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THEORY OF PROCESSES, I

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INTRODUCTION

For a variety of reasons it appears desirable to generalize the concept of an ordinary differential equation as far as possible. To indicate the motivation briefly, one might mention – in a purely subjective order of preference – discontinuous forcing terms and generalized solutions of various types, discontinuous feed-back, equations in contingents in connection with differential inequalities and control problems, difference-differential and functional-differential equations, and differential equations in function spaces as convenient interpretations of some partial differential equations.

Separately, each of these theories is, of course, perfectly adequate to its own main problem, and all use a similar terminology and arsenal of primitive notions. Considerations of generality (i.e., of economy and elegance) then emphasise the need of a wider theory, including all the preceding as special cases. Usually, this type of problem is solved by an approach primarily abstract, or rather axiomatic.

In selecting suitable properties to serve as axioms one returns, of course, to the basic model, viz. ordinary differential equations $\mathrm{d}x/\mathrm{d}\theta = f(x, \theta)$ in euclidean n-space. In the study of these objects, one is usually concerned with situations in which most of the following conditions are satisfied (pertaining to the initial value problem):

1° local existence of solutions,

2° indefinite prolongability of solutions (i.e. global existence),

3° unicity of solutions,

4° autonomness (i.e., $f(x, \theta)$ independent of θ).

There are, of course, many other quite reasonable requirements, both general and of special character; e.g., continuous dependence of solutions on initial data, $f(x, \theta)$ linear or periodic in x, $f(x, \theta)$ periodic in θ, boundedness of solutions, etc.

In several cases, the axiomatic approach to differential equation theory has already been exploited with considerable effect. Thus, corresponding to equations which satisfy 1° to 4°, A. A. MARKOV defined the so-called dynamical systems (in metric spaces; see [15, chap. V–VI] for a detailed exposition). Some results on dynamical systems without unicity were obtained in [2] and [14]; and concerning objects exhibiting 1° to 3°, the flows, in [8]. The Liapunov theory was extended by ZUBOV to objects with 1° and 2° only; see [19, chap. IV] and also [17] (terms: general systems and generalized flows, respectively). In overcoming 2°, the local dynamical systems, i.e. objects exhibiting 1°, 3° and 4°, were introduced and studied in some detail, [7], [9]. Finally, the present author obtained some unpublished results concerning the local flows, with properties 1° and 3°, and also concerning objects with property 3° only.

Paradoxically, the logical next step is to eliminate all of the reasonable properties 1° to 4°; let us attempt to motivate and then describe informally what remains.

In undefined (but, it is hoped, suggestive) terms, assume given a physical law; in some manner this should determine all of the possible behaviour of each member from a given set of individual physical systems. This behaviour is to be characterized by specifying the phases or states of the physical systems at various times. A familiar instance is Newton's law of motion for, say, one-dimensional movement of a particle in a given field of force; the individual physical systems are mass-invariant particles moving subject to the law; and the phase space is R^2, with coordinates interpreted as abscissa and velocity of a given particle at given time. (This is an example of a deterministic physical law; however, indeterministic laws will also be considered.)

For our purposes, the physical systems themselves are immaterial: we shall only be concerned with their behaviour, i.e. with their phase-time characterisations. There will be no restrictions on the nature of the physical law nor on that of the phase space; however the time-variable will be real-valued (occasionally even this can be avoided; see [9, chap. IX], or also topological transformation groups).

In this situation, the fact that it is possible for some system (among the given physical systems) to be in state x at time α and in state y at a previous time $\beta \leqq \alpha$ will be described by means of a relation between the pairs (x, α) and (y, β). The following two properties then appear both reasonable and fundamental.

(i) At a given time, each system can be in at most one state. The formal version of this condition will be termed the initial-value property.

(ii) This consists of two reciprocal requirements as follows. If it is possible for some system to change from state z at time γ to state y at $\beta \geqq \gamma$, and if it is possible for some (other) system to change from this state y at β to state x at $\alpha \geqq \beta$, then it is required that it also be possible for some further system to change from the original state z at γ to the final state x at α. Conversely, if a system changed from state z at γ to x at α, then, at each intermediate time β with $\alpha \geqq \beta \geqq \gamma$, it must have been in some state y; however little more than mere existence may be known about y (in particular, it need not be uniquely determined by (z, γ), β nor even by (x, α), (z, γ), $\beta)$. After formalization, this requirement will be named the compositivity property.

It is precisely the relation described above, between the phase-time pairs (x, α) and (y, β), which forms the abstract counterpart of the physical laws, and is the subject of axiomatisation. Thus if properties 1° to 3° are assumed, then one may introduce maps ${}_\alpha T_\beta$, the so-called movements, such that (x, α) is related to (y, β) iff $x = {}_\alpha T_\beta y$; or equivalently, a single map T, with $x = T(\alpha, y, \beta)$ representing the same relation. Conditions (i) and (ii), and also 4° if required, are reflected in appropriate properties of T. A similar treatment is possible even if 2° is omitted: one replaces the map T by a partial map. The method usually adopted in circumventing 3° is to use a multivalued map for T (loc. cit.). Thus the obvious suggestion for the most general case is to apply, in a similar manner, a multivalued partial map for T, or in other words, a relation between x and the triple (α, y, β). However, it appears preferable to consider the more symmetric "transferred" relation between pairs (x, α) and (y, β); this description is adopted in the present paper (the term process chosen for this concept is borrowed from mathematical statistics).

An entirely different approach is also possible. In place of the relations just described one may choose as primary objects of study the phase-time characteristics of the individual physical systems, the so-called solutions. For less imprecise formulations, connections with processes and further comments, see section 2.

Some further explanatory remarks are in place here. In all the reasonable requirements listed previously, the topology of R^n enters only into boundedness and continuous variation of solutions. Similarly, in local dynamical system theory it was useful to separate out the purely "dynamical" properties from the topological or mixed ones, [9]. In the present exposition it is also preferred to treat complex problems piecemeal; even though the topological considerations are of utmost importance, they are first separated from the abstract theory and deferred to a later paper (perhaps the analogy between this situation and that of abstract and topological groups is not too far-fetched). Secondly, in differential equation theory, theorems are often formulated in futuro, for positive time only. In analogy there were introduced the auxiliary concepts of unilateral (or semi-) dynamical systems and semi-flows; again, this will be paralleled here.

Finally, some technical remarks. Sections are divided into items, and some of these into sub-items; these are referred to by numbers, with item or section number given first in cross-references (thus item 3 of section 2 is referred to as 2.3 outside section 2, and as 3 within section 2; the references to its sub-item 4 are then 2.3.4° and 3.4° respectively). Within sections some displayed formulas are numbered separately, and referred to in similar fashion. Some conventions and notation, mainly concerning relations, are given in the Appendix, but used consistently without explicit reference; thus the reader may find it simpler to read the Appendix cursorily first. The extremely useful trick of treating the exercises also as a repository of minor results and examples is adopted here; the author sees no advantage in reserving this facility to monographs.

This first paper is rather singular in that the principal results are not the propositions it contains (however, theorem 4.12 does seem important) but rather the statement that the generalizations exhibited are interesting and useful. Most of the proofs are so trivial that it seemed unnecessary to present them otherwise than as, at most, suggestions.

1. PROCESSES

1. Assume given a set P, and also a subset $R \subset \mathsf{R}^1$. The set R is ordered by the natural order relation $\geqq$ inherited from R^1, and in particular one may speak of intervals in R (i.e. the non-void order-convex subsets of R^1; this includes the singletons i.e. the degenerate intervals).

Now consider any relation p on $P \times R$, i.e. between $P \times R$ and $P \times R$, with the property that

$$(x, \alpha)\, p\, (y, \beta) \quad \text{implies} \quad \alpha \geqq \beta \,. \tag{1}$$

Then p defines a system $\{{}_\alpha p_\beta \mid \alpha \geqq \beta \text{ in } R\}$ of relations ${}_\alpha p_\beta$ on P, in the following manner:

$$x \; {}_\alpha p_\beta \; y \quad \text{iff} \quad (x, \alpha)\, p\, (y, \beta)\,; \tag{2}$$

these ${}_\alpha p_\beta$ will be called the *individual relations* of p. It is obvious that, conversely, any system $\{{}_\alpha p_\beta \mid \alpha \geqq \beta \text{ in } R\}$ of relations on P defines, by (2), a relation p of the type described above, and that the two procedures are mutually inverse.

This notation and conventions will be used freely; thus e.g. a relation q may be defined merely by prescribing the individual relations ${}_\alpha q_\beta$, etc.

2. Definition. p is a *process on P over R* iff P is a set, $R \subset \mathsf{R}^1$, p is a relation on $P \times R$ with (1), and the following two conditions are satisfied:

1° ${}_\alpha p_a \subset 1$ for all $\alpha \in R$,

2° ${}_\alpha p_\beta \circ {}_\beta p_\gamma = {}_\alpha p_\gamma$ for all $\alpha \geqq \beta \geqq \gamma$ in R.

In this context, 1° may be termed the initial-value property, and 2° the compositivity property. A number of examples and intended applications is given in the exercises to this and the following section; however it is appropriate to exhibit the basic interpretations at this point.

3. (Example) Consider a differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = f(x, \theta) \tag{3}$$

in euclidean n-space, with continuous $f : D \to \mathsf{R}^n$ and D an open set in R^{n+1}; of course, this equation is completely determined by f. The *classical solutions* of (3) are partial maps $s : \mathsf{R}^1 \to R^n$ such that domain s is an interval in R^1, either degenerate or with

$$\frac{\mathrm{d}}{\mathrm{d}\theta} s\theta = f(s\theta, \theta) \quad \text{for} \quad \text{all} \quad \theta \in \text{domain } s$$

(here and in similar situations henceforth, the derivative is taken with respect to domain s; thus it may well be only a left derivative, etc.). With the differential equation one associates the process p, in R^n over R^1, defined by setting $x \,{}_\alpha p_\beta\, y$ (for x, y in R^n, $\alpha \geqq \beta$ in R^1) iff there is a classical solution of (3) assuming the values x and y at α and β respectively. Processes obtained in this manner, with the exhibited assumptions on f, will be termed *differential*.

4. (Example) In distant but possibly familiar analogy, let

$$f_k(x_{k+1}, x_k) = 0 \tag{4}$$

be a finite-difference equation, with partial maps $f_k : \mathsf{R}^{2n} \to \mathsf{R}^n$. The solutions of (4) are, of course, sequences $\{x_k \mid j \leqq k \leqq i \text{ in } \mathsf{C}^1\}$ such that (4) holds for $j \leqq k < i$. The associated process p, in R^n over C^1, is defined by setting $x \,{}_i p_j\, y$ iff $i \geqq j$ in C^1 and there is a solution as above with $x_i = x$, $x_j = y$.

5. In both these cases, conditions 1 (1) and 2.1° are satisfied trivially, and 2.2° follows from an appropriate concatenation property of solutions. These examples may also indicate the role of the sets P and R : P is a convenient set containing the carrier of the process, and may be enlarged without affecting the process essentially; and R serves only to specify the scope of the independent time variable, principally in the betweenness relation entering into the compositivity property 2.2°. It will be necessary to define these and a number of further notions related to processes formally.

6. Given a process p in P over R, occasionally P may be termed the *phase-space* of p; elements of P (of R) may be called phases (time-instants, respectively).

The *domain D*, *carrier C* and *parameter-domain B* of p are defined thus:

$$D = \text{domain } p\,, \quad C = \text{proj}_1 [D]\,, \quad B = \text{proj}_2 [D]\,.$$

(The symbols D, C, B, possibly with indices or primes, will usually be reserved for this purpose.) In vague terms, one may say that while R specifies the set in which the time-variable is allowed to vary, B is the set of time-instants actually needed in describing p; similarly for P and C.

It is immediate that

$$D \subset C \times B \subset P \times R\,;$$

iff $D = C \times B$ then p and D will be termed *cartesian*; iff $C = P$ then p and C will be termed *full*. Iff $\bar{R}$ is isolated in R^1, then p is said to have *discrete time*; and to have *continuous time* iff $R = \mathsf{R}^1$.

7. Even at this stage a reasonable amount of information concerning these concepts is available from the axioms 2.1°–2°. This is presented in the current and following items; throughout there is assumed given a process p in P over R.

By particular choice of time-instants in the compositivity property ($\gamma = \beta$, etc.) one obtains that $x\ {}_\alpha p_\beta\ y$ implies $x\ {}_\alpha p_\alpha\ x$ and $y\ {}_\beta p_\beta\ y$. There follows a useful description of the domain,

$$(5) \qquad D = \{(x, \alpha) \in P \times R : x\ {}_\alpha p_\alpha\ x\},$$

and also

$$\text{range}\ p = D = \text{domain}\ p\ ;$$

in particular, p is completely determined by its partialisation $p \mid D$. From (5) it also follows that

$$(6) \quad C = \{x \in P : x\ {}_\alpha p_\alpha\ x \text{ for some } \alpha \in R\}, \quad B = \{\alpha \in B : x\ {}_\alpha p_\alpha\ x \text{ for some } x \in P\}.$$

In particular, p is cartesian iff, for each $\alpha \in B$, ${}_\alpha p_\alpha$ is the identity map of C.

According to (5) (and from 2.1° and 2°), each process p is a (special type of) partial order on the corresponding domain; it is this partial order which is meant when one says that a function e.g. increases along p. More precisely, a partial map $f : P \times \times R \to A$ into a partially ordered set A is termed increasing along a process p in P over R iff $f \mid D$ is an increasing map between D (endowed with p as partial order) and A. Similarly for non-increasing maps, etc. As a familiar example, each Liapunov function of a differential equation is a non-increasing function into R^1 along the associated differential process.

8. Next, consider the relation r on B with $\alpha\ r\ \beta$ iff $x\ {}_\alpha p_\beta\ y$ for some $x, y \in P$ (on transferring p from $(P \times R) \times (P \times R)$ to $P \times R \times P \times R$, this reduces to $r = = \text{proj}_{2,4}[p]$). From compositivity, $\alpha\ r\ \beta$ implies $\alpha\ r\ \theta$ and $\theta\ r\ \beta$ for all $\theta \in R$ with $\alpha \geqq \theta \geqq \beta$; and then the relation $\sim$ obtained from r by transitivization and then symmetrization (i.e. $r_1 = r \cup r^2 \cup \ldots$ and $\sim = r_1 \cup r_1^{-1}$) also has this property. Now, $\sim$ is an equivalence relation on B (e.g. r is reflexive by (6)), and we have just shown that the equivalence classes modulo $\sim$ are intervals in R; these will be called the *interval-components* of p or B. Iff there is at most one interval-component of p, then p is said to be *extensive*.

