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EIGENVALUES OF OPERATORS IN L_p-SPACES IN MARKOV CHAINS WITH A GENERAL STATE SPACE

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1. INTRODUCTION AND NOTATION

The present paper is essentially an appendix to the preceding papers [3], [4]. All relevant definitions and assertions which are necessary for our present development may be found in detail in [4]; however, for the reader's convenience, some of them are also briefly listed here in Sections 1 and 2.

Let X be a general abstract space of points x, with a Borel σ-field $\mathscr{X}$ of subsets in it. Consider a (*sub-stochastic*) *transition function* p, that is a function $p = p(.\,,.)$ of two variables $x \in X$ and $A \in \mathscr{X}$ satisfying:

(i) $p(x, .)$ is a σ-additive non-negative measure on $\mathscr{X}$ for each $x \in X$, and $p(x, X) \leqq 1$,
(ii) $p(., A)$ is an $\mathscr{X}$-measurable function on X for each $A \in \mathscr{X}$.

Further, p is called a *stochastic transition function* if $p(x, X) = 1$ for each $x \in X$. The iterates $p^{(n)}$ of p are defined as usually by

$$p^{(n)}(x, A) = \int_X p^{(n-1)}(y, A)\, p(x, \mathrm{d}y)\,, \quad \text{with} \quad p^{(1)} = p\,.$$

Throughout the whole paper we shall assume that the transition function p is *irreducible*, which means that all of the measures

$$\nu_x = \sum_{n=1}^{\infty} 2^{-n} p^{(n)}(x, .)$$

on $\mathscr{X}$ have, for all $x \in X$, the same null sets.

Furthermore, we shall suppose that we have some *sub-invariant measure* μ for p, i.e. some σ-additive non-negative σ-finite measure μ on $\mathscr{X}$, which is not identically zero, and which satisfies

$$\int_X p(x, A)\, \mu(\mathrm{d}x) \leqq \mu(A) \quad \text{for all} \quad A \in \mathscr{X}\,.$$

Moreover, in the whole paper we assume that this μ has the same null sets as the measures ν_x. (This assumption causes no loss of generality, see [4].)

By the space $L_\alpha(\mu)$, for $1 \leqq \alpha < \infty$, we understand the well-known Banach space of all complex-valued $\mathcal{X}$-measurable functions f on X integrable in their α-th power with respect to the measure μ, the norm being given by $\|f\|_\alpha = [\int_X |f(x)|^\alpha \mu(\mathrm{d}x)]^{1/\alpha}$. Similarly, $L_\infty(\mu)$ is the Banach space of all complex-valued $\mathcal{X}$-measurable μ-essentially bounded f on X, with the norm $\|f\|_\infty = \operatorname{ess\,sup}_{\mu\ X} |f(x)|$. The notation $f \not\equiv 0$ will be used for the fact that $\mu(\{x; f(x) \neq 0\}) > 0$.

In the present paper we deal with the operator T_α defined in the space $L_\alpha(\mu)$, $1 \leqq \alpha \leqq \infty$, by the formula

$$T_\alpha f = \int_X f(y)\, p(., \mathrm{d}y)\,.$$

It is well known that T_α is a linear continuous operator in $L_\alpha(\mu)$ with the norm $\|T_\alpha\|_\alpha \leqq 1$, whenever μ is a sub-invariant measure for p. (Note that the form of the operators T_α is the same for all α, the index α being used only for distinguishing the Banach spaces in which they act.) It is also immediately seen that

$$T_\alpha^n f = \int_X f(y)\, p^{(n)}(., \mathrm{d}y)\,, \quad n = 1, 2, \ldots\,.$$

Finally note for clarity that by an *eigenvalue of* T_α on the unit circle we mean a complex number λ such that $|\lambda| = 1$ and $T_\alpha f = \lambda f$ μ-almost everywhere for some $f \not\equiv 0$, $f \in L_\alpha(\mu)$.

The purpose of the present paper is to find the eigenvalues of the operators T_α on the unit circle for different types of transition functions p. The results and methods are analogous to those in the previous papers [2] and [1] but, of course, they are much more general.

