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A GENERALISATION OF EHRESMANN'S JETS*)

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In this remark I merely show that a natural generalization of the notion „s-jet" leads to natural non-trivial problems.

0. In the study of the differentiable maps $f : M^n \to M^m$, M^n and M^m being differentiable manifolds, the fundamental notion is that of the jet of a map. The set of maps $f, g, \ldots : M^n \to M^m$ such that $j_p^s(f) = j_p^s(g)$, $p \in M^n$ being a fixed point, is decomposed in equivalence classes, f and g belonging to the same class if and only if $j_p^{s+1}(f) = j_p^{s+1}(g)$. If M^m carries some „structure" it is possible to consider a more profound classification of the maps. By a structure I mean something like this: The p^r-velocity in M^n at $x \in M^n$ is an r-jet of R^p into M^n with the source 0 and the target x; let $T_p^r(M^n, x)$ be the set of p^r-velocities in M^n at x. Now, let W be an affine or vector bundle over M^n, $W(x)$ being the fiber ober $x \in M^n$. The structure is the set of maps $\varphi(x) : T_p^r(M^n, x) \to W(x)$. For example, the affine connection on M^n provides such a structure, W being the affine tangent bundle and $p = 1$.

Let us restrict ourselves to the very simple case $M^n = R^n$, $M^m = R^m$, $n \leqq m$. Let $f, g : R^n \to R^m$ be maps such that $j_0^s(f) = j_0^s(g)$ is an invertible jet with the source $0 \in R^n$ and the target $0 \in R^m$. Let $\tau^n \subset R^m$ be given by $(\mathrm{d}f)_0\,(R^n)$. Introducing the coordinates x^i $(i = 1, \ldots, n)$ in R^n and y^α $(\alpha = 1, \ldots, m)$ in R^m such that τ^n is given by $y^{n+1} = \ldots = y^m = 0$, our maps are given by

$$y^\alpha = f^\alpha(x^i)\,, \quad y^\alpha = g^\alpha(x^i)\,. \tag{0.1}$$

Consider the numbers

$$c^\alpha_{a_1 \ldots a_{s+1}} = \left(\frac{\partial^{s+1}(f^\alpha - g^\alpha)}{\partial x^{a_1} \ldots \partial x^{a_{s+1}}}\right)_0\,, \quad a_i = 1, \ldots, n\,. \tag{0.2}$$

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Let $v_1, \ldots, v_{s+1}$ be vectors in τ^n, the coordinates of v_i being $(v_i^1, \ldots, v_i^n, 0, \ldots, 0)$. Define the vector $v_1 * v_2 * \ldots * v_{s+1}$ by

$$(0.3) \qquad [v_1 * v_2 * \ldots * v_{s+1}]^\alpha = \sum_{a_i = 1,\ldots,n} c^\alpha_{a_1 \ldots a_{s+1}} v_1^{a_1} \ldots v_{s+1}^{a_{s+1}},$$

$[w]^\alpha$ being the coordinates of the vector w. This definition does not depend on the considered coordinate systems.

Let L be a linear subspace of R^m through 0. We say that f, g belong to the same $(s + 1)$-jet mod L if $v_1 * \ldots * v_{s+1} \in L$ for each $(s + 1)$-tuple $v_1, \ldots, v_{s+1} \in \tau^n$. If $L_0 = 0 \in R^m$, then $j_0^{s+1}(f) = j_0^{s+1}(g)$ mod L_0 is, of course, equivalent to $j_0^{s+1}(f) = j_0^{s+1}(g)$.

