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PRODUCT OF SPECTRAL MEASURES

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One of the interesting questions of the spectral theory of operators can be formulated as follows: If A, B are commuting scalar-type spectral operators (in the sense of [3]), is every operator from the algebra generated by A and B a scalar-type spectral operator, too? Some significant results concerning this question are already known (see [5], [8]) but they were obtained by relatively complicated tools. The following approach to this problem seems to be more natural.

Let E, F be the resolutions of identity (i.e. spectral measures) for operators A, B, respectively, defined on the σ-algebra of Borel sets in the complex plane p. Suppose that on the σ-algebra of Borel subsets of the space $p \times p$ there is constructed a spectral measure G such that

$$G(\varrho \times \sigma) = E(\varrho) F(\sigma) \tag{1}$$

for every Borel set $\varrho, \sigma \subset p$. Then $A = \int_{p \times p} \lambda \, \mathrm{d}G(\lambda, \mu)$ and $B = \int_{p \times p} \mu \, \mathrm{d}G(\lambda, \mu)$. Hence for arbitrary polynomial f of two variables we obtain that $f(A, B) = \int_{p \times p} f(\lambda, \mu) \, \mathrm{d}G(\lambda, \mu)$. It is evident that if the construction of the spectral measure G is possible, then the solution of the given problem presents no difficulty. We can expect that the solution of the existence question of spectral measure G will enable us to solve some other problems.

In the present paper the conditions for the existence of the spectral measure G with required properties are given. Evidently, it is not necessary to suppose that E and F are resolutions of identity for some given operators and the complex plane p will be replaced by a more general case of a σ-compact Hausdorff space.

1. In the whole paper X will stand for some fixed Banach space, X' for the dual space to X and $B(X)$ for the algebra of all linear bounded mappings of X into X. Besides, we treat X as a subset of X'', i.e. we shall not distinguish between the points $x \in X$ and their canonical images in X''. Similarly for other spaces (X' etc.).

Let $\Gamma \subset X'$ be a total set and let $\mathscr{S}$ be an algebra of subsets of some set S. A spectral measure in X of class $(\mathcal{S}, \Gamma)$ is a $B(X)$-valued function E defined on $\mathcal{S}$ with the following properties:

(i) $\langle E(\ )\, x, x' \rangle$ is σ-additive on $\mathcal{S}$ for every $x \in X$ and every $x' \in \Gamma$ (Γ-weak σ-additivity);

(ii) there exists a constant K such that $\|E(\sigma)\| \leqq K$ for every $\sigma \in \mathcal{S}$ (uniform boundedness);

(iii) $E(\sigma_1 \cap \sigma_2) = E(\sigma_1)\, E(\sigma_2)$ for every $\sigma_1, \sigma_2 \in \mathcal{S}$ (multiplicativity) and $E(\emptyset) = 0$, $E(S) = \mathrm{I}$, where 0 is the zero-operator and I is the identity operator in $B(X)$.

Let S be a locally compact topological space and $\mathcal{S}$ be an algebra of its subsets. A spectral measure E in X of the class $(\mathcal{S}, \Gamma)$ is said to be regular if, for every $x \in X$ and $x' \in \Gamma$, $\langle E(\ )\, x, x' \rangle$ is a regular numerical measure on $\mathcal{S}$ (in the sense of [4; III.5.11]).

2. Let $\mathcal{R}$ and $\mathcal{S}$ be an algebra of subsets of a set R and S, respectively, and let $\Gamma \subset X'$ be total. We say that two commuting spectral measures E and F in X of the classes $(\mathcal{R}, \Gamma)$ and $(\mathcal{S}, \Gamma)$, respectively, fulfil the condition (B), if

(B), *the Boolean algebra of the projection operators generated by* $E(\varrho)$, $\varrho \in \mathcal{R}$, *and* $F(\sigma)$, $\sigma \in \mathcal{S}$, *is uniformly bounded, i.e. there exists a constant* $L > 0$ *such that* $\|\sum_{i=1}^{n} E(\varrho_i)\, F(\sigma_i)\| \leqq L$ *for every finite system of mutually disjoint sets* $\varrho_i \times \sigma_i$, *where* $\varrho_i \in \mathcal{R}$, $\sigma_i \in \mathcal{S}$.