2. KNOWN PRELIMINARIES

Recall that in [4] (see also [3]) we have shown that an irreducible transition function p with a sub-invariant measure μ belongs precisely to one of the following types: either $\sum_{n=1}^\infty p^{(n)}(x, A) = \infty$ for each $A \in \mathcal{X}$ such that $\mu(A) > 0$ and each x (p is then called *recurrent*), or $\sum_{n=1}^\infty p^{(n)}(x, A) < \infty$ for each $A \in \mathcal{X}$ such that $\mu(A) < \infty$ and μ-almost all x (p is *transient*). Further, a recurrent p belongs precisely to one of the following types: either

$$\text{(1)} \qquad \lim_{n\to\infty} n^{-1} \sum_{m=1}^n p^{(m)}(x, A)$$

exists and is positive for each x and each $A \in \mathcal{X}$ such that $\mu(A) > 0$ (p is called *positive-recurrent*), or the limit (1) is zero for each x and each $A \in \mathcal{X}$ such that $\mu(A) < \infty$ (p is *null-recurrent*).

In the whole present paper we shall assume that the following two conditions are satisfied.

Condition CD (cyclic decomposition). There exists a decomposition of X into $d + 1$ disjoint subsets $C_0, C_1, \ldots, C_{d-1}, D$ from $\mathscr{X}$ such that $\mu(D) = 0$, and $p(x, X - C_{j+1}) = 0$ for each $x \in C_j, j = 0, 1, \ldots, d - 1$ (where we also put $C_d = C_0$).

Condition PS (positivity for the same n). If $A_1, A_2 \subset C_j$ for some j, $A_1, A_2 \in \mathscr{X}$, and $\mu(A_1) > 0$, $\mu(A_2) > 0$, then for each x there exists some $n = n(x)$ such that $p^{(n)}(x, A_1) > 0$, $p^{(n)}(x, A_2) > 0$.

Now define the functions e_k, $k = 0, 1, \ldots, d - 1$, by

$$(2) \qquad \begin{aligned} e_k(x) &= e^{2\pi i jk/d} \quad \text{for} \quad x \in C_j\,, \quad j = 0, 1, \ldots, d - 1\,, \\ &= 0 \qquad\qquad \text{for} \quad x \in D\,. \end{aligned}$$

(In particular, $e_0(x) = 1$ for $x \in X - D$.)

Let us now recall several known results, which will be useful for our future development.

Lemma 1. *If the function $e_k \in L_\alpha(\mu)$ and p is stochastic, then*

$$(3) \qquad T_\alpha e_k = e^{2\pi i k/d} e_k \quad \mu\text{-almost everywhere}\,.$$

This lemma appears as Lemma 6 in [4].

Lemma 2. *For any complex $\mathscr{X}$-measurable function f on X we have, for all x and all $n = 1, 2, \ldots$,*

$$(4) \qquad \left|\int_X f(y)\, p^{(n)}(x, \mathrm{d}y)\right|^2 \leqq p^{(n)}(x, X) \int_X |f(y)|^2\, p^{(n)}(x, \mathrm{d}y)\,,$$

provided the integrals involved exist. If in (4) the case of equality occurs for some x and all $n = 1, 2, \ldots$, then f is constant μ-almost everywhere on each C_j, $j = 0, 1, \ldots$, $\ldots, d - 1$.

This lemma appears as Lemma 2 in [4].

Lemma 3. *If p is a recurrent transition function, then it is stochastic.*

This is an immediate consequence of Theorem 1 in [4].

Lemma 4. *If p is a positive-recurrent transition function, then $\mu(X) < \infty$.*

This assertion is a part of Corollary 2 in [4].

Lemma 5. *If p is either null-recurrent, or transient and such that $p(x, X) = 1$ for μ-almost all x, then $\mu(X) = \infty$.*

This lemma coincides with Theorem 10 in [4].