This notion is of some use in the theory of deformations of submanifolds of a manifold S endowed with a Lie group G which acts transitively on S. Let M_1, M_2 be two submanifolds of S, and $f : M_1 \to M_2$ be a diffeomorphism. Denote by $G(x)$ the isotropy group of the point $x \in S$, $\mathfrak{G}(x) \subset \mathfrak{G}$ being its Lie algebra; suppose $\dim \mathfrak{G}(x) = r$. Let $\mathfrak{G}^{(r)}$ be the manifold of r-dimensional subspaces of $\mathfrak{G}$, and consider the maps $\varphi_i : M_i \to \mathfrak{G}^{(r)}$ given by $\varphi_i(x) = \mathfrak{G}(x)$, $x \in V_i$. Let $M = \bigcup_{x \in S} \mathfrak{G}(x) \subset \mathfrak{G}^{(r)}$; each map $\gamma : S \to S$ given by $\gamma(x) = gx$, $g \in G$, provides a map $\Gamma : M \to M$ given by $\Gamma(\mathfrak{G}(x)) = \mathfrak{G}(gx)$. Denote by $\{\Gamma\}$ the set of such maps. We say that $f : M_1 \to M_2$ is the deformation of order r if, for each $x \in M_1$, there is an element $g_x \in G$ such that $j_x^r(\varphi_1) = j_x^r(\Gamma_x \varphi_2)$, $\Gamma_x \in \{\Gamma\}$ being induced by the map $\gamma(y) = g_x y$. It may well happen that, for some r, each diffeomorphism $f : M_1 \to M_2$ is the deformation of order r, however, f being the deformation of order $r + 1$, there is an element $g \in G$ such that $f(x) = g(x)$ for each $x \in M_1$. As the space $\mathfrak{G}^{(r)}$ has the structure of a vector space, we may apply the notion of our generalized jets to obtain non-trivial types of correspondences.

In what follows, I shall study two very simple examples of this general situation.

1. Let us consider two affine spaces A^n, A'^n and the vector spaces V^n, V'^n associated to them. Futher, let $M^r \subset A^n$, $M'^r \subset A'^n$ be two manifolds, and $f : \omega \to M'^r$ be a diffeomorphism of a neighborhood $\omega \subset M^r$ of a point $p \in M^r$. Denote by τ^r, τ'^r the tangent vector spaces of the manifolds M^r, M'^r at the points p and $f(p)$ resp.

Theorem 1. *Let us choose*

(1) *a diffeomorphism* $F : \Omega \to A'^n$, $\Omega \subset A^n$ *being a neighborhood of the point* p, *such that* $F|_{\Omega \cap \omega} = f$;

(2) *two vector fields* v, w *on* Ω *such that* $v_p, w_p \in \tau^r$. *The vector*

$$(1.1) \qquad v_p * w_p = [v, w']_p + [w, v']_p,$$

where

$$(1.2) \qquad v'_x = (\mathrm{d}F)_p^{-1} (dF)_x v_x, \quad w'_x = (\mathrm{d}F)_p^{-1} (\mathrm{d}F)_x w_x \quad \text{for} \quad x \in \Omega,$$

depends only on $j^2_p(f)$, $(\mathrm{d}F)_p$, v_p *and* w_p. *We have*

$$(1.3)\qquad v_p * w_p = w_p * v_p\,,\quad v_p * (\alpha w_p + \alpha' w'_p) = \alpha\,.\,v_p * w_p + \alpha'\,.\,v_p * w'_p$$
$$\text{for}\quad v_p, w_p, w'_p \in \tau^r\,;\quad \alpha, \alpha' \in R\,.$$

Proof. Choose the following ranges of indices

$$i, j, \ldots = 1, \ldots, n\,;\quad \alpha, \beta, \ldots = 1, \ldots, r\,;\quad A, B, \ldots = r+1, \ldots, n\,,$$

and use the summation convention.

In the spaces A^n and A'^n, let us choose the bases $M, J_1, \ldots, J_n$; $M', J'_1, \ldots, J'_n$ such that: (a) $p = M$, $f(p) = M'$; (b) $J_1, \ldots, J_r$ and $J'_1, \ldots, J'_r$ are the bases of τ^r and τ'^r resp.; (c) $(\mathrm{d}F)_p(z^i J_i) = z^i J'_i$ for each $z^1, \ldots, z^n \in R$. In some neighborhood of the point p, the manifold M^r is given parametrically by

$$(1.4)\qquad x^i = f^i(t^1, \ldots, t^r)\,.$$

Let us suppose that the point p corresponds to the values $t^1 = \ldots = t^r = 0$, i.e.

$$(1.5)\qquad (f^i)_0 = 0\,,\quad \left(\frac{\partial f^i}{\partial t^\alpha}\right)_0 = \delta^i_\alpha\,,$$

$(f^i)_0$ denoting $f^i(0, \ldots, 0)$, and δ^i_j being the Kronecker symbol. The other manifold M'^r and the map $f : \omega \to M'^r$ are given, at least locally, by the equations

$$(1.6)\qquad y^i = g^i(t^1, \ldots, t^r)$$

where

$$(1.7)\qquad (g^i)_0 = 0\,,\quad \left(\frac{\partial g^i}{\partial t^\alpha}\right)_0 = \delta^i_\alpha\,.$$