3. Let R and S be some σ-compact Hausdorff spaces and $T = R \times S$. Let $\mathcal{R}, \mathcal{S}, \mathcal{T}$ denote the σ-algebra of Baire subsets of the space R, S, T, respectively. The supposition "R, or S is σ-compact" is necessary and sufficient for the system of all Baire subsets of R, S, respectively, to be an algebra [6; Exercise 51.2].

We shall designate the set $\{\varrho \times \sigma : \varrho \in \mathcal{R}, \sigma \in \mathcal{S}\}$ by $\mathcal{P}$ and $\mathcal{Q}$ the algebra generated by $\mathcal{P}$. That is,

$$\mathcal{Q} = \{ \bigcup_{i=1}^{n} \varrho_i \times \sigma_i : \varrho_i \times \sigma_i \in \mathcal{P} \quad \text{mutually disjoint}\,, \quad n \in N \}\,,$$

where N stands for the set of all positive integers. It is known [6; Exercise 5.3] that $\mathcal{T}$ is the σ-algebra generated by $\mathcal{Q}$.

4. Let E and F be two commuting spectral measures in X of the classes $(\mathcal{R}, \Gamma)$ and $(\mathcal{S}, \Gamma)$, respectively. We define the $B(X)$-valued set-function G_0 on $\mathcal{P}$ by

$$G_0(\varrho \times \sigma) = E(\varrho)\, F(\sigma)\,, \quad \varrho \in \mathcal{R}\,, \quad \sigma \in \mathcal{S}\,.$$

Since G_0 is additive on $\mathcal{P}$, it has a unique additive extension on all $\mathcal{Q}$ (denoted by G_0 again) defined by

$$G_0(\tau) = \sum_{i=1}^{n} E(\varrho_i)\, F(\sigma_i) \tag{2}$$

whenever $\tau \in \mathcal{Q}$ has the form $\tau = \bigcup_{i=1}^{n} \varrho_i \times \sigma_i$ with mutually disjoint members. The additivity of G_0 and the commutativity and the multiplicativity of E and F imply the multiplicativity of G_0 on $\mathcal{Q}$.

Now we should like to prove that G_0 is Γ-weakly σ-additive on $\mathcal{Q}$ and extend it to some multiplicative and Γ-weakly σ-additive $B(X)$-valued function G on $\mathcal{T}$ thus to a spectral measure in X of the class $(\mathcal{T}, \Gamma)$ satisfying (1).

The uniform boundedness of spectral measure (i.e. condition (ii)) implies readily that condition (B) is necessary for the existence of such an extension.

5. We consider only spectral measures defined on the σ-algebra of all Baire sets in a given space. The following theorem states, however, that any spectral measure in X of the class $(\mathcal{R}, X')$ (it is the most interesting case from the point of view of spectral theory) can be uniquely extended to a regular spectral measure of the class $(\mathcal{R}_1, X')$, where $\mathcal{R}_1$ is the σ-algebra of all Borel subsets of the space R.

Theorem. *Let H_0 be a spectral measure in X of the class $(\mathcal{R}, X')$. Then in X there exists exactly one regular spectral measure H of the class $(\mathcal{R}_1, X')$ coinciding with H_0 on $\mathcal{R}$.*

Proof. Given $x \in X$, there exists a non-negative σ-additive measure ν_0 on $\mathcal{R}$ such that

$$\lim_{\nu_0(\sigma) \to 0} \|H_0(\sigma)\, x\| = 0 \tag{3}$$

(see [2] or [4; IV.10.5]). Realizing that ν_0 is a σ-additive non-negative measure on the algebra of Baire sets, it follows that it is regular on $\mathcal{R}$ [6; Theorem 52.G] and there exists exactly one regular non-negative σ-additive measure ν on $\mathcal{R}_1$ with $\nu(\varrho) = \nu_0(\varrho)$ for every $\varrho \in \mathcal{R}$ [6; Theorem 54.D].

Introduce a pseudo-metric in $\mathcal{R}_1$ by $d(\varrho_1, \varrho_2) = \nu(\varrho_1 \triangle \varrho_2)$, $\varrho_1, \varrho_2 \in \mathcal{R}_1$.