Lemma 6. *Let h be an $\mathscr{X}$-measurable function on X such that $h(x) \geqq 0$ and $\int_X h(y)\, p(x, \mathrm{d}y) \leqq h(x)$ for μ-almost all x. Then either $h(x) > 0$ for μ-almost all x or $h(x) = 0$ for μ-almost all x.*

Proof. If the last assertion of the lemma is not true, then $h(y) > 0$ for all y belonging to some set $A \in \mathscr{X}$ having the measure $\mu(A) > 0$. By our constant assumption on μ and ν_x, we have also $\nu_x(A) > 0$ for each x. This gives, for each x, the existence of some $n = n(x)$ such that $p^{(n)}(x, A) > 0$. Hence, for μ-almost all x,

$$h(x) \geqq \int_X h(y)\, p(x, \mathrm{d}y) \geqq \ldots \geqq \int_X h(y)\, p^{(n)}(x, \mathrm{d}y) > 0\,.$$

Lemma 7. *If $h \leqq T_\alpha h$ and $h \geqq 0$ μ-almost everywhere, where $h \in L_\alpha(\mu)$, $1 \leqq \alpha < \infty$, then $T_\alpha h = h$ μ-almost everywhere.*

Proof. Obviously, we obtain

$$\|h\|_\alpha^\alpha = \int_X [h(x)]^\alpha\, \mu(\mathrm{d}x) \leqq \int_X [(T_\alpha h)(x)]^\alpha\, \mu(\mathrm{d}x) = \|T_\alpha h\|_\alpha^\alpha \leqq \|h\|_\alpha^\alpha\,.$$

Since the two extreme terms here coincide, also the second and the third terms must be equal, which gives the desired conclusion.

Lemma 8. *If $T_\alpha h = h$ μ-almost everywhere for some function $h \in L_\alpha(\mu)$, $1 \leqq \alpha < \infty$, then h is constant μ-almost everywhere.*

Proof. Clearly it is sufficient to give the proof only for h real. Let $f^{(a)}$ be the function on X identically equal to a non-negative constant a. Then $\int_X f^{(a)}(y)\, p(x, \mathrm{d}y) = a p(x, X) \leqq f^{(a)}(x)$ for all x. Setting $g = h - f^{(a)}$ we have $g(x) \leqq h(x)$ and $\int_X g(y)\, p(x, \mathrm{d}y) \geqq g(x)$ for μ-almost all x. Finally, denote by g^+ the function defined by $g^+(x) = g(x)$ whenever $g(x) \geqq 0$, and by $g^+(x) = 0$ whenever $g(x) < 0$. It follows that $0 \leqq g^+(x) \leqq |h(x)|$ for all x, so that $g^+ \in L_\alpha(\mu)$, and it is easy to verify that $g^+(x) \leqq \int_X g^+(y)\, p(x, \mathrm{d}y)$ for μ-almost all x. Lemma 7 now implies $g^+(x) = \int_X g^+(y)\, p(x, \mathrm{d}y)$ for μ-almost all x, and by Lemma 6 either $g^+(x) = 0$ for μ-almost all x or $g^+(x) > 0$ for μ-almost all x. The first case yields, for μ-almost all x, the inequalities $g(x) \leqq 0$, $h(x) - f^{(a)}(x) \leqq 0$, $h(x) \leqq a$. The second case yields, for μ-almost all x, $g(x) > 0$, $h(x) - f^{(a)}(x) > 0$, $h(x) > a$. On choosing first $a = 0$ we see that the function h is either non-positive or positive, μ-almost everywhere. However, if h is positive it must be constant μ-almost everywhere, since a is an arbitrary non-negative number; if h is non-positive it suffices to change h into $-h$.