In particular, then, the parameter-domain of any process p in P over R decomposes into disjoint intervals of R; these have the property that if α, β are in distinct interval components, then $x\ {}_\alpha p_\beta\ y$ for no x, y in P. It follows that the process p may well be studied separately over each interval-component; this will be formulated more precisely in the following paper, using the concept of a direct sum.

Finally, consider a relation r' between a partially ordered set R' and R^1; if $r' \circ \mathrm{proj}_2$ non-decreases along a process p (i.e. $x\ {}_\alpha p_\beta\ y$ implies $r'\alpha \geqq r'\beta$) then $r' \mid B$ is a partial map; using the r and $\sim$ above it is easily shown that then $r' \mid I$ is non-decreasing for every interval-component I of p.

9. Given a process p in P over R, a relation s between P and R will be called a *solution* of p iff

$$s \times s \subset p \cup p^{-1} \tag{7}$$

and domain s is an interval in R. (We re-emphasise that degenerate intervals are allowed, and that domain s is to be an interval in R, not necessarily in R^1.) In this situation, for any $\alpha \in$ domain s, one may say that s is a solution through $(s\alpha, \alpha)$, or through $s\alpha$ at α.

Evidently, (7) may be formulated thus: $x\ s\ \alpha$ and $y\ s\ \beta$ with $\alpha \geqq \beta$ imply $x\ {}_\alpha p_\beta\ y$; in particular, from the initial-value property it follows that s is a partial map $R \to P$ with

$$(s\alpha)\ {}_\alpha p_\beta\ (s\beta) \quad \text{for all} \quad \alpha \geqq \beta \quad \text{in} \quad \text{domain}\ s\,. \tag{8}$$

Conversely, each partial map $s : R \to P$ with (8) and domain s an interval in R is a solution of p.

10. From the definitions it follows directly that $s \subset D$ for all solutions s of p; conversely, each element of D is contained in some solution (e.g. with degenerate domain; apply (8) and 2.1°). Thus D coincides with the set-union of all solutions of p. Furthermore, from the construction in 7 it follows that the domain of any solution of p is contained within some interval-component of p.

The *solution system* S of a process p in P over R is defined as consisting of all solutions of p. Obviously S has the following properties:

1° Each $s \in S$ is a partial map $R \to P$ with domain s an interval in R,

2° For any $s \in S$ and interval I in R intersecting domain s, also $s \mid I \in S$,

3° If $s_1, s_2 \in S$, the domains of s_i intersect and $s_1 \cup s_2$ is a partial map, then $s_1 \cup s_2 \in S$.

Here 2° may be termed the partialization property, and 3° the concatenation property. Of the slightly less elementary properties of S one may note the following:

4° If $\{s_i\}$ is a monotone family in S, then $\bigcup s_i \in S$.

5° Let $s : I \to P$ be a map with I an interval in R and the property that to any α, β in I there is an $s' \in S$ with $s\alpha = s'\alpha$, $s\beta = s'\beta$; then $s \in S$. In particular, from 4° and the Zorn lemma it follows that each solution of p is contained in at least one *maximal solution*.

11. Given a process p in P over R and an $(x, \alpha) \in D$, it is natural to inquire about the set of θ's such that $x' \, {}_\theta p_\alpha \, x$ for some $x' \in P$. Some information is directly available: according to the compositivity property, such θ's constitute an interval in R containing α as left end-point, and entirely contained within the interval-component of α. This suggests the following definition.

Let p be a process in P over R; the *escape time* $\varepsilon(x, \alpha)$ of $(x, \alpha) \in D$ is defined as

$$(10) \qquad \varepsilon(x, \alpha) = \sup \{\theta \in \mathrm{R}^1 : x' \, {}_\theta p_\alpha \, x \ \text{for some} \ x' \in P\}$$

(with the supremum taken in the extended real line); some properties of the escape times are immediate:

1° If $\alpha \leqq \theta < \varepsilon(x, \alpha)$ and $\theta \in R$, then there exists an $x' \in P$ with $x' \, {}_\theta p_\alpha \, x$; as a partial converse, if $x' \, {}_\theta p_\alpha \, x$ then $\alpha \leqq \theta \leqq \varepsilon(x, \alpha)$. In particular, $\alpha \leqq \varepsilon(x, \alpha)$.

2° Let I be the interval-component containing α; then $\varepsilon(x, \alpha) \leqq \sup I \leqq \sup B \leqq$ $\leqq \sup R \leqq +\infty$.

3° ε non-increases along p, i.e. $x \, {}_\alpha p_\beta \, y$ implies $\varepsilon(x, \alpha) \leqq \varepsilon(y, \beta)$. (In 3° and also often elsewhere, $\varepsilon(x, \alpha)$ is interpreted as the value at (x, α) of a map ε from D into the extended real line.)

12. Some special situations may now be separated out. There is still assumed given a process p in P over R, and, possibly, an $(x, \alpha) \in D$.

Iff $\alpha < \varepsilon(x, \alpha)$, then p is said to have *local existence at* (x, α); and iff this obtains for all elements of D, then p is said to have *local existence*. The negation of this property will also be named separately: (x, α) is an *end-pair* iff $\alpha = \varepsilon(x, \alpha)$; and iff this obtains at all elements of D, then the process p will be termed *trivial*.

Iff $\varepsilon(x, \alpha) = \sup B$, then p is said to have *prolongability at* (x, α); iff this occurs at all elements of D then p is said to have *prolongability*; and iff $\varepsilon(x, \alpha) = \sup R$ for all $(x, \alpha) \in D$, then the process p will be termed *global*, or to possess *global existence*.

The terminology chosen has an unfortunate defect: the reader is requested to resist the temptation of asking, of what is local existence asserted.

13. Next, some simple consequences will be exhibited. Note, e.g., that prolongability may, but local existence cannot, occur at an end-pair; and if B contains at least two interval-components, then p cannot have prolongability. If an interval-component I has $\sup I \in I$, then each $(x, \sup I) \in D$ is an end-pair. A relation p on $P \times R$ is a trivial process iff $p \subset 1$. A process p is cartesian with prolongability iff

$$1_C \subset ({}_\alpha p_\beta)^{-1} \circ {}_\alpha p_\beta \quad \text{for all} \quad \alpha \geqq \beta \quad \text{in} \quad B\,,$$

where 1_C is the identity map of C (for an assertion or globality merely replace B by R). If a process has prolongability, then it is extensive (in the sense of 8).

14. Lemma. *If p is a process and* $x\ {}_{\alpha}p_{\beta}\ y$ *with* $\alpha = \varepsilon(y, \beta)$, *then* (x, α) *is an end-pair.*

This follows directly from 11.1° and 3°. In particular, for processes with local existence, one has the following version of 11.1°: For given (x, α) and θ there is $x'\ {}_{\theta}p_{\alpha}\ x$ for some $x' \in P$ iff $\alpha \leqq \theta < \varepsilon(x, \alpha)$.

15. The definition of semi-flows and semi-dynamical systems (cf. 4.18; and also that of continuity of a process, to appear in the third paper of this series) is intimately connected with two concepts closely related to the process; it will be convenient to introduce these at the present point.

Assume given a process p in P over R. To p one assigns, first, the relation t between P and $R \times P \times R$ defined by

$$x\, t\, (\alpha, y, \beta) \quad \text{iff} \quad (x, \alpha)\, p\, (y, \beta)\,;$$

t will be termed the *relation associated to p.* Evidently t is obtained from p by transferring from $(P \times R) \times (P \times R)$ to $P \times (R \times P \times R)$; in particular, p is uniquely determined by t and conversely (and several authors prefer t to p as a medium of description). Obviously range $t = C$, and

$$\text{domain}\ t = \{(\alpha, y, \beta) : x\ {}_{\alpha}p_{\beta}\ y \quad \text{for some} \quad x \in P\}\,.$$

Then 11.1° may be formulated thus:

$$\{(\alpha, y, \beta) : (y, \beta) \in D,\ \beta \leqq \alpha < \varepsilon(y, \beta)\} \subset \subset \text{domain}\ t \subset \{(\alpha, y, \beta) : (y, \beta) \in D,\ \beta \leqq \alpha \leqq \varepsilon(y, \beta)\}\,. \tag{11}$$

The second concept is a relation d, to be called the *projection* of p, and defined as the relation between R and $P \times R$ with

$$\alpha\, d\, (y, \beta) \quad \text{iff} \quad x\ {}_{\alpha}p_{\beta}\ y \quad \text{for some} \quad x \in P\,.$$

Evidently d can be obtained from p or t via the projection $\text{proj}_{234} : P \times R \times P \times \times R \to R \times P \times R$ and appropriate transfers. Obviously

$$\text{range}\ d = B\,, \quad \text{domain}\ d = D\,, \quad \alpha\, d\, (y, \beta) \quad \text{iff} \quad (\alpha, y, \beta) \in \text{domain}\ t\,,$$

i.e. iff there is an $x \in P$ with $x\ {}_{\alpha}p_{\beta}\ y$.

16. A process p in P over R is said to possess *unicity* iff the associated relation t is a partial map $R \times P \times R \to P$. An evident necessary and sufficient condition for this is that every individual relation ${}_{\alpha}p_{\beta}$ be a partial map $P \to P$, i.e. that

$${}_{\alpha}p_{\beta} \circ ({}_{\alpha}p_{\beta})^{-1} \subset 1 \quad \text{for all} \quad \alpha \geqq \beta \quad \text{in} \quad R\,.$$

In this case the individual relations ${}_{\alpha}p_{\beta}$ are also called *motions* (of p or t).

An independent description of such processes will be useful. Let P be a set, $R \subset \mathrm{R}^1$, and let there be given a partial map $t : R \times P \times R \to P$; for all $\alpha \geqq \beta$ in R define partial maps ${}_\alpha p_\beta : P \to P$ by

$$ {}_\alpha p_\beta\, y = t(\alpha, y, \beta) . $$

Then t is the associated relation of a process in P over R (necessarily with unicity, and with the ${}_\alpha p_\beta$ as individual relations) iff the following two conditions are satisfied:

1° ${}_\alpha p_\beta\, x = x$ for all $(x, \alpha) \in P \times R$ such that ${}_\alpha p_\alpha\, x$ is defined,

2° ${}_\alpha p_\beta \circ {}_\beta p_\gamma\, x = {}_\alpha p_\gamma\, x$ for all $\alpha \geqq \beta \geqq \gamma$ in R whenever either side is defined.

In particular, this implies that ${}_\theta p_\alpha\, x$ with fixed $(x, \alpha) \in P \times R$ is defined for all $\theta \in R$ in an interval (possibly degenerate) with α as left-end point.

17. Next we shall exhibit a localisation of the unicity property, in a manner similar to that employed for local existence. Thus, let p be a process in P over R, and consider an $(x, \alpha) \in D$; now define the *extent of unicity*

$$ (12) \qquad \delta(x, \alpha) = \sup \{\lambda \in \mathrm{R}^1 : \alpha \leqq \theta \leqq \lambda,\ u\ {}_\theta p_\alpha\, x,\ v\ {}_\theta p_\alpha\, x \text{ imply } u = v\} $$

(the supremum is taken in the extended real line; observe that one may have $\delta(x, \alpha) = +\infty$ even if R is right-bounded). Some properties of this characteristic are immediate or easily established:

1° $\alpha \leqq \delta(x, \alpha) \leqq +\infty$

2° If $\alpha \leqq \beta \leqq \gamma < \delta(x, \alpha)$, then $y\ {}_\beta p_\alpha\, x$, $z\ {}_\gamma p_\alpha\, x$ imply $z\ {}_\gamma p_\beta\, y$

3° If $x\ {}_\alpha p_\beta\, y$ with $\beta \leqq \alpha < \delta(y, \beta)$ then $\delta(x, \alpha) = \delta(y, \beta)$ and $\varepsilon(x, \alpha) = \varepsilon(y, \beta)$; hence

4° δ non-decreases along p.

5° If $\delta(x, \alpha) < +\infty$, then there exist $\theta \in R$ with arbitrarily small $\theta - \delta(x, \alpha) \geqq 0$ and also u, v in P with $u \neq v$, $u\ {}_\theta p_\alpha\, x$, $v\ {}_\theta p_\alpha\, x$; in particular

$$ \alpha \leqq \delta(x, \alpha) \leqq \varepsilon(x, \alpha) \leqq \sup I $$

for the interval-component I containing α.

6° If $\delta(x, \alpha) = \varepsilon(x, \alpha) < +\infty$ then $\theta = \varepsilon(x, \alpha)$ has $\theta > \alpha$ and there exist $u \neq v$ in P with $u\ {}_\theta p_\alpha\, x$, $v\ {}_\theta p_\alpha\, x$. If $\alpha = \varepsilon(x, \alpha)$ then $\delta(x, \alpha) = +\infty$.

18. Some special situations may now be separated out; there is assumed given a process p in P over R, and, possibly, and $(x, \alpha) \in D$.

Iff $\alpha < \delta(x, \alpha)$, then p will be said to have *local unicity at* (x, α); and iff this for all elements of D, then p is said to have *local unicity*. Iff $\delta(x, \alpha) = +\infty$, then p is said to have (global) *unicity* at (x, α); evidently p has *unicity* in the sense of 16 iff it has unicity at all elements of D.

To each $(x, \alpha) \in D$ one may assign a uniquely determined solution $t_\alpha x$ of p as follows: set $(t_\alpha x)\,\lambda = x'$ iff $x'\ {}_\lambda p_\alpha\ x$ and also $\alpha \leqq \theta \leqq \lambda$, $u\ {}_\lambda p_\alpha\ x$, $v\ {}_\lambda p_\alpha\ x$ imply $u = v$. Then $t_\alpha x$ satisfies condition 1.9 (8) (cf. 17.2°), and its domain is an interval in R of the form $[\alpha, \alpha')$ or $[\alpha, \alpha']$ with

$$\alpha' = \min\left(\varepsilon(x, \alpha),\ \delta(x, \alpha)\right) = \begin{cases} \varepsilon(x, \alpha) & \text{if} \quad \delta(x, \alpha) = +\infty \\ \delta(x, \alpha) & \text{if} \quad \delta(x, \alpha) < +\infty \end{cases}$$

(cf. 17.5°). Hence $t_\alpha x$ is a solution of p (indeed, the only solution with the indicated domain), to be termed the *characteristic solution* of p *through* (x, α).