The map $F : \Omega \to A'^n$ be given by the equations

$$(1.8)\qquad y^i = h^i(x^1, \ldots, x^n)$$

with the obvious conditions

$$(1.9)\qquad (h^i)_0 = 0\,,\quad \left(\frac{\partial h^i}{\partial x^j}\right)_0 = \delta^i_j\,.$$

The condition $F = f$ on $\Omega \cap \omega$ is expressed by the identity

$$(1.10)\qquad g^i(t^1, \ldots, t^r) = h^i(f^1(t^1, \ldots, t^r), \ldots, f^n(t^1, \ldots, t^r))$$

for small $|t^\alpha|$. Derivating both sides of (1.10), we get

$$\frac{\partial g^i}{\partial t^\alpha} = \frac{\partial h^i}{\partial x^j}\frac{\partial f^j}{\partial t^\alpha}\,,\quad \frac{\partial^2 g^i}{\partial t^\alpha\,\partial t^\beta} = \frac{\partial^2 h^i}{\partial x^j\,\partial x^k}\frac{\partial f^j}{\partial t^\alpha}\frac{\partial f^k}{\partial t^\beta} + \frac{\partial h^i}{\partial x^j}\frac{\partial^2 f^j}{\partial t^\alpha\,\partial t^\beta}\,,$$

i.e.

$$\left(\frac{\partial^2 g^i}{\partial t^\alpha\,\partial t^\beta}\right)_0 = \left(\frac{\partial^2 h^i}{\partial x^\alpha\,\partial x^\beta}\right)_0 + \left(\frac{\partial^2 f^i}{\partial t^\alpha\,\partial t^\beta}\right)_0 . \tag{1.11}$$

The vector field v on Ω be $v = v^i(x^1, \ldots, x^n)\, J_i$, i.e.

$$v = v^i(x^1, \ldots, x^n)\, \frac{\partial}{\partial x^i} . \tag{1.12}$$

We obtain

$$(\mathrm{d}F)_x\, v_x = v^i(x^1, \ldots, x^n)\, \frac{\partial h^j}{\partial x^i}\, \frac{\partial}{\partial y^j}, \quad v' = v^i(x^1, \ldots, x^n)\, \frac{\partial h^j}{\partial x^i}\, \frac{\partial}{\partial x^j} \tag{1.13}$$

and analoguos equations for the vector field $w = w^i(x', \ldots, x^n)\, J_i$. Further,

$$[v, w'] = v^i\, \frac{\partial w^j}{\partial x^i}\, \frac{\partial h^k}{\partial x^j}\, \frac{\partial}{\partial x^k} + v^i w^j\, \frac{\partial^2 h^k}{\partial x^i\,\partial x^j}\, \frac{\partial}{\partial x^k} - w^i\, \frac{\partial v^k}{\partial x^j}\, \frac{\partial h^j}{\partial x^i}\, \frac{\partial}{\partial x^k}, \tag{1.14}$$

$$[w, v'] = w^i\, \frac{\partial v^j}{\partial x^i}\, \frac{\partial h^k}{\partial x^j}\, \frac{\partial}{\partial x^k} + v^j w^i\, \frac{\partial^2 h^k}{\partial x^i\,\partial x^j}\, \frac{\partial}{\partial x^k} - v^i\, \frac{\partial w^k}{\partial x^j}\, \frac{\partial h^j}{\partial x^i}\, \frac{\partial}{\partial x^k},$$

and we get

$$[v, w']_p + [w, v']_p = 2(v^i)_0\, (w^j)_0 \left(\frac{\partial^2 h^k}{\partial x^i\,\partial x^j}\right)_0 \frac{\partial}{\partial x^k} \tag{1.15}$$

as a consequence of (1.9). According to the supposition $(v^A)_0 = (w^A)_0 = 0$ and (1.11), we have

$$v_p * w_p = 2(v^\alpha)_0\, (w^\beta)_0 \left\{\left(\frac{\partial^2 g^i}{\partial t^\alpha\,\partial t^\beta}\right)_0 - \left(\frac{\partial^2 f^i}{\partial t^\alpha\,\partial t^\beta}\right)_0\right\} J_i , \tag{1.16}$$

the validity of the equations (1.3) being easy to see. Q.E.D.

Let us write $*_{f,A}$, $A = (\mathrm{d}F)_p$, instead of $*$ if there is the possibility of confusion.