Lemma 9. *Let* $\int_X f(y)\, p(x, \mathrm{d}y) = \lambda f(x)$ *for* μ*-almost all* x*, where* $|\lambda| = 1$*,* f *is* $\mathscr{X}$*-measurable, and* $|f(x)| = c \neq 0$ *for* μ*-almost all* x*. Then*

(*a*) $p(x, X) = 1$ *for* μ*-almost all* x*,*

(*b*) $\lambda^d = 1$*, i.e.* $\lambda = e^{2\pi ik/d}$ *for some* $k = 0, 1, \ldots, d - 1$*,*

(*c*) $f(x) = c_0\, e_k(x)$ *for* μ*-almost all* x*, with* e_k *being the function introduced in* (2) *and* c_0 *some constant.*

Proof. First, by our assumption we obtain easily that also

$$(5) \qquad \int_X f(y)\, p^{(n)}(x, \mathrm{d}y) = \lambda^n f(x) \quad \text{for} \quad n = 1, 2, \ldots, \quad \text{and} \quad \mu\text{-almost all } x\,.$$

Hence, if x is such that $|f(x)| = c$, we get by (5) and Lemma 2

$$(6) \qquad c^2 = |f(x)|^2 = |\lambda^n f(x)|^2 = \left|\int_X f(y)\, p^{(n)}(x, \mathrm{d}y)\right|^2 \leqq$$

$$\leqq p^{(n)}(x, X) \int_X |f(y)|^2\, p^{(n)}(x, \mathrm{d}y) = c^2 [p^{(n)}(x, X)]^2 \leqq c^2\,.$$

Since the two extreme terms in (6) coincide, all terms here must be equal. Therefore $p^{(n)}(x, X) = 1$; in particular, for $n = 1$, $p(x, X) = 1$. Thus, since $|f(x)| = c$ for μ-almost all x, the assertion (a) is proved.

Further, since we have equalities in (6), we get by Lemma 2 that f is constant μ-almost everywhere on each C_j, $j = 0, 1, \ldots, d - 1$. In other words, there exist some constants $c_0, c_1, \ldots, c_{d-1}$ such that

$$(7) \qquad f(x) = c_j \quad \text{for} \quad \mu\text{-almost all} \quad x \in C_j\,.$$

Taking some $x \in C_j$ for which the last equality holds and for which $p^{(d)}(x, X) = 1$ (which is possible in view of (a)), we obtain, using (5) for $n = d$, that

$$\lambda^d c_j = \lambda^d f(x) = \int_X f(y)\, p^{(d)}(x, \mathrm{d}y) = c_j \int_X p^{(d)}(x, \mathrm{d}y) = c_j\,,$$

which gives the assertion (b).

Finally, the assertion (c) is obtained easily from (5) for $n = 1, 2, \ldots, d - 1$, taking into account (7), (a), and (b).

Theorem 1. *Let the transition function* p *be positive-recurrent. Then the set of all eigenvalues of the operator* T_α $(1 \leqq \alpha < \infty)$ *on the unit circle consists precisely of the numbers* $e^{2\pi ik/d}$*,* $k = 0, 1, \ldots, d - 1$*, and every eigenfunction* $f \in L_\alpha(\mu)$ *for which*

$$(8) \qquad T_\alpha f = e^{2\pi ik/d} f \quad \mu\text{-almost everywhere}$$

is equal μ*-almost everywhere to some multiple of the function* e_k*.*

Proof. First, by Lemma 4, $e_k \in L_\alpha(\mu)$. Hence, by Lemma 3 and Lemma 1, each of the numbers $e^{2\pi ik/d}$ is an eigenvalue of T_α.

For the proof of the opposite assertion let us assume that $T_\alpha f = \lambda f$ μ-almost everywhere, where $|\lambda| = 1$, $f \not\equiv 0$, $f \in L_\alpha(\mu)$. Now, denoting by $|f|$ the function whose value at the point x is $|f(x)|$, we obtain $|f| = |\lambda f| \leqq T_\alpha |f|$. Hence, Lemma 7 gives $T_\alpha |f| = |f|$, and, by Lemma 8, $|f|$ is constant μ-almost everywhere. Thus we may use Lemma 9, and the theorem follows.

