R has a real spectrum and the spectrum of U lies on the boundary of the unit circle. Thus, an analog of expressing a complex number in the form $\lambda = \varrho e^{i\varphi}$ is under considerat-ion.

The polar decomposition of a normal operator (thus a scalar-type operator) in a Hilbert space is well-known [2].

The concepts and results from the theory of spectral operators used here are published in [1].

Notation. Denote p the complex plane, K the boundary of the unit circle in p, P the set of all non-negative real numbers, and $\tilde{P}$ the set of all positive real numbers. $\mathcal{B}_K$ will be the algebra of Borel sets on K and $\mathcal{B}_P$ the algebra of Borel sets on P.

Definition. The scalar-type operator S is said to be pseudo-unitary, if $\sigma(S) \subset K$.

Before proving our theorem we shall prove the following lemma.

Lemma. *Let A be a set and $\mathcal{B}$ be an algebra of its subsets. Let E be the spectral measure defined on $\mathcal{B}$. Let f and g be two bounded $\mathcal{B}$-measurable complex functions defined on A. Then*

$$(1) \qquad \int_A f(z)\, g(z)\, \mathrm{d}E(z) = \int_A f(z)\, \mathrm{d}E(z) \int_A g(z)\, \mathrm{d}E(z)\,.$$

The integral is that defined in [1].

Proof. Let f, g be $\mathscr{B}$-measurable finite-valued functions on A. Due to the multiplicative property of the spectral measure E (1) is valid.

If f, g are bounded $\mathscr{B}$-measurable functions on A, then there exist two sequences f_n, g_n of $\mathscr{B}$-measurable finite-valued functions such that $f_n \to f$, $g_n \to g$ uniformly on A. According to (1) and definition of the integral we have

$$\int_A f(z)\, g(z)\, \mathrm{d}E(z) = \lim_n \int_A f_n(z)\, g_n(z)\, \mathrm{d}E(z) =$$

$$= \lim_n \int_A f_n(z)\, \mathrm{d}E(z) \int_A g_n(z)\, \mathrm{d}E(z) = \int_A f(z)\, \mathrm{d}E(z) \int_A g(z)\, \mathrm{d}E(z)\,.$$

Theorem. *Let $\mathfrak{X}$ be a Banach space and $S \in B(\mathfrak{X})$ be a scalar type operator with the resolution of identity E.*

Then there exist operators R and U such that

(i) *$S = RU$ where R is scalar-type and U is pseudo-unitary*

(ii) *R commutes with U and the operators R and U commute with S*

(iii) $\sigma(R) = \{\varrho : \varrho = |\lambda|,\ \lambda \in \sigma(S)\}$

(iv) *if in addition $S^{-1} \in B(\mathfrak{X})$, then $E = E_R \otimes E_U$, where E_R and E_U are resolutions of identity for R and U, respectively. Under this assumption the decomposition $S = RU$ is unique and the spectrum of the operator U is given by*

$$\sigma(U) = \{\lambda : \lambda = \eta/|\eta|,\ \eta \in \sigma(S)\}\,.$$

Proof. If $\sigma \subset P$ and $\delta \subset K$, then $\sigma \times \delta$ denotes the set of all ordered pairs of numbers (ϱ, η), where $\varrho \in \sigma$, $\eta \in \delta$. Since every complex number λ can be written in the form $\lambda = \varrho e^{i\varphi}$, $\sigma \times \delta$ can be identified with the set of complex numbers of the form $\{\lambda : \lambda = \varrho e^{i\varphi},\ \varphi \in \sigma,\ e^{i\varphi} \in \delta\}$. Then the set $\tilde{P} \times K$ represents the entire complex plane without the origin.

We define

$$\begin{aligned} (2) \qquad & E_R(\sigma) = E(\sigma \times K) && \text{for} \quad \sigma \in \mathscr{B}_P\,, \\ & E_U(\delta) = E(\tilde{P} \times \delta) && \text{for} \quad 1 \notin \delta,\ \delta \in \mathscr{B}_K\,, \\ & E_U(\delta) = E(\tilde{P} \times \delta) + E(\{0\}) && \text{for} \quad 1 \in \delta,\ \delta \in \mathscr{B}_K\,. \end{aligned}$$

Thus, E_K is a spectral measure on $\mathscr{B}_P$. Let us prove that E_U is a spectral measure on $\mathscr{B}_K$. We have

$$E_U(\emptyset) = E(\tilde{P} \times \emptyset) = E(\emptyset) = 0\,,$$

$$E_U(K) = E(\tilde{P} \times K) + E(\{0\}) = E(p) = I\,.$$

If $\delta_1, \delta_2 \in \mathscr{B}_K$ and $1 \in \delta_1 \cap \delta_2$, then

$$
\begin{aligned}
E_U(\delta_1 \cap \delta_2) &= E(\tilde{P} \times \delta_1 \cap \delta_2) + E(\{0\}) = E(\tilde{P} \times \delta_1)\, E(\tilde{P} \times \delta_2) + E(\{0\}) = \\
&= E(\tilde{P} \times \delta_1)\, E(\tilde{P} \times \delta_2) + E(\{0\})\, E(\tilde{P} \times \delta_2) + E(\tilde{P} \times \delta_2)\, E(\{0\}) + \\
&\quad + E^2(\{0\}) = (E(\tilde{P} \times \delta_1) + E(\{0\}))\,(E(\tilde{P} \times \delta_2) + E(\{0\})) = \\
&= E_U(\delta_1)\, E_U(\delta_2)\,.
\end{aligned}
$$

An analogous calculation shows that E_U has all properties of a spectral measure. The uniform boundedness of E_U on $\mathscr{B}_K$ is obvious.