19. The next step is a result concerning a situation analogous to that of 14. Thus, let p be a process in P over R without global unicity at a given $(y, \beta) \in D$; for convenience, set $\alpha = \delta(y, \beta) < +\infty$ and $s = t_\beta y$. Then there are possible three mutually exclusive cases:

1° There exists an $x \in P$ with $x\ {}_\alpha p_\beta\ y$, whereupon either $\delta(x, \alpha) = \alpha$ (i.e. local unicity does not obtain at (x, α)), or

2° There exist $u_1 \neq u_2$ in P with

$$(u_j, \alpha)\ p\ (s\theta, \theta)$$

for $j = 1, 2$ and all θ in domain s.

3° One has $x\ {}_\alpha p_\beta\ y$ for no $x \in P$. Then $\delta(y, \beta) > \beta$, and there exist $\theta_i \searrow \alpha$ and $u_{i1} \neq u_{i2}$ with

$$(u_{ij}, \theta_i)\ p\ (s\theta, \theta)$$

for $j = 1, 2$, all i, and all θ in domain s. In particular, $\alpha \in \bar{B} - R$ (thus if R is closed in R^1, then 3° cannot occur).

Note that in case 1°, domain $t_\beta y$ is right-closed, and right-open in 2° and 3°. Fig. 1 may aid in visualising the situation. Also observe that if, for some process p, 2° and 3° are excluded in some manner, then local unicity of p implies global unicity of p; thus one has the situation familiar from differential equation theory.

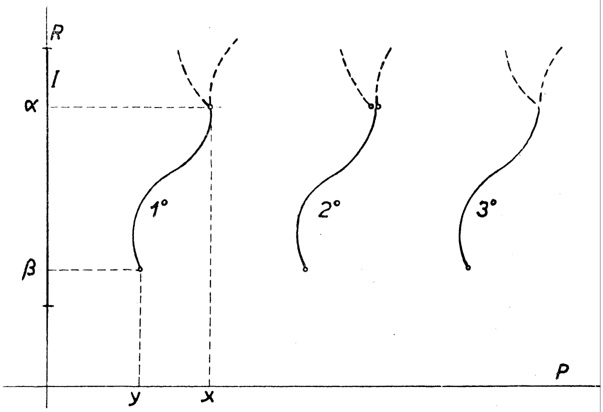


Fig. 1.

20. Processes are a particular type of relation, and thus the natural inclusion-relation between processes is well-defined. We shall now list the implications of a relation

$$p \subset p' \tag{13}$$

between a process p in P over R and a process p' in P' over R' (objects associated with p' will be distinguished by primes). Note first that (13) yields no a priori inclusion relations between P, R and P', R'. As for the concepts introduced in 1, 6 and 15, one has

$$_\alpha p_\beta \subset {_\alpha p'_\beta} \quad \text{for all} \quad \alpha \geqq \beta \quad \text{in} \quad R\,, \quad D \subset D'\,, \quad C \subset C'\,, \quad B \subset B'\,,$$
$$t \subset t'\,, \quad d \subset d'\,.$$

Each interval-component of p is contained within some interval-component of p'. If $R = R'$ (or $B = B'$) then $S \subset S'$ for the corresponding solution systems, i.e. each solution of p is a solution of p'; however in the case that $R \neq R'$ and $B \neq B'$ there seems to be, in general, no relation between S and S' (also see example 2.3).

For the escape times one has

$$\varepsilon(x, \alpha) \leqq \varepsilon'(x, \alpha) \quad \text{for all} \quad (x, \alpha) \in D\,;$$

thus if p has local existence at $(x, \alpha) \in D$ then so does p' (and conversely, each end-pair of p' in D is an end-pair of p). If $B = B'$ and p has prolongability at $(x, \alpha) \in D$, then p' also does; if $R = R'$, $D = D'$ and p is global, then so is p'.

For the extents of unicity one has

$$\delta(x, \alpha) \geqq \delta'(x, \alpha) \quad \text{for all} \quad (x, \alpha) \in D\,.$$

In this case, then, p has local or global unicity at $(x, \alpha) \in D$ if p' does; also note that this yields

$$\delta'(x, \alpha) \leqq \delta(x, \alpha) \leqq \varepsilon(x, \alpha) \leqq \varepsilon'(x, \alpha)$$

iff $\delta(x, \alpha) < +\infty$ (cf. 17.5°).

21. (Exercises) 1° Given a process p, one may define for $(x, \alpha) \in D$,
$\delta_1(x, \alpha) = \sup\,\{\lambda \in \mathrm{R}^1 : \lambda \geqq \theta \geqq \theta' \geqq \alpha,\ y\,{_\theta p_\alpha}\,x,\ y\,{_\theta p_{\theta'}}\,y' \text{ imply } y'\,{_{\theta'} p_\alpha}\,x\}\,.$
Interpret and study the characteristic δ_1, in particular in connection with the characteristic ε and δ, and also with negative unicity (cf. 3.14).

2° Treat similarly
$\delta_2(x, \alpha) = \sup\,\{\lambda \in \mathrm{R}^1 : \lambda \geqq \theta \geqq \alpha,\ y\,{_\lambda p_\theta}\,u_i,\ u_i\,{_\theta p_\alpha}\,x \text{ for } i = 1, 2 \text{ imply } u_1 = u_2\}\,.$

3° In the situation of example 3 show that the domain of p is domain $f = D$; and that the interval-components of p are the components of $B = \mathrm{proj}_2\, D$ as a subspace of R^1 (hint: p has local existence). Formulate unicity and globality of p in terms of the differential equation (3).

4° In example 3 verify that the classical solutions of (3) are solutions of p in the sense of 9. Hence show that p determines (3) completely in the sense that f can be reconstructed from p. (Hint: f may be defined by (3) using solutions of p.)

5° Continuing in 4°, prove that also conversely each solution of p is a classical solution of (3). (Suggestion: It is required to show that $(d/d\theta)\, s\theta = f(s\theta, \theta)$ for each non-degenerate solution s of p; by definition, whenever $\theta_n \to \theta$ in domain s, there exist classical solutions s_n with $s\theta_n = s_n\theta_n$, $s\theta = s_n\theta$. Then, if $\theta_n \neq \theta$,

$$\frac{s\theta_n - s\theta}{\theta_n - \theta} = \frac{s_n\theta_n - s_n\theta}{\theta_n - \theta} = f(s_n\theta_n', \theta_n')$$

by a mean-value theorem, for some $\theta_n' \to \theta$. Since f was assumed continuous, one need only prove $s_n\theta_n' \to s\theta$, e.g. using local boundedness of f.)

The following examples are intended to indicate applications of the concept of a process to situations rather different than those of example 3 (also see the exercises to section 2). In 6° – 8° we shall need the notation x_λ for the λ-translate of a map x, i.e.

$$x_\lambda\theta = x(\theta + \lambda) \quad \text{for} \quad \theta + \lambda \in \text{domain } x\,.$$

6° A *difference-differential equation* (with retarded time, [6, chap. V]) in R^1,

$$\frac{dx}{d\theta} = f(x\theta, x(\theta - \tau), \theta)\,, \tag{14}$$

is specified by a continuous $f : R^3 \to R^1$ and $\tau > 0$. The solutions of (14) are continuous maps $s : [\beta - \tau, \alpha] \to R^1$ with $-\infty \leqq \beta \leqq \alpha \leqq +\infty$ and such that

$$\frac{d}{d\theta} s\theta = f(s\theta, s(\theta - \tau), \theta) \quad \text{for} \quad \beta < \theta \leqq \alpha\,;$$

finally, the initial value problem for (14) is to find, to given $\beta \in R^1$ and continuous $y : [-\tau, 0] \to R^1$, a solution s of (14) as above and with $y \subset s_\beta$, i.e. such that

$$s\theta = y(\theta - \beta) \quad \text{for} \quad \beta - \tau \leqq \theta \leqq \beta\,.$$

This initial value problem may be conveniently and adequately described by means of a process p in $C^1[-\tau, 0]$ over R^1 as follows: For x, y in $C^1[-\tau, 0]$ and $\alpha \geqq \beta$ in R^1 let $x\; {}_\alpha p_\beta\; y$ iff $x \subset s_\alpha$, $y \subset s_\beta$ for some solution s of (14). Prove that the relations ${}_\alpha p_\beta$ on $C^1[-\tau, 0]$ thus defined are the individual relations of a process p. (Hint: the non-trivial part of the verification consists in showing that, if s is a solution as above and $s' : [\gamma - \tau, \beta] \to R^1$ a second solution with $s\theta = s'\theta$ for $\beta - \tau \leqq \theta \leqq \beta$, then $s \cup s'$ is again a solution of (14), and in particular has a derivative at β.)

7° Describe in what sense the preceding problem reduces to that of 1.3 on replacing $\tau > 0$ by $\tau = 0$. (Hint: prove first, that a solution s of (14) has a continuous right derivative at β; and second that a continuous function with a continuous right derivative has a continuous derivative.)

8° In the situation of 6° show that every solution x of (14) defines a solution s of p by $s\theta = x_\theta \mid [-\tau, 0]$; conversely, prove that every solution of p has this form. Assuming that local existence of solutions obtains for (14) [6, V, § 2], verify that p describes (14) completely (hints: parallel 5° and 4°). Describe the relations between unicity and prolongability for (14) and for p.

9° Generalize 6°–8° to *functional-differential equations* with retarded time [10] of the form

$$\frac{dx}{d\theta} = f(x_\theta \mid [-\tau, 0], \theta)$$

with continuous $f : C^1[-\tau, 0] \times \mathsf{R}^1 \to \mathsf{R}^1$ and $\tau > 0$; and also to partial maps f and n dimensions. Finally, obtain similar results with $[-\tau, 0]$ replaced by $(-\infty, 0]$.

10° Consider a *numerical process*, defined [1, II, 2.1] by specifying sequences of (normed) linear spaces X_k and of partial maps $f_k : X_1 \times \ldots \times X_k \to X_{k+1}$ ($k = 1, 2, \ldots$). The solutions of such a numerical process $\mathcal{N} = (\{X_k\}, \{f_k\})$ are sequences $\{s_k\}_{k=1}^{w}$ with $w \leqq +\infty$ and $s_{k+1} = f_k(s_1, \ldots, s_k)$ for all $k < w$.

To define an appropriate process, take the set P of all finite sequences $\{x_k\}_1^n$ with $n \in \mathsf{C}^1$ and $x_k \in X_k$ for $1 \leqq k \leqq n$; and then define relations ${}_ip_j$ on P as follows: let $x\ {}_ip_j\ y$ iff $i \geqq j \geqq 1$ in C^1, $x = \{x_k\}_1^i$, $y = \{y_k\}_1^j$, and also

$$\text{(15)} \qquad x_k = y_k \quad \text{for} \quad 1 \leqq k \leqq j\,, \quad x_{k+1} = f_k(x_1, \ldots, x_k) \quad \text{for} \quad j \leqq k < i\,.$$

Show that these ${}_ip_j$ are the individual relations of a process p in P over C^1; and that the solutions of p coincide with "solutions starting from a k-th step" of $\mathcal{N}$.

11° Modify the preceding construction for the case that the second relation in (15) is replaced by

$$| x_{k+1} - f_k(x_1, \ldots, x_k) | \leqq 1 \quad \text{for} \quad j \leqq k < i\,;$$

also relate with "approximative solutions" of $\mathcal{N}$ with given error bounds $\{\delta_k\}_1^w$. (Hint: introduce new norms in X_k, trivial iff $\delta_k = 0$.)

12° Consider *stochastic processes* defined as follows (cf. [4, XV, §§ 13–14]; the example is taken over from [9, IX, 1.9.4]). Let B be a Boolean σ-algebra, and L_B the set of σ-additive measure functions m on B; endowed with the obvious linear operations, partial order and norm $\|m\| = \sup_{x, y \in B} |mx - my|$, L_B becomes a Banach lattice. Also let P be the set of probability distributions on B, i.e. of $m \in L_B$ with $m \geqq 0$, $mI = 1$. Then a stochastic process of Markovian type on B over $R \subset \mathsf{R}^1$ is a system $\{{}_\alpha T_\beta \mid \alpha \geqq \beta \text{ in } R\}$ of linear maps ${}_\alpha T_\beta : L_B \to L_B$ with ${}_\alpha T_\beta[P] \subset P$ (i.e., the ${}_\alpha T_\beta$ are positive and norm-preserving), and such that

$${}_\alpha T_\alpha = 1\,, \quad {}_\alpha T_\beta \circ {}_\beta T_\gamma = {}_\alpha T_\gamma$$

for all $\alpha \geqq \beta \geqq \gamma$ in R. Verify that the ${}_\alpha T_\beta$ are the motions (cf. 16) of a process on P over R, which is full, cartesian, global and with unicity.

13° Let p be a process in P over R, measurable with respect to an atomic Boolean σ-algebra B of subsets of P (in the sense that ${}_\alpha p_\beta^{-1}[X] \in B$ for all $X \in B$, $\alpha \geqq \beta$ in R). With p, B one may associate a "nonlinear stochastic" process defined as follows (the summations below are to extend over atoms Y of B): For $\alpha \geqq \beta$ in R and probability distribution x on B one defines ${}_\alpha T_\beta\, x$ iff

$$0 < \sum_Y x\, {}_\alpha p_\beta^{-1}[Y] < +\infty\,, \tag{16}$$

whereupon the value of ${}_\alpha T_\beta\, x$ at any $X \in B$ is to be

$$({}_\alpha T_\beta\, x)\, X = \frac{\sum_{Y \subset X} x\, {}_\alpha p_\beta^{-1}[Y]}{\sum_Y x\, {}_\alpha p_\beta^{-1}[Y]}\,.$$

Then, evidently, ${}_\alpha T_\beta\, x$ is again a probability distribution on B. Show that the ${}_\alpha T_\beta$ are the individual relations of a process with unicity in the set of all distributions on B, over R. (Hint: now the compositivity property reduces, after some manipulation, to

$$x\, {}_\beta p_\gamma^{-1} \circ {}_\alpha p_\beta^{-1}[Z] = \sum_{Y \subset {}_\alpha p_\beta^{-1}[Z]} x\, {}_\beta p_\gamma^{-1}[Y]\,;$$

and this follows from ${}_\beta p_\gamma^{-1} \circ {}_\alpha p_\beta^{-1}[Z] = {}_\beta p_\gamma^{-1}[{}_\alpha p_\beta^{-1}[Z]]$ and σ-additivity of x.) Also observe that, for $\beta = \alpha$ in the parameter-domain of p, one has

$$0 < \sum_Y x\, {}_\alpha p_\alpha^{-1}[Y] \leqq \sum_Y x Y = 1\,,$$

so that, under appropriate continuity conditions on p, (16) subsists even for small $|\alpha - \beta|$, and hence the constructed process has local existence.