According to regularity we may conclude that $\mathcal{R}$ is a dense subset of $\mathcal{R}_1$ in the introduced pseudo-metric [6; Theorem 50.D]. Moreover, by (3), $H_0(\ )\, x$ is a uniformly continuous map of $\mathcal{R}$ into X. Thus, according to a well known theorem (e.g. [4; I.6.17]) there exists exactly one uniformly continuous map h_x on $\mathcal{R}_1$ satisfying $h_x(\varrho) = H_0(\varrho)\, x$, $\varrho \in \mathcal{R}$. It can be proved easily that, for every $\varrho \in \mathcal{R}_1$, $h_x(\varrho)$ depends linearly and continuously on $x \in X$. Thus, for every $\varrho \in \mathcal{R}_1$, there exists an operator $H(\varrho) \in B(X)$ such that $H(\varrho)\, x = h_x(\varrho)$, $\varrho \in \mathcal{R}_1$. Since $H(\ )\, x$ is uniformly continuous on $\mathcal{R}_1$, it follows that

$$\lim_{\nu(\varrho) \to 0} \|H(\varrho)\, x\| = 0\,. \tag{4}$$

This relation implies easily that $H(\ )\, x$ is regular and σ-additive on $\mathcal{R}_1$ so that $\langle H(\ )\, x, x' \rangle$ is also regular and σ-additive on $\mathcal{R}_1$ for all $x \in X$, $x' \in X'$.

We must still prove the multiplicativity of H on $\mathscr{R}_1$. Let $\tau_1, \tau_2 \in \mathscr{R}_1$ be two arbitrary sets. Let $\varepsilon > 0$ and $x \in X$ be arbitrary. The uniform continuity implies that there exists $\delta > 0$ such that $\|H(\sigma_1) x - H(\sigma_2) x\| < \varepsilon$ for every $\sigma_1, \sigma_2 \in \mathscr{R}_1$ for which $\nu(\sigma_1 \triangle \sigma_2) < \delta$. The density of $\mathscr{R}$ in $\mathscr{R}_1$ implies the existence of sets $\varrho_1, \varrho_2 \in \mathscr{R}$ such that $\nu(\tau_1 \triangle \varrho_1) < \frac{1}{2}\delta$, $\nu(\tau_2 \triangle \varrho_2) < \frac{1}{2}\delta$. Since $(\tau_1 \cap \tau_2) \triangle (\varrho_1 \cap \varrho_2) \subset (\tau_1 \triangle \varrho_1) \cup \cup (\tau_2 \triangle \varrho_2)$, we have also $\nu((\tau_1 \cap \tau_2) \triangle (\varrho_1 \cap \varrho_2)) < \delta$ and therefore $\|H(\tau_1) x - - H(\varrho_1) x\| < \varepsilon$, $\|H(\tau_2) x - H(\varrho_2) x\| < \varepsilon$ and $\|H(\tau_1 \cap \tau_2) x - H(\varrho_1 \cap \varrho_2) x\| < \varepsilon$. These relations and multiplicativity of H on $\mathscr{R}$ imply

$$\|H(\tau_1 \cap \tau_2) x - H(\tau_1) H(\tau_2) x\| \leqq \|H(\tau_1 \cap \tau_2) x - H(\varrho_1 \cap \varrho_2) x\| + \\ + \|H(\varrho_1)\| \|H(\tau_2) x - H(\varrho_2) x\| + \|H(\tau_2)\| \|H(\tau_1) x - H(\varrho_1) x\| < (1 + 2K) \varepsilon,$$

where K is the constant from (ii). Hence, $H(\tau_1 \cap \tau_2) = H(\tau_1) H(\tau_2)$ for all $\tau_1, \tau_2 \in \mathscr{R}_1$.

We have proved that H is the spectral measure in X of the class $(\mathscr{R}_1, X')$ with the required properties.

6. Theorem. *Let E and F be two commuting spectral measures in X' of the classes $(\mathscr{R}, X)$ and $(\mathscr{S}, X)$, respectively. Let them satisfy condition* (B).