Define

$$
\varphi(\lambda) = \begin{cases} \lambda/|\lambda| & \text{for} \quad \lambda \neq 0\,, \quad \lambda \in p \\ 1 & \text{for} \quad \lambda = 0 \end{cases}
$$

and

$$
(3) \qquad R = \int_P \varrho = \varrho \,\mathrm{d}E_R(\varrho) \quad \text{and} \quad U = \int_K z \,\mathrm{d}E_U(z)\,.
$$

According to Lemma 6 in [1] R and U are scalar-type operators with resolutions of identity E_R and E_U, respectively. U is pseudo-unitary and $\sigma(R) \subset P$. From the definition of R and U it follows that

$$
R = \int_p |\lambda| \,\mathrm{d}E(\lambda) \quad \text{and} \quad U = \int_p \varphi(\lambda)\,\mathrm{d}E(\lambda)\,.
$$

Using the lemma we have

$$
\begin{aligned}
RU &= \int_p |\lambda| \,\mathrm{d}E(\lambda) \int_p \varphi(\lambda)\,\mathrm{d}E(\lambda) = \int_{\sigma(S)} |\lambda| \,\mathrm{d}E(\lambda) \int_{\sigma(S)} \varphi(\lambda)\,\mathrm{d}E(\lambda) = \\
&= \int_{\sigma(S)} |\lambda|\, \varphi(\lambda)\,\mathrm{d}E(\lambda) = \int_{\sigma(S)-\{0\}} |\lambda|\, \lambda/|\lambda| \,\mathrm{d}E(\lambda) + \int_{\{0\}} \lambda \,\mathrm{d}E(\lambda) = \\
&= \int_p \lambda \,\mathrm{d}E(\lambda) = S\,.
\end{aligned}
$$

Therefore (i) is proved. From the lemma there follows the commutativity of the operator R with U and of the operators R and U with S, and hence (ii). To prove (iii) consider that R is a continuous function of S.

Let now $S^{-1} \in B(\mathfrak{X})$. Then $E(\{0\}) = 0$ and in (2) we have $E_U(\delta) = E(\tilde{P} \times \delta)$ for every $\delta \in \mathscr{B}_K$.

$$
\begin{aligned}
(4) \quad E_R(\sigma)\, E_U(\delta) &= E(\sigma \times K)\, E(\tilde{P} \times \delta) = E((\sigma \times K) \cap (\tilde{P} \times \delta)) + E(\{0\}) = \\
&= E((\sigma \cap \tilde{P}) \times \delta) \cup \{0\}) = E(\sigma \times \delta)\,.
\end{aligned}
$$

Let pairs of operators R_1, U_1 and R_2, U_2 satisfy (i) to (iii) and let their resolutions of identity be E_R, E_U and F_R, F_U, respectively. Due to (4) we have $E = E_R \otimes E_U$ and $E = F_R \otimes F_U$.

If $\sigma \in \mathscr{B}_P$ and $\delta \in \mathscr{B}_K$, the identities

$$\begin{aligned} E(\sigma \times K) &= E_R(\sigma)\, E_U(K) = E_R(\sigma) \\ E(\sigma \times K) &= F_R(\sigma)\, F_U(K) = F_R(\sigma) \\ E(P \times \delta) &= E_R(P)\, E_U(\delta) = E_U(\delta) \\ E(P \times \delta) &= F_R(P)\, F_U(\delta) = F_U(\delta) \end{aligned}$$

are valid. This implies $E_R(\sigma) = F_R(\sigma)$ and $E_U(\delta) = F_U(\delta)$; thus $R_1 = R_2$ and $U_1 = U_2$.

Finally,

$$U = \int_K \lambda \, \mathrm{d}E(\lambda) = \int_P \varphi(\lambda) \, \mathrm{d}E(\lambda) = \int_P \lambda/|\lambda| \, \mathrm{d}E(\lambda) \, .$$

The function $\varphi(\lambda)$ is continuous on some neigborhood of the spectrum of the operator S and therefore $U = \varphi(S)$, $\sigma(U) = \sigma(\varphi(S)) = \varphi(\sigma(S)) = \{\lambda : \lambda = \eta/|\eta|, \eta \in \sigma(S)\}$, and the proof is completed.

Note. As we can see in the proof of the theorem, the operator U need not be uniquely determined if $S^{-1} \notin B(\mathfrak{X})$. Defining the resolution of identity E_U as in the proof of the theorem we have

$$\sigma(U) = \{\lambda : \lambda = \eta/|\eta|, \; \eta \in \sigma(S) - \{0\}\} \cup \{1\} \, .$$

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