14° Let G be an *oriented graph*, [3, chap. I]; on its set V of vertices define relations ${}_i p_j$ (for $i, j \in \mathsf{C}^1$), by letting $x\, {}_i p_j\, y$ iff there is an oriented path in G from x to y incident with precisely $i - j + 1$ vertices. Verify that the ${}_i p_j$ are the individual relations of a process p in V over C^1. (Also see 4.17.22°.)

15° Let $q \in C^1[0, +\infty)$, and consider the following incomplete *boundary value problem* for u:

$$\frac{d^2 u}{d\theta^2} + q\,.\,u = 0\,, \quad u0 = 0\,. \tag{17}$$

For $\alpha \geqq \beta > 0$ define a relation ${}_\alpha p_\beta$ on R^1 by letting $x\, {}_\alpha p_\beta\, y$ iff $x = u\alpha$, $y = u\beta$ for some solution u of (17). Verify that these ${}_\alpha p_\beta$ are the individual relations of a process p in R^1 over $(0, +\infty)$. Observe that p adequately describes all the boundary value problems obtained on augmenting (17) by a second boundary condition $u\alpha = \gamma$ ($\alpha > 0$, γ in R^1); and that in quite reasonable cases p does not have unicity.

2. SOLUTION SYSTEMS

1. In this section we shall consider the family of solutions of a process from a different point of view, by studying a slightly more general concept.

Definition. S is a *solution system* in P over R iff P is a set, $R \subset \mathsf{R}^1$, and S satisfies conditions 1.10.1°–3°; members of S will be termed *solutions* of S. (The use of the term "solution" here is quite formal; in particular it is not implied that the members of S are solutions of differential equations or processes in the sense used previously.)

2. With each such solution system S one may associate a process in P over R, at present conveniently denoted by $pr\ S$, by letting $(x, \alpha)\ pr\ S\ (y, \beta)$ iff $\alpha \geqq \beta$ in R and

$$x = s\alpha\,, \quad y = s\beta \quad \text{for some} \quad s \in S\,.$$

Quite obviously $S_1 \subset S_2$ between solution systems implies $pr\ S_1 \subset pr\ S_2$ (not necessarily both in the same or over the same sets).

It should be emphasised that this is one of the most important methods of constructing processes (or rather, solution-complete processes, see 4). In particular, each of the examples described in item 10 is also an instance of a process; thus one may speak of the process associated with a regulated system as in 10.14°, by invoking automatically the above construction of $pr\ S$.

3. Secondly, in 1.10 there has been assigned a solution system to each process p; denote it by $sol\ p$ (both in P over R, say). For processes over a fixed set R, evidently $p_1 \subset p_2$ implies $sol\ p_1 \subset sol\ p_2$ (cf. 1.20); and furthermore,

$$(1) \qquad p \supset pr\ sol\ p\,, \quad S \subset sol\ pr\ S\,.$$

In the terminology of [4, IV, § 6] this shows that, for a fixed set $R \subset \mathsf{R}^1$, the two maps *sol* and *pr* define a Galois connection between the class of all solution systems over R and that of processes over R (ordered by set-inclusion and inverse relation-inclusion, respectively). In particular – and this is also obvious directly – one has that always

$$(2) \qquad sol\ p = sol\ pr\ sol\ p\,, \quad pr\ S = pr\ sol\ pr\ S$$

4. In the same situation, consider the first relation in (1); iff equality obtains, i.e. iff $p = pr\ sol\ p$, then the process p will be called *solution-complete*. Thus a process p is solution-complete iff, whenever $x\ {}_\alpha p_\beta\ y$, there exists a solution s of p with $x = s\alpha$, $y = s\beta$. The concatenation property 1.10.3° then yields a formally stronger version: If p is a solution-complete process and

$$(x_{i+1}, \alpha_{i+1})\ p\ (x_i, \alpha_i) \quad \text{for} \quad i = 1, 2, \ldots, n-1\,,$$

then there exists a solution s of p with $x_i = s\alpha_i$ for all i.

From the construction of characteristic solutions in 1.18 it follows directly that every process with unicity is solution-complete; and so is every process with discrete time, as may be shown directly. However, not all processes are solution-complete, as indicated in the following

5. (Example) Let P be the set of all rationals in R^1; and consider the process p in P over R^1 defined by its individual relations thus: $x \,_\alpha p_\alpha\, x$ for all $(x, \alpha) \in P \times \mathrm{R}^1$, and for $\alpha > \beta$

$$x \,_\alpha p_\beta\, y \quad \text{iff} \quad |x - y| < \alpha - \beta\,.$$

It is then easily verified that p is indeed a process (however $<$ cannot be replaced by $\leqq$; also, p is global cartesian full). From 1.9 (8) it is seen that each solution of p is Lipschitzian and hence continuous in the natural topologies; also, P is totally disconnected, so that each solution of p is constant. However, there exist very many $x \neq y$ in P with $x \,_\alpha p_\beta\, y$, and therefore p is not solution-complete.

Also note that if this process is altered to a process p' merely by restricting its time-variable to P (thus $p' \subset p$, p' is in P over P), then p' is solution-complete (cf. 10.5°), and no non-constant solution of p' is a solution of p.

6. The following assertions are direct consequences of the Galois connection exhibited in item 3. For any solution system S, in P over R, the process $pr\, S$ is solution-complete; and every solution-complete process is of this form. In particular, $pr\, sol\, p$ is solution-complete, and in fact it is the largest solution-complete process $q \subset p$ over R; thus $pr\, sol\, p$ may be termed the solution-complete (lower) *modification* of p.

Analogous assertions hold for *process-complete* solution systems, defining these by the requirement $S = sol\, pr\, S$. Then 1.10.4° is necessary, and 1.10.5° necessary and sufficient, for process-completeness of a solution system S.

In particular, given a solution system S in P over R, the process-complete (upper) *modification* $S_0 = sol\, pr\, S$ of S may be described as follows: S_0 consists of all maps $s : I \to P$ with I an interval in R and such that, for any $\alpha \geqq \beta$ in I, there is an $s' \in S$ with $s\alpha = s'\alpha$, $s\beta = s'\beta$.

7. The requirements on a solution system suggest the following method of constructing solution systems from more elementary objects.

As usual, let P be a set and $R \subset R^1$, but now assume that S is any system of partial maps satisfying the first requirement (i.e. each $s \in S$ is a partial map $s : R \to P$ with domain s an interval in R); such a system might be termed a *solution system sub-base.*

Now let S_0 consist of all interval-partializations of members of S, i.e.

$$S_0 = \{s \mid I : s \in S,\ I \text{ interval in } R\}\ ;$$

if $S_0 = S$, then S will be termed a *solution system base.* As the next step, let S_1

consist of the appropriate concatenations of members of S_0, i.e. $s \in S_1$ iff $s = \bigcup_{i=1}^{n} s_i$ with s a partial map, $s_i \in S_0$, domain s_{i+1} intersects domain s_i for $i = 1, 2, \ldots, n - 1$. Then, evidently, S_1 is the least solution system in P over R containing S.

8. The Galois connection between solution systems and processes was exhibited, among other reasons, to emphasise that these two concepts are at commensurable levels of generality. It is recognised that a reader familiar with differential equation theory may well prefer the approach via (bases of) solution systems, and consider the processes as an auxiliary concept obtained by means of the procedures described in 7 and 3. If this attitude is adopted, the theory of processes as presented here could be, at the very least, interpreted as the theory of solution-complete processes (cf. 6).

9. Finally, some terminological notes. Iff p is a process and its solution-complete modification (i.e. the process *pr sol p*) has some property $\mathscr{P}$, then it will be said that p has property *$\mathscr{P}$ of solutions*; thus one may speak of local existence of solutions, unicity of solutions, etc. Observe e.g. that a process with unicity necessarily has unicity of solutions, but that the converse assertion does not hold (thus the process described in example 5 has unicity of solutions but not unicity). For some of these questions apply 1.20 to the relation *pr sol p* $\subset p$.

10. (Exercises) 1° Prove that the domains of p and *pr sol p* coincide. More generally, define suitably the domain of a solution system S, and ascertain whether then the domains of S and *pr S*, and also of p and *sol p*, coincide.

2° Let $S \subset S'$ be solution systems in P over R; show that *pr S* $=$ *pr S'* is equivalent with $S' \subset$ *sol pr S*.

3° Verify that the processes defined in 1,3, 1.21.5° and 1.21.6° are all solution-complete. Interpret 1.21.2° and 1.21.4° as asserting process-completeness of some solution systems.

4° For processes, localize solution-completeness to individual pairs $(x, \alpha) \in D$, by defining appropriately the extent of solution-completeness $\lambda(x, \alpha)$; obtain the elementary properties of λ, including $\delta \leqq \lambda$, and $\lambda \leqq \varepsilon$ or $\lambda = +\infty$.

The next two exercises are due to I. Vrkoč.

5° Prove that every process p over a countable set $R \subset \mathrm{R}^1$ is solution-complete. (Hint: Assuming $x\, {}_{\alpha}p_{\beta}\, y$, well-order the interval $[\beta, \alpha]$ in $R : \theta_1 = \beta$, $\theta_2 = \alpha$, $\theta_3, \theta_4, \ldots$ Then, using compositivity, define maps s_n with domain $s_n = \{\theta_k \mid 1 \leqq k \leqq n\}$, $s_n\alpha = x$, $s_n\beta = y$, $s_n \times s_n \subset p \cup p^{-1}$, $s_n \subset s_{n+1}$. Finally verify that $\bigcup s_n$ is a solution of p as required.)

6° Prove that every process in a finite set P is solution-complete. (Hint: modify the proof procedure suggested in 5° to a transfinite induction.)

7° Suggest reasonable definitions of positive and negative unicity for solution systems S, and compare with the corresponding concepts applying to $pr\ S$.

8° For solution system with positive unicity (cf. 7°) show that property 1,10.4° is equivalent with process-completeness.

11. (Exercises) To avoid repetition, in these exercises Q denotes a set, L a Banach space, $d/d\theta$ strong differentiation in L, $\int d\theta$ (strong) Lebesgue integration in L.

1° Given a partial map $f : L \times \mathsf{R}^1 \to L$ (not necessarily continuous), define a system S of partial maps $\mathsf{R}^1 \to L$ thus: $s \in S$ iff domain s is an interval in R^1, either a singleton θ with $(s\theta, \theta) \in$ domain f, or non-degenerate with

$$\frac{d}{d\theta} s\theta = f(s\theta, \theta) \tag{3}$$

for all $\theta \in$ domain s. Prove that S is a solution system with property 1.10.4°; its members may be termed the *classical solutions* of (3).

2° Modify 1° by requiring all $s \in S$ to be continuous, but with (3) only almost everywhere in domain s (the *generalized solutions* of (3), [5, II, § 1]).

3° If one replaces $d/d\theta$ in (3) by some "generalized differentiation procedure" D_θ, under what natural conditions on D_θ can one still obtain the conclusions of 1° and 2°? (E.g. may one take for D_θ a first-order linear differential operator, or the right derivative, or for $L = \mathsf{R}^1$ the upper derivative?)

4° Given a partial map $f : L \times \mathsf{R}^1 \to L$, define a system S of partial maps $\mathsf{R}^1 \to L$ thus: $s \in S$ iff domain s is an interval in R^1 and

$$s\alpha - s\beta = \int_\beta^\alpha f(s\theta, \theta)\, d\theta \tag{4}$$

for all $\alpha \geqq \beta$ in domain s. Prove that S is a solution system with property 1.10.4°; its members are termed the *Carathéodory solutions* of (3) [16, I, § 1.5]. Show that these are precisely the absolutely continuous generalized solutions of (3).

5° Verify that the domains of the solution systems described in 1° and 4° (cf. 10.1°) coincide with domain f.

6° Replace $\int d\theta$ in (4) by an appropriate "generalized integration procedure".

7° Extend example 1.3 to phase-spaces which are differentiable n-manifolds of class C^1; also define the corresponding Carathéodory and generalized solutions.

8° Modify the procedures indicated in 5°–10° so as to apply to difference- and functional-differential equations (cf. 1.21.3° and 1.21.5°).

9° *Let orientor field* mean a partial map F of $L \times \mathsf{R}^1$ into the system of subsets of L. Define the solutions of an orientor field F as continuous partial maps $s : \mathsf{R}^1 \to L$

with domain s an interval and such that each accumulation point of

$$\frac{s\lambda - s\theta}{\lambda - \theta} \quad \text{as} \quad \lambda \to \theta \quad \text{in} \quad \text{domain}\, s$$

is in $F(s\theta, \theta)$ for all $\theta \in \text{domain}\, s$, [18].

Prove that the solutions of F constitute a solution system with property 1.10.4°; also define generalized solutions of F (loc. cit.) and Carathéodory solutions (e.g. via 4°).

10° Show that the differentiable solutions of an orientor field F from a solution system base; member of the corresponding solution system (cf. 7; these are the piecewise differentiable solutions of F) may be termed the classical solutions of F. Apply to implicit differential equations $f(\mathrm{d}x/\mathrm{d}\theta, x, \theta) = 0$ with given partial map $f : L \times \times L \times \mathsf{R}^1 \to Q$, and to differential inequalities.