Then in X' there exists exactly one spectral measure G of the class $(\mathscr{T}, X)$ satisfying (1).

Proof. Let G_0 be the function on $\mathscr{Q}$ defined by (2). For arbitrary $x' \in X'$ and $x \in X$ we define

(5) $$\lambda_{x',x}(\pi) = \langle G_0(\pi) x', x \rangle, \quad \tau \in \mathscr{Q}.$$

Let $\{\varrho_n\}$ be a sequence of mutually disjoint sets from $\mathscr{R}$, $\varrho = \bigcup_{n=1}^{\infty} \varrho_n$ and $\sigma \in \mathscr{S}$. Then

$$\lambda_{x',x}(\varrho \times \sigma) = \langle G_0(\varrho \times \sigma) x', x \rangle = \langle E(\varrho) F(\sigma) x', x \rangle = \\ = \sum_{n=1}^{\infty} \langle E(\varrho_n) F(\sigma) x', x \rangle = \sum_{n=1}^{\infty} \lambda_{x',x}(\varrho_n \times \sigma).$$

Hence $\lambda_{x',x}(\varrho \times \sigma)$ is σ-additive as a function of ϱ for every fixed $\sigma \in \mathscr{S}$. Similarly it can be proved that $\lambda_{x',x}(\varrho \times \sigma)$ is σ-additive as a function of $\sigma \in \mathscr{S}$ for every fixed $\varrho \in \mathscr{R}$. Condition (B) implies the existence of a constant $L > 0$ such that $\|G_0(\pi)\| \leqq \leqq L$, $\pi \in \mathscr{Q}$. Thus $|\lambda_{x',x}(\pi)| \leqq L\|x'\| \|x\|$ for all $\pi \in \mathscr{Q}$, $x' \in X'$, $x \in X$. All assumptions of Theorem 2 from [9] are fulfilled so that $\lambda_{x',x}$ is σ-additive on $\mathscr{Q}$ and there exists exactly one σ-additive function $\mu_{x',x}$ on $\mathscr{T}$ such that

(6) $$\mu_{x',x}(\pi) = \lambda_{x',x}(\pi), \quad \pi \in \mathscr{Q}.$$

At the same time

(7) $$|\mu_{x',x}(\tau)| \leqq L\|x'\| \|x\|$$

remains valid for every $\tau \in \mathscr{T}$.

Using (3) we conclude that for every $\tau \in \mathscr{T}$, $\mu_{x',x}(\tau)$ is a bounded linear form on $X' \times X$. Thus, there exists a unique operator $G(\tau) \in B(X')$ such that $\langle G(\tau)\, x', x\rangle = \mu_{x',x}(\tau)$, $x' \in X'$, $x \in X$. (5) and (6) imply that G defined in this way satisfies (1). It follows from the definition of G that G is X-weakly σ-additive on $\mathscr{T}$. We shall finish the proof by showing that

$$G(\tau_1 \cap \tau_2) = G(\tau_1)\, G(\tau_2) \tag{8}$$

whenever τ_1 and τ_2 are in $\mathscr{T}$.

By a straightforward computation using the commutativity of E and F and the additivity of G we may conclude that (8) is valid for every $\tau_1, \tau_2 \in \mathscr{Q}$.

Let τ_1 be an arbitrary element of $\mathscr{Q}$. We denote $\mathscr{M}_1$ the system of all sets $\tau_2 \in \mathscr{T}$ such that (8) is valid. It follows that $\mathscr{Q} \subset \mathscr{M}_1$. Due to the σ-additivity of $\langle G(\ )\, x', x\rangle$, $\mathscr{M}_1$ is a monotone system. Consequently, $\mathscr{T} \subset \mathscr{M}_1$ [6; Theorem 6B].

Now, let τ_2 be an arbitrary element of $\mathscr{T}$. We denote $\mathscr{M}_2$ the system of all sets $\tau_1 \in \mathscr{T}$ such that (8) is valid. It follows from the preceding section that $\mathscr{Q} \subset \mathscr{M}_2$. The fact that $\mathscr{M}_2$ is a monotone system implies that $\mathscr{T} \subset \mathscr{M}_2$.

We have proved that (8) is valid for every $\tau_1, \tau_2 \in \mathscr{T}$.