Theorem 2. *Let the transition function p be null-recurrent or transient. Then the operator T_α $(1 \leqq \alpha < \infty)$ has no eigenvalues on the unit circle.*

Proof. Suppose, on the contrary, that $T_\alpha f = \lambda f$ μ-almost everywhere for some $f \in L_\alpha(\mu)$, $f \not\equiv 0$, $|\lambda| = 1$. Then $|f| = |\lambda f| \leqq T_\alpha |f|$, which gives, by Lemma 7, $T_\alpha |f| =$ $= |f|$, and, by Lemma 8, $|f|$ is equal μ-almost everywhere to some constant $c \neq 0$. Hence we may use Lemma 9(a), obtaining $p(x, X) = 1$ for μ-almost all x, which further shows, by Lemma 5, that $\mu(X) = \infty$. Thus $\int_X |f(x)|^\alpha \mu(dx) = c^\alpha \mu(X) = \infty$, but this contradicts the assumption $f \in L_\alpha(\mu)$.

4. EIGENVALUES OF T_∞

Lemma 10. *If the transition function p is recurrent, and if $h \leqq T_\infty h$ μ-almost everywhere, with h being some real function in $L_\infty(\mu)$, then $T_\infty h = h$ μ-almost everywhere.*

Proof. Setting $g = T_\infty h - h$, we have $g \in L_\infty(\mu)$, $g \geqq 0$, and $T_\infty h = h + g$. We obtain successively $T_\infty^2 h = T_\infty h + T_\infty g, \ldots, T_\infty^{n+1} h = T_\infty^n h + T_\infty^n g$. On adding these equalities we get

$$\sum_{r=1}^{n+1} T_\infty^r h = \sum_{r=0}^{n} T_\infty^r h + \sum_{r=0}^{n} T_\infty^r g\,,$$

that is

$$\sum_{r=0}^{n} T_\infty^r g = T_\infty^{n+1} h - h\,. \tag{9}$$

Consider now the set $N_k = \{y;\, g(y) \geqq k^{-1}\}$, k being a positive integer. We have

(10)

$$\sum_{r=0}^{n} (T_\infty^r g)(x) = \sum_{r=0}^{n} \int_X g(y)\, p^{(r)}(x, \mathrm{d}y) \geqq \sum_{r=0}^{n} \int_{N_k} g(y)\, p^{(r)}(x, \mathrm{d}y) \geqq k^{-1} \sum_{r=0}^{n} p^{(r)}(x, N_k)\,.$$

Therefore, by (10) and (9), we obtain

$$\sum_{r=0}^{n} p^{(r)}(x, N_k) \leqq k \Big\| \sum_{r=0}^{n} T_\infty^r g \Big\|_\infty \leqq k(\|T_\infty^{n+1} h\|_\infty + \|h\|_\infty) \leqq 2k \|h\|_\infty < \infty$$

for each positive integer n and for μ-almost all x, which gives

$$\sum_{r=0}^{\infty} p^{(r)}(x, N_k) < \infty \quad \text{for } \mu\text{-almost all } x .$$

However, since p is recurrent, this may occur only if $\mu(N_k) = 0$. Now, k was arbitrary, and hence $\mu(\{y; g(y) > 0\}) = \mu(\bigcup_{k=1}^{\infty} N_k) = 0$; this means that $g = 0$ μ-almost everywhere, and the assertion follows.

Lemma 11. *If the transition function p is recurrent, and if $T_\infty h = h$ μ-almost everywhere, with $h \in L_\infty(\mu)$, then h is constant μ-almost everywhere.*

The proof follows the same pattern as that of Lemma 8, only $L_\alpha(\mu)$ is replaced by $L_\infty(\mu)$, and Lemma 10 is used in place of Lemma 7.

Theorem 3. *Let the transition function p be recurrent. Then the set of all eigenvalues of the operator T_∞ on the unit circle consists precisely of the numbers $e^{2\pi ik/d}$, $k = 0, 1, \ldots, d - 1$, and every eigenfunction $f \in L_\infty(\mu)$ for which*

$$T_\infty f = e^{2\pi ik/d} f \quad \mu\text{-almost everywhere} \tag{11}$$

is equal μ-almost everywhere to some multiple of the function e_k.