11° Consider a *regulated system*, conventionally written as

(5) $$\frac{\mathrm{d}x}{\mathrm{d}\theta} = f(x, u, \theta)\,, \quad u \in U\,,$$

completely specified by a partial map $f : L \times Q \times \mathsf{R}^1 \to L$ and by a system U of partial maps $\mathsf{R}^1 \to Q$; on its own, U is to constitute a solution system base in Q over R^1 in the sense of 7 (its members are termed the regulators or forcing terms of (5)). The classical solutions of (5) are defined as partial maps $s : \mathsf{R}^1 \to L$ with domain s an interval in R^1, etiher a singleton θ with $(s\theta, u\theta, \theta) \in \text{domain}\, f$ for some $u \in U$ or with

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\, s\theta = f(s\theta, u\theta, \theta)$$

for some regulator $u \in U$ and all $\theta \in \text{domain}\, s$.

Prove that the classical solutions of (5) constitute a solution system base. Interpret the solution system obtained from this base as the classical solutions of a regulated system as in (5) with a suitably extended set of regulators. Define, and prove similar results for, the Carathéodory and generalized solutions. (Rather extensive generalizations of the important concept of a regulated system will be given in the second paper of this series.)

3. CATEGORIES OF PROCESSES

1. The category Proc of processes may now conveniently be introduced as follows (for the concept of categories see e.g. [13]). Its class of objects is to consist of all triples (P, R, p) with p a process in P over R; the morphisms

(1) $$r : (P, R, p) \to (P', R', p') \quad \text{in} \quad \mathsf{Proc}$$

are to be those relations r between $P' \times R'$ and $P \times R$ which satisfy the morphism condition

$$(2) \qquad r \circ p \circ r^{-1} \subset p' ;$$

and composition in Proc is to be the set-theoretical composition of relations (as in the Appendix). Finally, the natural order-relation between processes is carried over into a quasi-ordering relation on the objects of Proc; i.e., one defines $(P, R, p) \subset (P', R', p')$ iff $p \subset p'$ (no inclusion relation between P, R and P', R' is implied; however then the inclusion $D \subset D'$ between the corresponding solution spaces does extend to a morphism in Proc).

2. Assuming (1), obviously one has

$$r[D] \subset D' .$$

Next, r fulfils (2) iff $r \mid D$ does, so that the morphism condition depends only on the behaviour of r on D. As concerns this behaviour, there is a rather unexpected consequence of (2): If r satisfies (2), then $r \mid D$ is a partial map. To see this, merely observe that (2) is equivalent with isotonicity of $r \mid D$ as a relation between D' and D partially ordered by p and p' respectively (cf. 1.7), and then apply the lemma on isotone relations.

3. Thus in treating morphisms in Proc it suffices to consider only the special relations just described. Explicitly, given the objects from (1), a partial map $r : D \to D'$ is a morphism in Proc iff

$$(x, \alpha)\, p\, (y, \beta) \quad \text{implies} \quad (r(x, \alpha))\, p'\, (r(y, \beta))$$

whenever (x, α), (y, β) are in domain r.

Some minor modifications will also be useful. A relation r' between P' and P will be termed *admissible* relative to $(P, R, p) \to (P', R, p')$ in Proc iff the relation $r' \times 1$ between $P' \times R$ and $P \times R$ is a morphism in Proc. From 2 it then follows that $r' \mid C$ is a partial map into C', and the morphism condition reduces to

$$x \,{}_{\alpha}p_{\beta}\, y \quad \text{implies} \quad (r'x)\,{}_{\alpha}p'_{\beta}\,(r'y)$$

for all x, y in domain r'. Similar remarks are to apply to relations r'' between R' and R, admissible relative to $(P, R, p) \to (P, R', p')$ in Proc (using $1 \times r''$); by convention, morphisms in Proc may also be termed admissible.

Useful sub-categories may be obtained from Proc by restricting its class of objects, e.g. to processes with unicity or to full processes, or by restricting or even fixing the phase spaces P; and also by restricting its morphisms, e.g. to maps between the phase spaces.

4. The following lemma describes the action of a morphism in Proc on the solutions of the processes involved; later, this will be seen to be the basis of the abstract counterpart of transformation theory for differential equations.

Fundamental lemma. *Given a morphism* (1), *if s is a solution of p with domain $r[s]$ an interval in R', then $r[s]$ is a solution of p'.*

As a partial converse, given the objects in Proc *indicated in* (1) *and a relation r between $P' \times R'$ and $P \times R$ with the property that $r[s]$ is a solution of p for each solution s of p, then r is a morphism in* Proc *provided that p is solution-complete.*

In verifying the condition from 1.9 for $r[s]$ to be a solution of p', i.e.

$$r[s] \times r[s] \subset p' \cup p'^{-1},$$

possibly only the meaning of $r[s]$ requires any explanation. Here r is a relation between $P' \times R'$ and $P \times R$, and s is a particular type of subset of $P \times R$; thus $r[s]$ is the set of all $(x', \alpha') \in P' \times R'$ with

$$(x', \alpha')\, r\, (s\alpha, \alpha) \quad \text{for some} \quad \alpha \in R .$$

5. In this and the following two items there is assumed given a morphism

$$r : (P, R, p) \to (P', R', p') \quad \text{in} \quad \mathsf{Proc} .$$

According to 2 one need only consider the case of r a partial map; and since this maps into the product $P' \times R'$, one has partial maps r', r'' with

(3) $$r = (r', r''), \quad r' : P \times R \to P', \quad r'' : P \times R \to R' .$$

From $r[D] \subset D'$ it then follows that

$$r'[D] \subset C', \quad r''[D] \subset B' ;$$

also, directly from 1.1(1), r'' non-decreases along p.

In the situation of 4, $r[s]$ is the set of all points

$$(r'(s\theta, \theta),\ r''(s\theta, \theta)) \quad \text{for} \quad \theta \in \text{domain } s ;$$

and the fundamental lemma also asserts, loosely speaking, that $r'(s\theta, \theta)$ depends only on the value of $r''(s\theta, \theta)$. Furthermore,

$$\text{domain } r[s] = \text{range } r''[s] ;$$

thus if s and r'' are continuous in some convenient topologies, and if both R and R' are intervals in R^1, then necessarily domain $r[s]$ is an interval.

6. Now assume in addition that domain $r \supset D$. Then $u\ {}_{\theta}p_{\alpha}\ x$ implies

$$r''(x, \alpha) \leqq r''(u, \theta) \leqq \limsup r''(u, \theta) \leqq \varepsilon' \circ r(x, \alpha),$$

taking the limsup over all (u, θ) with $u\ {}_{\theta}p_{\alpha}\ x$ and $\theta \nearrow \varepsilon(x, \alpha)$ (recall that r'' non-decreases along p). (According to 1.11, ε' may be conceived as a mapping, where upon $\varepsilon' \circ r$ denotes the corresponding composition of maps; a similar remark applies to analogous situations elsewhere.) In particular, if $r''(u, \theta)$ is independent of u, then

$$r''(\varepsilon(x, \alpha) - 0) \leqq \varepsilon' \circ r(x, \alpha). \tag{4}$$

Hence, if domain $r \supset D$ and r'' strictly increases along p, then (4) may be written as $\varepsilon \leqq r''^{-1} \circ \varepsilon' \circ r$, and

1° If p has local existence at $(x, \alpha) \in D$, then p' has local existence at $r(x, \alpha)$ (and conversely, each counter-image of an end-pair is an end-pair);

2° If $r''(u, \theta)$ is independent of u and $\sup_B r'' = \sup B'$, and if p has prolongability at $(x, \alpha) \in D$, then p' has prolongability at $r(x, \alpha)$ (for globality, replace B and B' by R and R' respectively).

7. To obtain results on unicity assume that $r''(x, \alpha)$ is independent of x (i.e. $r'' = \varrho \circ \mathrm{proj}_2$ for some $\varrho : R \to R'$), and instead of domain $r \supset D$ require that r be one-to-one. It then follows easily that

$$r''(\delta(x, \alpha) + 0) \geqq \delta' \circ r(x, \alpha) \tag{5}$$

for $(x, \alpha) \in D$. Hence, assuming r is one-to-one and $r''(x, \alpha)$ independent of x,

1° If r'' is right-continuous at 0, then p has local unicity at $(x, \alpha) \in D$ if p' has local unicity at $r(x, \alpha)$;

2° If r'' is bounded on bounded subsets, then p has global unicity at $(x, \alpha) \in D$ if p' has global unicity at $r(x, \alpha)$.

For the purposes of the following section, the reader may find it useful to apply the results of items 5 to 7 to the two special cases of admissible relations indicated in 3; and in particular to observe that the condition on domain $r[s]$ appearing in the fundamental lemma is satisfied automatically if either $r = r' \times 1$, or $r = 1 \times r''$ with $R = R' = \mathsf{R}^1$ and r'' a continuous map $\mathsf{R}^1 \to \mathsf{R}^1$.

8. To each process one can assign, in a one-to-one manner, an object rather richer in content; its appearance may already be recognised in the definition of solutions 1.9 (8).

If p is a process, then the symmetrisation q of p, i.e.

$$q = p \cup p^{-1}, \tag{6}$$

will be termed the *bi-process* induced by p. Naturally, various properties of p imply some properties of q; and an independent description of these objects is in place. We only remark that the bi-processes which are induced by processes as in (6) indeed have the properties 1° to 3° listed below.

9. Let q be a relation on $P \times R$ (as usual P is a set, $R \subset \mathsf{R}^1$); and define the individual bi-relations of q as follows: for any α, β in R (not necessarily with $\alpha \geqq \beta$) let ${}_\alpha q_\beta$ be the relation on P with

$$x \; {}_\alpha q_\beta \; y \quad \text{iff} \quad (x, \alpha) \; q \; (y, \beta) \, .$$

(Of course, these describe the relation q completely.) Then q is called a *bi-process* in P over R iff its bi-relations satisfy these three conditions:

1° ${}_\alpha q_\alpha \subset 1$ for all $\alpha \in R$,

2° ${}_\alpha q_\beta \circ {}_\beta q_\gamma = {}_\alpha q_\gamma$ for all α, γ in R and all $\beta \in R$ between α and γ,

3° ${}_\alpha q_\beta = ({}_\beta q_\alpha)^{-1}$ for all α, β in R

(the initial-value, compositivity and symmetry properties, respectively). Note that 2° need only be required for $\alpha \geqq \beta \geqq \gamma$, and 3° for $\alpha \geqq \beta$.

Given a bi-process q in P over R, one defines a process p in P over R by specifying its individual relations, ${}_\alpha p_\beta = {}_\alpha q_\beta$ for $\alpha \geqq \beta$ in R; then (6) is fulfilled. The process p is said to be induced by q, or obtained by restriction of q. It may be useful to observe that $x \; {}_\alpha q_\beta \; y$ iff either $\alpha \geqq \beta$ and $x \; {}_\alpha p_\beta \; y$, or $\beta \geqq \alpha$ and $y \; {}_\beta p_\alpha \; x$; and that a process is a bi-process iff it is trivial.

10. For bi-processes one may now repeat, formally, the definitions (cf. 1.6 to 1.10) of the objects D, C, B, interval-component, solution, S; each of these coincides with the corresponding object of the induced process. As an example, a relation s between P and R is defined to be a solution of a bi-process q in P over R iff domain s is an interval in R and

$$s \times s \subset q \quad (q = q \cup q^{-1})\,;$$

according to (6), s is a solution of q iff it is a solution of the induced process.

Next, for each bi-process one may repeat, formally again, the definitions (cf. 1.12 to 1.16) of the associated bi-relation, bi-projection, characteristic solution, local and global unicity. Now, however, the resulting concepts are distinct from those of the induced process.

Finally, in complete analogy with 1, one may introduce the category Bipr of bi-processes, and again prove the fundamental lemma 4 in this new setting. However it is no longer true that a morphism is necessarily a partial map (one cannot apply the lemma on isotone relations, since bi-processes are not partial orderings); and indeed counter-examples are easily constructed. That the fundamental lemma remains valid for bi-processes is therefore rather surprising. Henceforth we assume that all this has been performed; note that, as yet, the escape times, local existence, etc., of a bi-process have not been defined.

It may be observed that the morphism condition in Bipr, i.e. $r : (P, R, q) \to$ $\to (P', R', q')$ in Bipr, in terms of the processes p, p' induced by q, q' respectively, is

$$r \circ p \circ r^{-1} \subset p' \cup p'^{-1} .$$

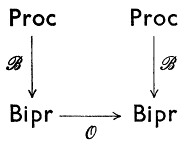
11. The procedure described in 8, of assigning the induced bi-process to a given process, is easily recognised as the action of a covariant functor $\mathscr{B}$: Proc $\to$ Bipr on the objects of these categories. The procedure of restricting a bi-process to a process is the inversion of $\mathscr{B}$ on the class of objects, but this is not the action of a functor Bipr $\to$ Proc (loosely speaking, a relation may be admissible in Bipr but not in Proc, e.g., the change of orientation of R^1 described below).

12. Next we shall describe, rather formally, the procedure of orientation change. Let r be any relation between $P' \times R'$ and $P \times R$; here P, P' are sets, R, R' subsets of R^1. Define a new relation $-r$ between $P' \times -R'$ and $P \times -R$, (here e.g. $-R =$ $= \{-\alpha : \alpha \in R\}$, etc.) in the evident manner:

$$(x', -\alpha') -r (x, -\alpha) \quad \text{iff} \quad (x', \alpha') \, r \, (x, \alpha) .$$

It is quite obvious that $-(r_1 \circ r_2) = (-r_1) \circ (-r_2)$, $(-r)^{-1} = -r^{-1}$, $- -r = r$, etc.

Now define a self-inverse covariant functor $\mathscr{O}$: Bipr $\to$ Bipr, the orientation-changing functor, by setting $\mathscr{O}(P, R, p) = (P, -R, -p)$ for objects, and $\mathscr{O}r = -r$ for morphisms. Furthermore, the diagram



may be made a commutative square on introducing an orientation-changing functor $\mathscr{O}$: Proc $\to$ Proc defined by

$$\mathscr{O}(P, R, p) = (P, -R, -p^{-1}) , \quad \mathscr{O}r = -r .$$

In greater detail, a process p over R and its orientation-changed process $p' = -p^{-1}$ over $-R$ are related thus:

$${}_{\alpha}p'_{\beta} = ({}_{-\alpha}p_{-\beta})^{-1} \quad \text{for} \quad \alpha \geqq \beta \quad \text{in} \quad -R ;$$

and similarly for bi-processes, where the right side may also be written as ${}_{-\alpha}p_{-\beta}$ (for all α, β in $-R$).