7. Corollary. *Suppose that E and F are two commuting spectral measures in X of the classes $(\mathscr{R}, X')$ and $(\mathscr{S}, X')$, respectively. Let them satisfy condition* (B). *Then in X'' there exists exactly one spectral measure G of the class $(\mathscr{T}, X')$ such that*

$$G(\varrho \times \sigma) = E(\varrho)''\, F(\sigma)''\,, \quad \varrho \in R\,, \quad \sigma \in \mathscr{S}\,. \tag{9}$$

Proof. We denote E'' the mapping of $\mathscr{R}$ into $B(X'')$ defined by $E''(\varrho) = E(\varrho)''$, $\varrho \in \mathscr{R}$. F'' is defined analogously. Then E'' and F'' are commuting spectral measures in X'' of the classes $(\mathscr{R}, X')$ and $(\mathscr{S}, X')$, respectively (see e.g. [4; VI.3.3]). E'' and F'' satisfy condition (B), hence by the theorem 6 there exists exactly one spectral measure G in X'' of the class $(\mathscr{T}, X')$ satisfying (9).

8. Theorem. *Let E and F be two commuting spectral measures in X of the classes $(\mathscr{R}, X')$ and $(\mathscr{S}, X')$, respectively.*

Then in X there exists a spectral measure G of the class $(\mathscr{T}, X')$ such that (1) *is satisfied if and only if, for every $x \in X$, the set $N(x) = \{G_0(\pi)\, x : \pi \in \mathscr{Q}\}$ is relatively weakly compact in X. (G_0 is the function* (2) *from* 4.)

Proof. Let $N(x)$ be relatively weakly compact for every $x \in X$. Then $N(x)$ is a bounded set and the system $\{G_0(\pi) : \pi \in \mathscr{Q}\}$ of continuous linear operators is uniformly bounded in the uniform operator topology (see e.g. [4; II.3.27 and II.1.11]). We conclude that E and F satisfy condition (B). By Corollary 7 there exists exactly one spectral measure H in X'' of the class $(\mathscr{T}, X')$ such that $H(\pi) = G_0(\pi)''$, $\pi \in \mathscr{Q}$.

We want to prove that there exists a spectral measure G in X of the class $(\mathscr{T}, X')$ such that $G(\pi)'' = H(\tau)$, $\tau \in \mathscr{T}$.

For $x \in X$ we denote $\overline{N(x)}$ the closure of $N(x)$ in the weak topology of the space X. By hypothesis and Eberlein-Šmulian theorem [4; V.6.1] $\overline{N(x)}$ is weakly compact. We denote $\mathscr{M}$ the system of all sets τ for which there exists $G(\tau) \in B(X)$ such that $G(\tau)'' = H(\tau)$ and $G(\tau)\, x \in \overline{N(x)}$ for every $x \in X$. It is obvious that $\mathscr{Q} \subset \mathscr{M}$.

Let $\{\tau_n\}$ be a monotone sequence of sets from $\mathscr{M}$ and let $\tau = \lim_n \tau_n$. There exists a sequence $\{G(\tau_n)\}$ of operators from $B(X)$ such that $G(\tau_n)'' = H(\tau_n)$ and $G(\tau)\, x \in \overline{N(x)}$, $x \in X$. Since H is a spectral measure it implies the existence of $\lim_n \langle H(\tau_n)\, x'', x' \rangle$ for every $x' \in X'$, $x'' \in X''$. Thus, the sequence $\{H(\tau_n)\, x''\}$ is weakly* fundamental for every $x'' \in X''$. Therefore the sequence $\{G(\tau_n)\, x\}$ is weakly fundamental. Consequently the sequence $\{G(\tau_n)\, x\}$ is weakly convergent and since $\lim_n \langle H(\tau_n)\, x'', x' \rangle = \langle H(\tau)\, x'', x' \rangle$, there exists $G(\tau) \in B(X)$ such that $\langle G(\tau)\, x, x' \rangle = \lim_n \langle G(\tau_n)\, x, x' \rangle$ for every $x' \in X'$ and $G(\tau)'' = H(\tau)$. Obviously, $G(\tau)\, x \in \overline{N(x)}$. Thus, $\tau \in \mathscr{M}$ and the system $\mathscr{M}$ is monotone. The fact that $\mathscr{Q}$ is a ring implies that $\mathscr{T} \subset \mathscr{M}$ [6; Theorem 6B].