The proof follows the same pattern as that of Theorem 1, only $L_\infty(\mu)$, Lemma 10 and Lemma 11 are used in place of $L_\alpha(\mu)$, Lemma 7, and Lemma 8, respectively.

Theorem 4. *Let the transition function p be transient and stochastic. Then each number $e^{2\pi ik/d}$, $k = 0, 1, \ldots, d - 1$, is an eigenvalue of the operator T_∞; namely,*

$$T_\infty e_k = e^{2\pi ik/d} e_k . \tag{12}$$

The proof is immediate by Lemma 1.

Example 1. Under the assumptions of Theorem 4, the operator T_∞ may have also other eigenvalues on the unit circle in addition to the eigenvalues $e^{2\pi ik/d}$, $k = 0, 1, \ldots, d - 1$. This is seen by the following example (given as Example 1 in [1]), even for a denumerable space X : Let $X = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, and let

$$p(j, j-1) = \tfrac{2}{3},\; p(j, j+1) = \tfrac{1}{3} \quad \text{for} \quad j < 0 ,$$
$$p(0, -1) = p(0, 0) = p(0, 1) = \tfrac{1}{3} ,$$
$$p(j, j-1) = \tfrac{1}{3},\; p(j, j+1) = \tfrac{2}{3} \quad \text{for} \quad j > 0 ,$$

$p(j, k) = 0$ otherwise. Puting $f(k) = 3(-1)^{|k|} - 2(-\frac{1}{2})^{|k|}$ for every integer k, we have $f \in L_\infty(\mu)$, $T_\infty f = -f$, so that -1 is an eigenvalue of T_∞, though $d = 1$.

Similarly, T_∞ may have also other eigenvectors associated to the eigenvalues

$e^{2\pi ik/d}$ in addition to the eigenvectors e_k, shown in (12). This may be seen again with the aid of the preceding transition function p. Namely, putting $g(k) = 2^{-k}$, $g(-k) = 2 - 2^{-k}$ for $k \geqq 0$, we have $g \in L_\infty(\mu)$ and $T_\infty g = g$, in addition to $T_\infty e_0 = e_0$. (See Example 2 in [1].)

Example 2. It is easy to find a sub-stochastic transition function, which is transient but not stochastic, such that the corresponding operator T_∞ has no eigenvalues on the unit circle. For example, choose some p such that $p(x, X) \leqq r$ for all x, where $r < 1$. Then it is immediately seen that $\|T_\infty\|_\infty \leqq r$ so that, by a well-known theorem, each eigenvalue λ of T_∞ satisfies $|\lambda| \leqq r$.

Example 3. On the other hand, if p is transient and not stochastic, the corresponding operator T_∞ may still have some eigenvalues on the unit circle; this may be seen by the following example. First, choose for each $n = 1, 2, \ldots$ some number b_n, $0 < b_n < 1$, such that the infinite product $\prod_{n=1}^{\infty} b_n = b$ exists, and $0 < b < 1$. (E.g., we may put $b_n = \exp[-1/n^2]$, so that $\prod_{n=1}^{\infty} b_n = \exp[-\sum_{n=1}^{\infty} 1/n^2] = \exp[-\pi^2/6]$.) Further, choose also a_n such that $0 < a_n < 1 - b_n$. Now, take $X = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, and define the transition function p by

$$p(0, 1) = p(0, -1) = \tfrac{1}{4},$$
$$p(n, n + 1) = p(-n, -n - 1) = b_n \quad \text{for} \quad n = 1, 2, \ldots,$$
$$p(n, n - 1) = p(-n, -n + 1) = a_n \quad \text{for} \quad n = 1, 2, \ldots,$$

$p(j, k) = 0$ otherwise. Note that, in particular, $p(x, X) < 1$ for all $x \in X$.