13. It is now obvious that the self-inverse orientation-change of R^1, suitably restricted to a subset $R \subset \mathsf{R}^1$, is an admissible map in Proc and Bipr relative to $(P, R, p) \to (P, -R, -p^{-1})$. Results concerning the objects D, B, etc., associated with p then follow immediately; e.g. the (otherwise obvious) relation between solutions of p and those of $-p^{-1}$ follow from a two-fold application of the fundamental lemma: s is a solution of p iff s',

$$s'(-\theta) = s\theta \quad \text{for} \quad \theta \in \text{domain } s\,,$$

is a solution of $-p^{-1}$.

14. Next there will be defined the escape times and extents of unicity for bi-processes, and some further terminology introduced.

Let q be a bi-process, p the induced process, and $p' = -p^{-1}$ the orientation-changed process; let ε, δ correspond to p, and ε', δ' to p'. For any (x, α) in the domain of q (i.e., of p) set

$$\varepsilon^+(x, \alpha) = \varepsilon(x, \alpha)\,, \quad \varepsilon^-(x, \alpha) = -\varepsilon'(x, -\alpha)\,,$$

$$\delta^+(x, \alpha) = \delta(x, \alpha)\,, \quad \delta^-(x, \alpha) = -\delta'(x, -\alpha)\,.$$

These will be called, respectively, the *positive* and *negative* escape times and extents of unicity, of both q and p.

Using these one can define the positive and negative variants of local existence, prolongability, globality, local unicity and unicity, for both q and p. The conjunction of e.g. positive and negative prolongability of q will be termed prolongability of q (but not of p, since this has already been defined), or also *bilateral* prolongability of q and p. Similar conventions are to apply to the remaining concepts, viz. local existence (at a pair), prolongability (at a pair), globality, local or global unicity (at a pair). Finally, an end-pair of p will be termed an end-pair of q; and (x, α) is a *start-pair* of q or p iff $(x, -\alpha)$ is an end-pair of p'.

The direct descriptions, in terms of p, of these concepts are quite straightforward; e.g.

$$\varepsilon^-(x, \alpha) = \inf\{\theta \in \mathsf{R}^1 : x\, {}_\alpha p_\theta\, x' \quad \text{for some} \quad x' \in P\}\,,$$

$$\delta^-(x, \alpha) = \inf\{\lambda \in \mathsf{R}^1 : \alpha \geqq \theta \geqq \lambda,\ x\, {}_\alpha p_\theta\, u,\ x\, {}_\alpha p_\theta\, v \quad \text{imply} \quad u = v\}\,,$$

(x, α) is a start-pair iff $x\, {}_\alpha p_\theta\, x'$ for no $\theta < \alpha$, $x' \in P$.

15. (Exercises) 1° Describe property 1.1 (1) as a morphism condition. (Hint: define a suitable process in R^1 and consider $\text{proj}_2 : P \times R \to \mathsf{R}^1$.)

2° Let p' and p'' be processes in P' and P'' respectively, both over R, and consider the relations ${}_\alpha p_\beta$ on $P = P' \times P''$ defined by

$$(x', x'')\, {}_\alpha p_\beta\, (y', y'') \quad \text{iff} \quad x'\, {}_\alpha p'_\beta\, y'\,, \quad x''\, {}_\alpha p''_\beta\, y''\,.$$

Show that these ${}_\alpha p_\beta$ are the individual relations of a process p in P (the *direct product* of p' and p''); obtain results on the solutions of p, p', p''.

3° In the situation of 2° show that the natural projections

$$\mathrm{proj}_1 : P \to P' , \quad \mathrm{proj}_2 : P \to P''$$

are admissible in Proc relative to suitable morphisms.

4° Let p be a process in P over R, and π a process in R over R' with the property that $\xi \, {}_\alpha\pi_\beta \, \eta$ implies $\xi \geqq \eta$. For $\alpha \geqq \beta$ in R' define relations ${}_\alpha q_\beta$ on $P \times R$ by

$$(x, \xi) \, {}_\alpha q_\beta \, (y, \eta) \quad \text{iff} \quad x_\xi \, p_\eta \, y \, , \quad \xi \, {}_\alpha\pi_\beta \, \eta \, .$$

Show that these ${}_\alpha q_\beta$ are the individual relations of a process q in $P \times R$ (the *skew-product* of p and π), and obtain results on the solutions of q, p, π. Also treat admissibility of various projections of $P \times R \times R'$.

5° Let p, q be differential processes in R^n, associated with

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = f(x, \theta) \, , \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = g(y, \theta)$$

respectively (the assumptions on f, g are as in 1.3); also, let partial $r = (r', r'')$: : domain $f \to$ domain g be a differentiable map. Prove that r is admissible relative to

$$(\mathsf{R}^n, \mathsf{R}^1, p) \to (\mathsf{R}^n, \mathsf{R}^1, q) \quad \text{in} \quad \mathsf{Proc}$$

iff

$$\left((d_x r'', f) + \frac{\partial r''}{\partial \theta} \right) g \circ r = (\mathrm{d}_x r', f) + \mathrm{d}_\theta r' \, ,$$

where d_x, d_θ denote the appropriate differential operators. (Hint: since p, q are solution-complete by 2.8.3°, the admissibility of r may be characterised as solution-preservation, as in the fundamental lemma.)

6° Consider the direct product, the skew-product and orientation change for differential and functional-differential processes.

7° Assume given $P, R \subset \mathsf{R}^1$, and a system $\{{}_\alpha q_\beta \mid \alpha, \beta \text{ in } R\}$ of relations on P with the initial value and symmetry properties as in 9.1° and 9.3°. Show that the ${}_\alpha q_\beta$ possess a strong form of the compositivity property (cf. 9.2°), viz. ${}_\alpha q_\beta \circ {}_\beta q_\gamma = {}_\alpha q_\gamma$ for unrestricted α, β, γ in R, iff they are the individual bi-relations of a global bi-process with unicity.

8° In the situation of 1.21.9° show that a necessary condition for negative local existence to obtain at a pair $(x, \alpha) \in C^1[-\tau, 0] \times \mathsf{R}^1$ (i.e. that (x, α) is not a start-pair) is that x have a continuous derivative in some left-neighbourhood of 0.

1. In this section several interesting types of processes will be examined. In each case (excepting only that of 14) there will also be exhibited a characterisation in terms of the admissible relations introduced in 3.3; the object of this is to bring to bear all the results of the preceding section.

To illustrate, the processes to be described in items 2.4 and 13 are instances of the following situation: a process p in P over R is said to *admit the relation* r if r is admissible (in the sense of 3.3) relative to

$$(1) \qquad (P, R, p) \to (P, R, p) \quad \text{in} \quad \mathsf{Proc}\,.$$

Thus, at least for morphisms, the condition is that

$$x\,{}_{\alpha}p_{\beta}\,y\,, \quad (x', \alpha')\,r\,(x, \alpha)\,, \quad (y', \beta')\,r\,(y, \beta) \quad \text{imply} \quad x'\,{}_{\alpha'}p'_{\beta'}\,y'\,.$$

The fundamental lemma 3.4 then gives a necessary and sufficient condition for r to reproduce the solutions of p; one also has that the domain of p is r-invariant in the sense that $r[D] \subset D$, etc. The definition may, of course, be carried over to the category Bipr.

2. Let p be a process in P over R; then p is said to *admit the period* τ iff $\tau \in \mathsf{R}^1$ and

$$(2) \qquad {}_{\alpha-\tau}p_{\beta-\tau} = {}_{\alpha}p_{\beta} = {}_{\alpha+\tau}p_{\beta+\tau} \quad \text{for all} \quad \alpha \geqq \beta \quad \text{in} \quad R\,.$$

Obviously this condition is equivalent to the requirement that the two assignments $\theta \to \theta \pm \tau$ be maps $R \to R$ admissible relative to (1). In particular, if s is a solution of p, then so are the partial maps s_τ and $s_{-\tau}$ obtained by "translation":

$$(3) \qquad s_\tau\theta = s(\theta + \tau) \quad \text{for all} \quad \theta + \tau \in \text{domain}\, s\,;$$

and for solution-complete processes, this condition characterises admissibility of period τ.

Next, $\theta \in R$ implies $\theta \pm \tau \in R$; also, from 3.6 (4) and 3.7 (5) it follows that

$$(4) \qquad \varepsilon(x, \alpha + \tau) = \varepsilon(x, \alpha) + \tau\,, \quad \delta(x, \alpha + \tau) = \delta(x, \alpha) + \tau\,.$$

Hence local existence, etc., obtains at $(x, \alpha + \tau)$ iff it obtains at (x, α).

3. Relative to a process p admitting period τ, define *τ-periodic pairs* (or pairs admitting period τ) as the pairs (x, α) with

$$(x, \alpha + |\tau|)\, p\, (x, \alpha)$$

(and then, from (2), also $(x, \alpha)\, p\, (x, \alpha - |\tau|)$). Let (x, α) be a τ-periodic pair; then (4) yields the following results:

1° If $\tau \neq 0$ then $\varepsilon(x, \alpha) = +\infty$,

2° If $0 < |\tau| < \delta(x, \alpha)$ then $\delta(x, \alpha) = +\infty$, whereupon the characteristic solution s through (x, α) is positively $|\tau|$-periodic in the sense that $s\theta = s(\theta + |\tau|)$ for all $\theta \geqq \alpha$ in R.

Next, it is easily shown that the set of periods admitted by any given process p is an additive subgroup T of R^1. The set of periods admitted by a pair, relative to p, is by definition a subset of T, and indeed an additive subgroup of T (but it may well be a proper subgroup, e.g. trivial).

If a process p admits a period τ, then the orientation-changed process $-p^{-1}$ also admits period τ (and the τ-periodic pairs correspond via $(x, \alpha) \leftrightarrow (x, -\alpha)$). Thus one may also speak without ambiguity of bi-processes admitting a period.

4. A process p in P over R is termed *stationary* iff it admits all periods $\theta \in R$ (nothing is required of θ's not in R). Reducing the definition, the condition is that

$$_{\alpha-\theta}p_{\beta-\theta} = {}_\alpha p_\beta = {}_{\alpha+\theta}p_{\beta+\theta} \quad \text{for all } \alpha \geqq \beta \text{ and } \theta \text{ in } R\,.$$

In particular, R is an additive subgroup of R^1. If s is a solution of p then so are all s_θ for arbitrary $\theta \in R$ (cf. (2); and for solution-complete processes, this latter condition characterises stationarity). The following lemma is easily established.

5. Lemma. *Let p be a stationary process in P over R. Then*

1° *p is cartesian; moreover $D = C \times R$, and hence $B = R$ if $p \neq 0$;*

2° *p is either extensive or trivial;*

3° *for every $(x, \alpha) \in D$,*

$$\varepsilon(x, \alpha) = \varepsilon(x, 0) + \alpha\,; \quad \delta(x, \alpha) = \delta(x, 0) + \alpha\,. \tag{5}$$

(In proving this, 1° is straightforward; for 2° obtain a description of the interval-component containing 0 showing that it is an additive subgroup of R^1, hence trivial or unbounded; and for 3° apply (4).) From 1° it follows that $x \in C$ iff $(x, 0) \in D$.

Thus if local existence obtains at (x, α), then it also obtains at all (x, α') for arbitrary $\alpha' \in R$; and similarly for prolongability, local and global unicity, τ-periodicity. In particular, p has local existence iff $\varepsilon(x, 0) > 0$ for all $x \in C$, globality iff $\varepsilon(x, 0) = +\infty$ for all $x \in C$ etc. Hence all these properties concern individual elements $x \in C$ rather than pairs $(x, \alpha) \in D$; by convention, they are thus reduced, and we speak e.g. of local existence at a point $x \in C$ of the escape time $\varepsilon_x = \varepsilon(x, 0)$ of an $x \in C$, etc. Since change of orientation preserves stationarity, this convention may be made to apply to the negative and bilateral variants of the mentioned properties; thus e.g. one has the negative unicity extent δ_x^- of an $x \in C$, etc.

6. In items 6 to 9 we shall examine the effect of stationarity on several of the concepts introduced previously.

First, the individual relations of a stationary process p in P over R satisfy ${}_\alpha p_\beta = {}_{\alpha-\beta}p_0$ for all $\alpha \geqq \beta$ in R, and hence depend on a single parameter $\alpha - \beta \geqq 0$; an independent description of such objects is quite feasible.

Let there be given a set P, an additive subgroup R of R^1 (we shall write $R_+ = \{\alpha \in R : \alpha \geqq 0\}$), and also a system $\{q_\alpha \mid \alpha \in R_+\}$ of relations q_α on P with the following two properties:

1° $q_0 \subset 1$,

2° $q_\alpha \circ q_\beta = q_{\alpha+\beta}$ for α, β in R_+,

(the initial value and semigroup property, respectively). Then

$$ {}_\alpha p_\beta = q_{\alpha-\beta} \quad \text{for} \quad \alpha \geqq \beta \quad \text{in} \quad R $$

defines the individual relations of a stationary process in P over R. Conversely, given a stationary process p in P over R, (6) defines q_θ for $\theta \in R_+$ unequivocally and then the system $\{q_\alpha \mid \alpha \in R_+\}$ satisfies conditions 1°–2°.

7. Next we shall introduce a concept whose relation to the q_α's of 6 is similar to that of a process or its associated relation to the individual relations. Thus, let p be a stationary process in P over R, and define q_θ by (6) for all $\theta \in R_+$. Then the relation q between P and $P \times R_+$ with

$$ x\, q\, (y, \alpha) \quad \text{iff} \quad x\, q_\alpha\, y $$

will be termed the *dynamical relation* associated with p (or with $\{q_\alpha \mid \alpha \in R_+\}$). This seems to be the most economical manner of specifying stationary processes.

In this situation one has $x\, {}_\alpha p_\beta\, y$ or $x\, t\, (\alpha, y, \beta)$ iff $x\, q_{\alpha-\beta}\, y$ or $x\, q\, (y, \alpha - \beta)$; also $\alpha\, d\, (y, \beta)$ iff $(y, \alpha - \beta) \in$ range q, range $q = C$, etc. The analogue of 1.15 (11) is

$$ \{(x, \alpha) \in C \times R_+ : 0 \leqq \alpha < \varepsilon_x\} \subset \text{domain } q \subset \{(x, \alpha) \in C \times R_+ : 0 \leqq \alpha \leqq \varepsilon_x\}\,. $$

The process p has local existence iff the former of these inclusions is an equality; it is global iff domain $q = C \times R_+$; it has unicity iff q is a partial map $P \times R_+ \to P$ (or $C \times R_+ \to C$), or also iff each q_α is a partial map $P \to P$ (or $C \to C$).