Thereby we have proved that on $\mathscr{T}$ there exists a function G with values in $B(X)$ such that $G(\tau)'' = H(\tau)$. Thus, G is the required spectral measure.

On the other hand, let there exist a spectral measure G in X of the class $(\mathscr{T}, X')$ satisfying relation (1). Then $G(\pi) = G_0(\pi)$, $\pi \in \mathscr{Q}$. If x is an arbitrary element of X then $G(\ )\, x$ is a σ-additive X-valued vector measure. Following [2], the set $N_1(x) = \{G(\tau)\, x : \tau \in \mathscr{T}\}$ is relatively weakly compact in X; hence $N(x)$ (as a subset of $N_1(x)$) is relatively weakly compact in X, too.

9. Theorem. *Let X be a sequentially weakly complete Banach space. Let E and F be two commuting spectral measures in X of the classes $(\mathscr{R}, X')$ and $(\mathscr{S}, X')$, respectively, satisfying condition* (B).

Then in X there exists exactly one spectral measure G of the class $(\mathscr{T}, X')$ with property (1).

Proof. The assumptions of Corollary 7 are satisfied so that there exists exactly one spectral measure H in X'' of the class $(\mathscr{T}, X')$ such that $H(\pi) = G(\pi)''$, $\pi \in \mathscr{Q}$.

We denote $\mathscr{M}$ the system of all sets $\tau \in \mathscr{T}$ for which there exists $G(\tau) \in B(X)$ such that $G(\tau)'' = H(\tau)$. Obviously $\mathscr{Q} \subset \mathscr{M}$. Let $x \in X$ be arbitrary. Let $\{\tau_n\}$ be a monotone sequence of sets from $\mathscr{M}$ and let $\tau = \lim_n \tau_n$. Then the sequence $\{G(\tau_n)\, x\}$ is weakly fundamental for the same reasons as in the proof of Theorem 8. Due to the weak completeness of the space X, $\{G(\tau_n)\, x\}$ is weakly convergent so that there exists $G(\tau) \in B(X)$ such that $\langle G(\tau)\, x, x' \rangle = \lim_n \langle G(\tau_n)\, x, x' \rangle$, for every $x' \in X'$, and $G(\tau)'' = H(\tau)$. Hence, $\tau \in \mathscr{M}$.

We may apply now the trick used several times already and we obtain that on $\mathscr{T}$ there exists exactly one $B(X)$-valued function G that is a spectral measure in X of the class $(\mathscr{T}, X')$ such that (1) is valid.

10. If X is a Hilbert space then two commuting spectral measures E and F with values in $B(X)$ satisfy condition (B) in all cases [12]. For Banach spaces in general, however, condition (B) need not be satisfied as S. KAKUTANI proved in [7]. Moreover, even the reflexivity of the space X does not guarantee the validity of the condition (cf. [10]). (It is, however, satisfied if X is an L^p-space; cf. [11].) The supposition that this condition holds, is therefore certainly not redundant.

11. In the sense of the terminology introduced in [3], an operator $A \in B(X)$ is said to be a spectral operator of the scalar type of the class $(\mathscr{S}, \Gamma)$, if there exists a spectral measure E of the class $(\mathscr{S}, \Gamma)$ in X such that

$$A = \int_p \lambda \, \mathrm{d}E(\lambda) . \tag{10}$$

We assume that $\mathscr{S}$ is an algebra of Borel subsets of the complex plane p. It is known [3] that the spectral measure E is determined uniquely by the operator A and is called the resolution of identity for A.

Theorems on the product of spectral measures allow us to obtain in a simple way results concerning computation with such operators. In the sequel we denote $\mathscr{S}$ the σ-algebra of all Borel sets in the plane p.

Let $A, B \in B(X')$ be commuting operators of scalar type of the class $(\mathscr{S}, X)$. Let the resolutions of identity E, F for A, B satisfy condition (B). Then every operator from the algebra generated by operators A, B is a scalar-type operator of the class (S, X).