We shall now construct a function $f \in L_\infty(\mu)$ satisfying $T_\infty f = f$. First, setting $f(0) = 0$, $f(1) = 1$, $f(-1) = -1$, we have clearly $(T_\infty f)(0) = f(0)$. Further, $T_\infty f = f$ means, for $n = 1, 2, \ldots$,

$$f(n) = p(n, n + 1) f(n + 1) + p(n, n - 1) f(n - 1) = b_n f(n + 1) + a_n f(n - 1), \tag{13}$$

that is

$$f(n + 1) = \frac{1}{b_n} [f(n) - a_n f(n - 1)]. \tag{14}$$

Therefore, evidently, the values $f(n + 1)$, $n = 1, 2, \ldots$, can be computed successively from (14). Finally, put $f(-n) = -f(n)$ for $n = 2, 3, \ldots$

Now, we shall prove

$$f(n) > f(n - 1) \geqq 0 \quad \text{for} \quad n = 1, 2, \ldots \tag{15}$$

Clearly, these inequalities (15) are true for $n = 1$. Further, if (15) is true for some n, then (13) and (15) gives

$$f(n) \leqq b_n f(n+1) + (1 - b_n) f(n-1) < b_n f(n+1) + (1 - b_n) f(n),$$

that is $f(n) < f(n+1)$, which shows the validity of (15) in general. On the other hand, (14) and (15) entail, for $n = 1, 2, \ldots$,

$$(16) \quad f(n+1) \leqq \frac{f(n)}{b_n} \leqq \frac{f(n-1)}{b_n b_{n-1}} \leqq \ldots \leqq \frac{1}{b_n b_{n-1} \ldots b_1} \leqq \frac{1}{\prod_{n=1}^{\infty} b_n} = \frac{1}{b} < \infty .$$

Thus $0 \leqq f(n) \leqq b^{-1}$ for $n = 0, 1, 2, \ldots$, and, more generally, $-b^{-1} \leqq f(n) \leqq b^{-1}$ for all $n \in X$. Therefore, on gathering the results, we have $f \in L_\infty(\mu)$, $f \not\equiv 0$, $T_\infty f = f$, so that the number 1 is the eigenvalue of T_∞.

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Резюме

СОБСТВЕННЫЕ ЗНАЧЕНИЯ ОПЕРАТОРОВ В ПРОСТРАНСТВАХ L_p ДЛЯ ЦЕПЕЙ МАРКОВА С ПРОИЗВОЛЬНОЙ СИСТЕМОЙ СОСТОЯНИЙ

ЗБЫНЕК ШИДАК (Zbyněk Šidák), Прага

Рассматривается неприводимая субстохастическая переходная функция $p = p(x, A)$ в произвольном пространстве X состояний x, для которой существует субинвариантная мера μ. Обозначим через $L_\alpha(\mu)$ $(1 \leqq \alpha < \infty)$ пространство всех комплексных функций f на X, для которых $\|f\|_\alpha = [\int_X |f(x)|^\alpha \mu(\mathrm{d}x)]^{1/\alpha}$ конечна; $L_\infty(\mu)$ будет аналогичное пространство тех f, для которых $\|f\|_\infty = \operatorname{ess\,sup}_{\mu \; X} |f(x)|$ конечна. Определим оператор T_α $(1 \leqq \alpha \leqq \infty)$ в пространстве $L_\alpha(\mu)$ соотношением $T_\alpha f = \int_X f(y)\, p(., \mathrm{d}y)$.

При некоторых предположениях (тех же самых, как в [3], [4], но очень широких) доказывается: Для положительной возвратной p с периодом d множество всех собственных значений T_α $(1 \leqq \alpha < \infty)$ на единичной окружности совпадает с множеством $\{e^{2\pi ik/d};\ k = 0, 1, \ldots, d - 1\}$, и собственные подпространства, принадлежащие к этим значениям, одномерны; аналогичный результат верен для T_∞ и возвратной p. Для нулевой возвратной и для невозвратной p оператор T_α $(1 \leqq \alpha < \infty)$ не имеет никаких собственных значений на единичной окружности. Для невозвратной стохастической p все числа $e^{2\pi ik/d}$, $k = 0, 1, \ldots, \ldots, d - 1$, являются собственными значениями T_∞, и нельзя утверждать больше.