Obviously, all this may be carried over to bi-processes, by replacing R_+ by R, the semigroup condition 6.2° by the group condition $q_\alpha \circ q_\beta = q_{\alpha+\beta}$ for α, β in R either both non-negative or both non-positive, and defining the corresponding dynamical bi-relation.

8. Given a stationary bi-process p in P over R and an $x \in C$, consider the periods admitted by x (i.e. by the pair $(x, 0) \in D$, cf. 5); these constitute an additive subgroup of R^1, and hence one has the following alternatives.

9. Lemma (and **definition**). *For each x in the carrier of a stationary bi-process p in P over R there obtains precisely one of the following alternatives*:

1° $x \, {}_{\theta}p_0 \, x$ *iff* $\theta = 0$; *then x is termed aperiodic*;

2° *there is a least positive $\theta \in R$ with $x \, {}_{\theta}p_0 \, x$; then x is said to have primitive period θ*;

3° *the set of all $\theta \in R$ with $x \, {}_{\theta}p_0 \, x$ is a non-trivial subgroup of R, dense in* R^1; *then x is termed feebly critical*;

4° $x \, {}_{\theta}p_0 \, x$ *for all $\theta \in R$, and x is termed critical.*

If all elements of the carrier of p are aperiodic, then p itself is called *aperiodic*; and similarly for primitive periods, feeble criticality, and criticality. An elementary example of a feebly critical stationary process appears in [9, II, 4.7].

10. Let p be a stationary process in P over R. A subset $X \subset P$ is termed *positively invariant* iff ${}_{\theta}p_0 \, [X] \subset X$ for all $\theta \geqq 0$ in R; *negatively invariant* iff it is positively invariant relative to the orientation changed process, i.e. iff $({}_{\theta}p_0)^{-1} \, [X] \subset X$ for all $\theta \geqq 0$ in R; and (bilaterally) *invariant* iff it is both positively and negatively invariant.

It is then easily shown that positive invariance is preserved under general unions and intersections (e.g. P and θ are positively invariant) so that each subset $X \subset P$ has a positively invariant hull and kernel; and that a set is positively invariant iff its complement is negatively invariant (hence similar assertions also hold for negative invariance and invariance). There even are constructive descriptions: e.g. the positively invariant hull of an $X \subset P$ is $\bigcup_{\theta \geqq 0} {}_{\theta}p_0 \, [X]$.

Next, a subset $X \subset P$ is termed *time-convex* iff

$$\bigcup_{\alpha \geqq \beta \geqq \gamma} ({}_{\alpha}p_{\beta})^{-1} \, [X] \cap {}_{\beta}p_{\gamma}[X] \subset X \, ,$$

i.e. iff $x_1, x_2 \in X$ imply $x \in X$ for all $x \in P$ with $x_1 \, {}_{\alpha}p_{\beta} \, x \, {}_{\beta}p_{\gamma} \, x_2$. It can be shown that the time-convex subsets coincide with the intersections of positively and negatively invariant subsets (i.e. with the differences of positively invariant subsets).

11. Speaking rather vaguely, every process may be made stationary; or better, the study of general processes can be reduced to that of stationary processes. Since the penalty for this is passage to a different phase-space, it need not always be useful to perform the reduction. The procedure is the abstract analogue of the familiar transition from a non-autonomous differential equation $\mathrm{d}x/\mathrm{d}\theta = f(x, \theta)$ in n-space to the autonomous

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = f(x, \xi) \, , \quad \frac{\mathrm{d}\xi}{\mathrm{d}\theta} = 1$$

in $(n + 1)$-space.

Lemma (and **definition**). *Let p be a process in P over R, let R_0 be the additive subgroup generated by R in* R^1. *Then the relations ${}_\alpha q_\beta$ ($\alpha \geqq \beta$ in R_0) on $P \times R$ defined by*

$$(x, \xi)\,{}_\alpha q_\beta\,(y, \eta) \quad \textit{iff} \quad x\,{}_\xi p_\eta\,y\,, \quad \xi - \alpha = \eta - \beta$$

are the individual relations of a stationary aperiodic process q in $P \times R$ over R_0, called the stationarization of p.

Obviously q describes p completely: one has $x\,{}_\alpha p_\beta\,y$ iff $(x, \alpha)\,{}_{\alpha+\theta}p_{\beta+\theta}\,(y, \beta)$ for some (or all) $\theta \in R_0$. Also observe that the carrier of q is precisely the domain of p, that the escape times and extents of unicity of a pair (x, α) relative to p coincide with those of the point (x, α) relative to q, etc.

Now assume that $R = R_0$, i.e. that R itself is a subgroup of R^1. Then, for every solution s of p and $\alpha \in R$, the relation

$$\{(s\theta, \theta, \theta - \alpha) \mid \theta \in \text{domain } s\}$$

is a solution of q, and conversely, every solution of q has the indicated form.

In this connection, subsets of $P \times R_0$ invariant relative to q are termed the *integral sets* of p (often in the context of integral manifolds), and one may even define positive and negative integrality.

12. Let G be a group (written additively, even though commutativity is not assumed), and let there be given a process p in G over R. Then p is called *additive*, or compatible with the group structure, iff

$$x_i\,{}_\alpha p_\beta\,y_i \quad \text{for} \quad i = 1, 2 \quad \text{implies} \quad (x_1 - x_2)\,{}_\alpha p_\beta\,(y_1 - y_2)$$

(whereupon also $x_i\,{}_\alpha p_\beta\,y_i$ for $i = 1, \ldots, n$ implies $(\sum m_i x_i)\,{}_\alpha p_\beta\,(\sum m_i y_i)$ for all integers $m_1, \ldots, m_n$).

This condition may be described in terms of admissible relations as follows. Construct an auxiliary process q in $G \times G$ over R as the direct product of p with itself (cf. 3.15.2°); and also the natural group mapping $r : G \times G \to G$, $r(x, y) = x - y$. Then p is additive iff r is admissible relative to $(G \times G, R, q) \to (G, R, p)$ in Proc.

In particular, if s_1, s_2 are solutions of p with a common domain, then $s_1 - s_2$ is also a solution of p (and for solution-complete processes in a group, this condition characterises additivity). Hence and from 1.10.5°, each interval-component of p is the domain of a significant solution of p, the constant 0. Next, each α-cut of D, i.e. the set $\{x \in G : (x, \alpha) \in D\}$, is a subgroup of G.

(7) For each $\alpha \in R$ one has $\alpha \in B$ iff $(0, \alpha) \in D$, whereupon $\varepsilon(0, \alpha) = \max \varepsilon(x, \alpha) = \sup I$, $\delta(0, \alpha) = \min \delta(x, \alpha)$

with I the interval-component containing α (and letting (x, α) vary only within D).

In particular, prolongability occurs at all $(0, \alpha) \in D$ if p is extensive; p is trivial iff all $(0, \alpha) \in D$ are end-pairs; p has local or global unicity iff $\delta(0, \alpha) > 0$ or $\delta(0, \alpha) = +\infty$, respectively, for all $\alpha \in B$.

Additivity is preserved on changing orientation; hence one may speak about linear bi-processes.

13. A *pseudo-scalar product* in a group G is a bilinear map $[\ ,\ ] : G \times G \to \mathsf{R}^1$ such that $[a, x] = 0$ for all $x \in G$ implies $a = 0$. Two processes p, q in a group G will be termed *adjoint* (relative to a pseudo-scalar product $[\ ,\ ]$ in G) iff

$$x\ {}_\alpha p_\beta\ y\,, \quad u\ {}_\alpha q_\beta\ v \quad \text{imply} \quad [x, u] = [y, v]\,.$$

In the following theorem it will be shown that unicity and existence, in opposite orientations, are related in an interesting manner.

14. Theorem. *Let p, q be adjoint additive processes, p full. If p has positive global existence, then q has negative global unicity.*

Proof. Take any $\alpha \geqq \beta$ and v with $0\ {}_\alpha q_\beta\ v$; by the assumptions on p, to any y there is an x with $x\ {}_\alpha p_\beta\ y$. Adjointness then implies $0 = [x, 0] = [y, v]$, and then $v = 0$ since y was arbitrary. Thus $0\ {}_\alpha q_\beta\ v$ only for $v = 0$, i.e. q has infinite extent of negative unicity as asserted.

Corollary. *Each global full additive bi-process, self-adjoint relative to some pseudo-scalar product, necessarily has bilateral unicity.*

15. A process p in P over R is said to be *f-symmetric* iff f is a *symmetry* of C, in the sense that f is a map $C \to C$ such that $f \circ f$ is the identity of C, and p admits the relation f; the latter condition is, of course, that

$$x\ {}_\alpha p_\beta\ y \quad \text{imply} \quad (fx)\ {}_\alpha p_\beta\ (fy)\,.$$

Similarly, p is termed *f-antisymmetric* iff f is a symmetry of C and f is admissible relative to $(P, R, p) \to (P, -R, -p^{-1})$ in Proc (for $-p^{-1}$ see 3.12); the latter condition reduces to

$$x\ {}_\alpha p_\beta\ y \quad \text{implies} \quad (fy)\ {}_{-\beta} p_{-\alpha}\ (fx)\,.$$

For any solution s of p one then has that $f \circ s$ is again a solution of p in the first case, and that $f \circ s \circ o$ is a solution of p in the second (with o the orientation change of R^1, i.e. the assignment $\theta \to -\theta$); for p solution-complete, these conditions characterise the corresponding relation between p and f. Furthermore,

$$\varepsilon(fx, \alpha) = \varepsilon(x, \alpha)\,, \quad \delta(fx, \alpha) = \delta(x, \alpha)$$

in the first case; and in the second,

$$\varepsilon^+(fx, \alpha) = \varepsilon^-(x, \alpha), \quad \varepsilon^-(fx, \alpha) = \varepsilon^+(x, \alpha),$$
$$\delta^+(fx, \alpha) = \delta^-(x, \alpha), \quad \delta^-(fx, \alpha) = \delta^+(x, \alpha).$$

Both f-symmetry and antisymmetry are preserved under orientation change; thus the property may well be applied to bi-processes.

16. Finally, consider processes with discrete time (cf. 1.6), say p in P over R. The condition on R may be expressed, less concisely but possibly more naturally, by saying that the permissible time-instants constitute a sequence in R^1 without any accumulation points in R^1; essentially, then, they may be identified with some consecutive integers.

Slightly more formally, there is a partial map $\phi : \mathsf{C}^1 \to R$ which is strictly increasing, onto, and with domain ϕ an interval in C^1. The corresponding partial map $1 \times \phi : P \times \mathsf{C}^1 \to P \times R$ may be made an isomorphism in Proc by defining an appropriate process in P over C^1, induced by p via ϕ; and this latter may then be identified with p.

Thus, let p be a process in P over C^1, and consider the following individual relations of p:

$$q_n = {}_n p_{n-1}, \quad q_n' = {}_n p_n \quad (\text{for } n \in \mathsf{C}^1).$$

The q_n may be termed the *transition relations* of p. These individual relations describe p completely, since for $n > m$ in C^1 one has

$${}_n p_m = q_n \circ q_{n-1} \circ \dots \circ q_{m+1};$$

and if p is full, then of course all $q_n' = 1$.

17. Now we will review some of the concepts introduced in preceding sections in the present special case. There is sup X = max X for every bounded subset X of C^1; this may be applied to 1.11 (10) and 1.17 (12) to obtain the following assertion: if $\varepsilon(x, \alpha) < +\infty$ then there exists an end-pair (indeed an end-pair y, β with $x\ {}_\alpha p_\beta\ y$, $\beta = \varepsilon(x, \alpha)$). Hence, every process with local existence and discrete time is global. Results similar but more significant, viz. prolongability theorems, will be obtained later by replacing the time-discreteness assumption by various other versions on the processes concerned.

Returning to the time-discrete process p in P over C^1, a similar application to the definition of time-extents yields that in 1.19, only case 1° is possible; thus every process with local unicity and discrete time has global unicity. In this connection, in example 1.4, if the finite-difference equation (4) has the special form $x_{i+1} - x_i = f_i(x_i)$, then obviously the corresponding process does have local unicity.

Next, it is quite obvious that every process with discrete time is solution-complete. In the fundamental lemma 3.4 as applying to processes over C^1, the condition on domain $r[s]$ is equivalent with the following: with $r = (r', r'')$,

$$|r''(x, n) - r''(y, n - 1)| \leqq 1 \quad \text{if} \quad x_n p_{n-1}\, y\,.$$

In terms of the transition relations, p admits period $\tau \in \mathsf{C}^1$ iff

$$q_{n-\tau} = q_n = q_{n+\tau} \quad \text{for all} \quad n \in \mathsf{C}^1\,;$$

and hence p is stationary iff the q_n are independent of n (whereupon ${}_n p_m = q_0^{n-m}$ for $n > m$).

18. It is now in place to exhibit the precise connections between the concepts introduced in this paper and several related concepts defined previously (for references see the introduction). However, all continuity requirements will be ignored; the same effect may be obtained by taking the discrete topologies for P and R^1.

1° A *dynamical system* in P is the dynamical bi-relation (cf. item 7) associated with a full global stationary bi-process with unity in P over R^1. Thus if $f : P \times \mathsf{R}^1 \to$ $\to P$ is the dynamical system and p the process, then one has

$$x\; {}_\alpha p_\beta\; y \quad \text{iff} \quad x = f(y, \alpha - \beta)\,.$$

A *unilateral dynamical system* (in another terminology, a *semi-dynamical system*) in P is obtained from the above on replacing bi-relation and bi-process with relation and process respectively (the bilateral requirements of globality and unicity are of course replaced automatically by the positive variants, see 3.10 and 3.14).

2° A *local dynamical system* in P is the dynamical bi-relation (cf. item 7) associated with a full stationary bi-process with unicity and local existence in P over R^1. In the notation of [9], if $\mathsf{T} : P \times \mathsf{R}^1_+ \to P$ is the local dynamical system and p the process, then one has

$$x\; {}_\alpha p_\beta\; y \quad \text{iff} \quad x = y\, \mathsf{T}(\alpha - \beta)\,.$$

A *local semi-dynamical system* in P is obtained from this on replacing bi-relation and bi-process by relation and process respectively.