According to [3] E and F commute. We construct (by Theorem 6) spectral measure G in X' of the class $(\mathscr{T}, X)$, where $\mathscr{T}$ is the system of Borel sets in $p \times p$ such that $G(\sigma_1 \times \sigma_2) = E(\sigma_1) F(\sigma_2)$, $\sigma_1, \sigma_2 \in \mathscr{S}$. Since $A = \int_{p \times p} \lambda \, \mathrm{d}G(\lambda, \mu)$ and $B = \int_{p \times p} \mu \, \mathrm{d}G(\lambda, \mu)$ it follows that

$$f(A, B) = \int_{p \times p} f(\lambda, \mu) \, \mathrm{d}G(\lambda, \mu) \tag{11}$$

for arbitrary polynomial f of two variables. By the equation (11) we define $f(A, B)$ even for arbitrary bounded Borel measurable function f of two variables.

The resolution of identity H_f for the operator $f(A, B)$ is given by

$$H_f(\sigma) = G(\{(\lambda, \mu) : f(\lambda, \mu) \in \sigma\}) , \quad \sigma \in \mathscr{S} , \tag{12}$$

(see [3; Lemma 6]).

Under the supposition that X is a weakly complete Banach space, the assertions referred to formerly are valid also for scalar-type spectral operators $A, B \in B(X)$ of the class $(\mathscr{S}, X')$. Hence, if A and B commute and their resolutions of identity E and F satisfy condition (B) then all operators from the algebra generated by A, B

are scalar-type spectral operators of the class $(\mathcal{S}, X')$. This follows from Theorem 9. Operators $f(A, B)$ and their resolutions of identity are given by (11) and (12). From this statement we can obtain the result of S. R. Foguel (proved in [5] by other methods) which guarantees that $A + B$ and AB are scalar-type spectral operators if A and B are commuting scalar-type spectral operators, $A, B \in B(X)$, X a weakly complete Banach space and resolutions of identity for A and B satisfy condition (B).

As we can see from the above mentioned process this result remains valid also for unbounded scalar-type spectral operators introduced in [1] (they are operators of the form (11) such that E has not a compact support).

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Резюме

ПРОИЗВЕДЕНИЕ СПЕКТРАЛЬНЫХ МЕР

ИГОРЬ КЛУВАНЕК и МАРТА КОВАРЖИКОВА (Igor Kluvánek, Marta Kovářiková), Кошице

Пусть R и S — σ-компактные локально компактные хаусдорфовы пространства и $T = R \times S$. Обозначим через $\mathcal{R}, \mathcal{S}, \mathcal{T}$ соответственно σ-алгебры борелевых подмножеств пространства R, S, T. X — пространство Банаха.

Теорема. *Пусть E и F — две коммутирующие спектральные меры класса* $(\mathscr{R}, X)$ *и* $(\mathscr{S}, X)$ *в* X'. *Пусть булева алгебра, порожденная проекторами* $E(\varrho)$, $F(\sigma)$, $\varrho \in \mathscr{R}$, $\sigma \in \mathscr{S}$, *равномерно ограничена.*

Тогда в X' *существует одна и только одна спектральная мера G класса* $(\mathscr{T}, X)$, *выполняющая равенство*

$$G(\varrho \times \sigma) = E(\varrho)\, F(\sigma)\,, \quad \varrho \in \mathscr{R}\,, \quad \sigma \in \mathscr{S}\,. \tag{1}$$

Теорема. *Пусть E и F — две коммутирующие спектральные меры класса* $(\mathscr{R}, X')$ *и* $(\mathscr{S}, X')$ *в X.*

Тогда в X существует спектральная мера G класса $(\mathscr{T}, X')$, *выполняющая* (1) *тогда и только тогда, когда для всяких* $x \in X$ *множество всех сумм* $\sum_{i=1}^{n} E(\varrho_i) \cdot F(\sigma_i)\, x$ *для любых* $\varrho_i \in \mathscr{R}$, $\sigma_i \in \mathscr{S}$, *где* $\varrho_i \times \sigma_i$ *попарно не пересекаются и n — любое натуральное число, относительно слабо компактно в X.*