3° A *flow* in P over R ($R \subset \mathsf{R}^1$) is the associated bi-relation (cf. 3.10) to a full global bi-process with unicity in P over R. Thus if ${}_\alpha T_\beta : P \to P$ are the motions of the flow and p the process, then one has

$$x\; {}_\alpha p_\beta\; y \quad \text{iff} \quad x = {}_\alpha T_\beta\; y\,.$$

A *semi-flow* in P over $R \subset \mathsf{R}^1$ is obtained from this on replacing the terms bi-relation and bi-process by relation and process respectively.

4° In analogy with the generalization 2° of 1°, one may define a *local flow* (and a *local semi-flow*) in P over $R \subset \mathsf{R}^1$ as the associated bi-relation (or relation) of a full bi-process (or process) with unicity and local existence in P over R.

5° A *generalized flow* in P is the relation associated with a full global process in P over R^1.

6° An *unilateral dynamical system without unicity* in P is the dynamical relation associated with a full global stationary process in P over R^1.

12. (Exercises) 1° Let p be a bi-process with unicity and $0 \neq \tau \in \mathsf{R}^1$. Show that every solution of p can be extended to a τ-periodic solution iff p admits period τ, p is bilaterally global and all pairs in D are τ-periodic.

2° Let p be a process with unicity admitting a period τ. Prove that there exists a positively $|\tau|$-periodic solution through (x, α) iff is a τ-periodic pair.

3° Show that a process p admitting a period $\tau \neq 0$ is completely characterised by the individual relations ${}_{\alpha}p_{\beta}$ with indices restricted as follows: $0 \leqq \beta < \tau$ and either $\beta \leqq \alpha \leqq \tau$ or $\alpha = \beta + \tau$.

4° Given a process p in P admitting period τ, consider the process q in P over C^1 obtained by "sampling" p in the sense that ${}_{n}q_{m} = {}_{n\tau}p_{m\tau}$. Show that q is stationary, and examine other relations between q and p.

5° Show that, relative to a stationary process, the set of points of prolongability, of local existence and of start-points are all negatively invariant.

The following three exercises are connected with the classification introduced in lemma 9 of points in the carrier of a stationary process.

6° Prove that the classification of 9 is independent of orientation change.

7° Every stationary deterministic physical law on a countable phase space and with continuous time is either completely immobile or admits at most a type of Brownian movement; more precisely, every point in the countable carrier of a stationary bi-process with unicity over R^1 is either critical or feebly critical or a start-end point.

8° Consider a stationary process with unicity. Show that the set of τ-periodic points is positively invariant, and obtain results on the critical points, etc., as in 9. Also classify further the aperiodic points, as indicated by the following terms: leading to a critical (feebly critical, primitively periodic) point, ultimately aperiodic.

9° Generalize lemma 11 as follows. First show that the skew-product of processes p and π as in 3.15.4° is stationary iff π is stationary; then interpret the construction of lemma 9 as that of a suitable skew-product.

10° Prove that (x, ξ) is a τ-periodic point relative to the skew-product of a process p with a stationary process π iff ξ is τ-periodic relative to π. Then apply to lemma 11.

The following four exercises are connected with theorem 14.

11° Formulate adjointness in terms of admissibility relations. (Hint: use the process in R^1 associated with $\mathrm{d}x/\mathrm{d}\theta = 0$.)

12° Show that for linear processes in an abelian group with finite rank, local existence implies uniform local existence, in the sense that $\inf_x \varepsilon(x, \alpha) > 0$ for each $\alpha \in B$ (with (x, α) varying in D only).

13° Using 7° prove a *local* variant of theorem 14: for adjoint linear processes, q and full p, in an abelian group with finite rank, positive local existence of p implies negative local unicity of q. (Suggestion: from the proof read off that $\beta < \delta_q^-(0, \alpha)$ implies $\inf_x \varepsilon_p^+(x, \alpha) < \alpha$.)

14° Consider the differential equation $\mathrm{d}x/\mathrm{d}\theta = f(x, \theta)$ in R^1 with (discontinuous)

$$f(x, \theta) = \begin{cases} 2x/\theta & \text{for} \quad \theta \neq 0\,, \\ 0 & \text{for} \quad \theta = 0\,, \end{cases}$$

and define the associated process p as in example 1.3. Verify that p is indeed a process, cartesian, full, transitive, linear, self-adjoint (for this also see 21°), but without local existence or local unicity.

The remaining exercises are concerned with the differential process p in R^n associated with a differential equation $\mathrm{d}x/\mathrm{d}\theta = f(x, \theta)$, under assumptions as in 1.3. Then necessary and sufficient conditions on f for p to be of the special types investigated in the present section may be obtained from 3.15.2°; or, possibly more satisfactorily, from the fundamental lemma in the manner indicated in 3.15.2°.

15° p admits period τ iff $f(x, \theta)$ is τ-periodic in θ.

16° p is stationary iff $f(x, \theta)$ is independent of θ.

17° x_0 is a critical point (with p stationary) iff $f(x_0, \theta) = 0$; p is r-symmetric (for differentiable r) iff

$$f(rx, \theta) = (\mathrm{d}_x rx, f(x, \theta))\,,$$

where d_x is the differential operator as in 2.15.5°.

18° p is r-antisymmetric (for differentiable r) iff

$$-f(rx, -\theta) = (\mathrm{d}_x rx, f(x, \theta))\,.$$

19° p is additive iff $f(x, \theta)$ is linear in x.

20° Let p and q be additive differential processes in R^n, associated with

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = A(\theta)\,x\,, \quad \frac{\mathrm{d}x}{\mathrm{d}\theta} = B(\theta)\,x$$

respectively. Then p, q are adjoint relative to the pseudoscalar product defined by a non-singular matrix U, i.e. $[x, y] = (x, Uy)$, iff $B(\theta) = -UA^{\mathrm{T}}(\theta)\,U^{-1}$; in particular for the natural scalar product $(\ ,\ )$ one has the usual condition $B(\theta) = -A^{\mathrm{T}}(\theta)$.

21° Verify that the process of 1.21.14° is stationary with discrete time. Show that the correspondence thus set up is an iso-functor between graphs (and edge-preserving maps) and stationary processes over C^1 (and admissible carrier maps). (Hint: interpret ${}_1p_0$ as the set of edges.)

22° Let G be a separated topological group, p an additive process in G over an R closed in R^1, and assume that all solutions of p are continuous. Prove that, for each $\alpha \in B$, either $\delta(0, \alpha) = \alpha$ or $\delta(0, \alpha) = +\infty$. (Hint: apply the remark in 1.19 and 4.12 (7).)

APPENDIX

The purpose of this section is only to introduce, informally, some possibly not quite standard notation and conventions.

Euclidean n-space is denoted by R^n, the set of real integers by C^1. The cartesian product of sets X, Y is denoted by $X \times Y$; the notation for the natural projections is as in

$$\mathrm{proj}_1 : X \times Y \to X\,, \quad \mathrm{proj}_2 : X \times Y \to Y\,.$$

The obvious natural maps between

$$(X \times Y) \times Z\,, \quad X \times (Y \times Z)\,, \quad X \times Y \times Z$$

will all be termed *transfers*; e.g. R^{n+m} is obtained by a suitable transfer from $\mathsf{R}^n \times \mathsf{R}^m$.

A *relation* between sets X and Y is merely a subset of $X \times Y$ (thus we do not distinguish between relations and their graphs); a relation on X is a relation between X and X. For relations one may then use much of the notation and concepts carried over from sets (also see [4], chap. X, § 2); however, if the elements x, y are in a relation r, we prefer to write $x\ r\ y$ instead of $(x, y) \in r$. Thus there is relation-inclusion as a partial ordering, with $r \subset r'$ iff always $x\ r\ y$ implies $x\ r'\ y$; and also joins and meets of relations

$$r \cup r' \quad \text{and} \quad \bigcup r_i\,; \quad r \cap r' \quad \text{and} \quad \bigcap r_i\,.$$

Between given sets there is a least relation 0 and a greatest relation I. The relation inverse to r is denoted by r^{-1}; thus $x\ r\ y$ iff $y\ r^{-1}\ x$. If the set X is obvious from the context or given in advance, the identity relation on X is denoted by 1: thus $x\ 1\ y$ iff $x = y \in X$. For some pairs of relations r, r' the composition $r \circ r'$ is defined (with $x\ r \circ r'\ y$ iff $x\ r\ u$ and $u\ r'\ y$ for some u) in such a manner as to form a category.

Let r be a relation between X and Y, and r' a relation between X' and Y'. The cartesian (or rather direct) *product* $r \times r'$ is the relation between $X \times X'$ and $Y \times Y'$ defined in the obvious manner: $(x, x')\ r \times r'\ (y, y')$ iff $x\ r\ y$ and $x'\ r'\ y'$. If $Y = Y'$

in this situation, then one may also define the *relational product* (r, r'); this is the relation between $X \times X'$ and Y with $(x, x')(r, r')\, y$ iff $x\, r\, y$ and $x'\, r'\, y$.

Next, all maps can be represented, canonically, by relations: with each map $f : X \to Y$ one associates, in a one-to-one manner, the relation denoted again by f between Y and X (in this order), and defined by $y\, f\, x$ iff $y = fx$. The reason for this choice is that then the natural composition of maps corresponds to the natural composition of the corresponding relations.

Several concepts commonly applied to maps are then carried over to relations by analogy. Thus, if r is a relation between P and Q, then for each subset X of Q one defines

$$r[X] = \{y \in P : y\, r\, x \quad \text{for some} \quad x \in X\}\,, \quad r \mid X = r \cap (P \times X)\,,$$

the *image* of X under r and the *partialization* of r to X, respectively. Also

$$\text{range}\, r = r[Q] = \text{proj}_1\, [r]\,, \quad \text{domain}\, r = r^{-1}[P]\, \text{proj}_2\, [r]\,.$$

In the same situation, the relation r will be termed a *partial map* iff $r \circ r^{-1} \subset 1$ (this is usefully abbreviated to: partial map $r : Q \to P$). Next, r is a multiple-valued map iff $1 \subset r^{-1} \circ r$ (or equivalently, domain $r = Q$); one-to-one iff $r^{-1} \circ r \subset 1$ (or equivalently, r^{-1} is a partial map); onto iff $1 \subset r \circ r^{-1}$ (or equivalently, range $r = P$, or r^{-1} is multiple-valued).

Lemma. *Let r be a relation between partially ordered sets; if r is isotone in the sense that*

$$x\, r\, y\,, \quad x'\, r\, y' \quad \text{and} \quad y \geqq y' \quad \text{imply} \quad x \geqq x'\,,$$

then r is a partial map.
(This is trivial: $x\, r\, y$ and $x'\, r\, y$ imply, via $y \geqq y$, that $x \geqq x'$ and symmetrically $x' \geqq x$.)

Finally, a relation r on a set is termed reflexive iff $1 \subset r$, symmetric iff $r = r^{-1}$, transitive iff $r \circ r \subset r$. The symmetrization of r is then $r^{\mathsf{S}} = r \cup r^{-1}$, the transitivization of r is $r^{\mathsf{T}} = r \cup (r \circ r) \cup (r \circ r \circ r) \cup \ldots$; easily, $r^{\mathsf{ST}} = r^{\mathsf{TS}}$. It may be useful to emphasise that the composition operation is $\circ$ even for partial maps; that the value of a partial map at an element is written as fx, with round parentheses used primarily to comply with the usual conventions on precedence.

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Резюме

ТЕОРИЯ ПРОЦЕССОВ I

ОТОМАР ГАЕК (Otomar Hájek), Прага

Настоящая работа — первая из серии работ по аксиоматической теории дифференциальных уравнений. Здесь сформулированы основные определения, элементарыне свойства и фундаментальные интерпретации. Следующие работы относятся к некоторым конструкциям процессов, напр., наименьшей нижней грани семейства процессов как абстрактной форме регулированной системы, к категорическим конструкциям (произведение процессов, фактор-процесс, итд.); далее, к непрерывным процессам и вопросам введения топологии в семействе процессов — абстрактная форма понятия возмущения системы; и, наконец к некоторым вопросам теории линейных процессов.

Пусть даны множество состояний P и область определения переменного времени $R \subset \mathsf{R}^1$; тогда процесс p в P над R определяется как отношение в $P \times R$, т.е. между парами (x, α) и (y, β) элементов множества $P \times R$, такое, что 0° $(x, \alpha)\, p\, (y, \beta)$, если только $\alpha \geqq \beta$, 1° из $(x, \alpha)\, p\, (y, \beta)$ при $\alpha = \beta$ следует $x = y$ (начальное условие), 2° p транзитивно, и, наоборот, из $(x, \alpha)\, p\, (z, \gamma)$ следует для любого $\beta \in R$ между α и γ существование $y \in P$ с $(x, \alpha)\, p\, (y, \beta)\, p\, (z, \gamma)$ (условие композитивности).

Дифференциальное уравнение $\mathrm{d}x/\mathrm{d}\theta = f(x, \theta)$ в пространстве R^n отождествляется с процессом p в R^n над R^1, определенным так: полагаем $(x, \alpha)\, p\, (y, \beta)$, если существует решение s данного уравнения с $x = s\alpha$, $y = s\beta$. Аналогичным образом определяются процессы, присоединеные к дифференциальным уравнениям с разрывными правыми частьми, к дифф. уравнениям в пространствах Банаха, к уравнениям в контингентах, к дифф. уравнениям, содержащим регуляторы, к обобщенным решениям дифф. уравнений. Для дифф. уравнений с запаздывающим аргументом (и вообще, к функционально-дифференциальным уравнениям с запаздыванием) строится соответствующий процесс в функциональном пространстве. (См. задачи к п. 1 и 2.)

На абстрактные процессы перенесены в п. 1 некоторые основные понятия из теории дифф. уравнений, напр., определены решения процесса, единственность, итд. Система всех решений процесса изучена абстрактно в п. 2. В п. 3 введена категория процессов, и ее морфизмы интерпретированы как аналоги трансформаций дифф. уравнений; также наблюдается влияние морфизмов на единственность и продолжаемость. В п. 4 указаны специальные классы процессов, соответствующие, напр., стационарным (автономным) системам, линейным дифф. системам, дифф. системам с симметрией.