Theorem 2. *Be given manifolds M^r, N^r in A^n and M'^r, N'^r in A'^n. Let $p \in M^r$, $q \in N^r$ be fixed points and $\omega \subset M^r$, $\omega' \subset N^r$ neighborhoods of p and q resp. Be given diffeomorphisms $f : \omega \to M'^r$, $f' : \omega' \to N'^r$, $\varphi : \omega \to N^r$; $\varphi(p) = q$. Without loss of generality, we may restrict ourselves to the case $\omega' = \varphi(\omega)$, all considerations being local. Consider the map $\varphi' : f(\omega) \to f'(\omega')$ given by the commutative diagram*

$$\begin{array}{ccc} M^r \supset \omega & \xrightarrow{\quad f \quad} & f(\omega) \subset M'^r \\ \Big\downarrow \varphi & & \Big\downarrow \varphi' \\ N^r \supset \varphi(\omega) = \omega' & \xrightarrow[\quad f' \quad]{} & f'(\omega') \subset N'^r . \end{array}$$

Denote by $i : M^r \to M^r$, $i' = M'^r \to M'^r$ *the identity maps. Let us suppose*

$$j_p^2(i) = j_p^2(\varphi)\,, \quad j_{f(p)}^1(i') = j_{f(p)}^1(\varphi')\,. \tag{1.17}$$

Then

$$j_{f(p)}^2(i') = j_{f(p)}^2(\varphi') \tag{1.18}$$

implies

$$v_p *_{f,A} w_p = v_p *_{f',A} w_p \tag{1.19}$$

for each $v_p, w_p \in \tau^r$ *and each* $A : V^n \to V'^n$ *such that* $A|_{\tau^r} = (\mathrm{d}f)_p$. *If* (1.19) *is satisfied for each* $v_p, w_p \in \tau^r$ *and at least one* A, *we have* (1.18).

Proof. The proof follows directly from the explicit formula (1.16).

Theorem 3. *Let* $M^r \subset A^n$, $M'^r \subset A'^n$ *be manifolds and* $f : \omega \to M'^r$ *be a diffeomorphism,* $\omega \subset M^r$ *being a neighborhood of the point* $p \in M^r$. *Let* $A : V^n \to V'^n$ *be a non-singular linear transformation such that* $A|_{\tau^r} = (\mathrm{d}f)_p$, *and let* $0 \neq v_p \in \tau^r$ *be a fixed vector. The vector* $V = v_p *_{f,A} v_p$ *has the following geometrical signification:*

Let $\gamma : (-1, 1) \to M^r$ *be any curve through* p; *suppose e.g.,* $\gamma(0) = p$; *which is tangent to* v_p; *i.e. the vectors* v_p *and* $(\mathrm{d}\gamma)_0\,(1)$ *are linearly dependent. There is* $\varepsilon > 0$ *such that* $\gamma\{(-\varepsilon, \varepsilon)\} \subset \omega$. *Let us define the curve* $\gamma' : (-\varepsilon, \varepsilon) \to A^n$ *by the formula* $\gamma'(t) = (A^{-1}f\gamma)\,(t)$ *for* $t \in (-\varepsilon, \varepsilon)$. *Of course,* $j_0^1(\gamma) = j_0^1(\gamma')$. *There are three possible cases:*

A. $V = 0$. *Then* $j_0^2(\gamma) = j_0^2(\gamma')$.

B. $V \neq 0$, V *and* v_p *being linearly dependent. Then* $j_0^2(\gamma) \neq j_0^2(\gamma')$, *but there is a small number* ε_1, $0 < \varepsilon_1 < \varepsilon$, *and a diffeomorphism* $\delta : (-\varepsilon_1, \varepsilon_1) \to (-\varepsilon, \varepsilon)$ *such that* $j_0^2(\gamma) = j_0^2(\gamma'')$ *where* $\gamma''(t) = (\gamma'\delta)\,(t)$ *for* $t \in (-\varepsilon_1, \varepsilon_1)$.

C. V *and* v_p *are linearly indepedent. Then* $j_0^2(\gamma) \neq j_0^2(\gamma')$ *and there are no* ε_1 *and* δ *satisfying the condition* B. *Let* A^{n-1} *be any hyperplane in* A^n *which does not contain the vectors* V, v_p *in its vector space, and let* $\pi : A^n \to A^{n-1}$ *be the parallel projection in the direction* V. *Then* $j_0^2(\pi\gamma) = j_0^2(\pi\gamma')$.

Moreover, in the case B *there is no projection* π *satisfying the condition* C.

Proof. The proof of this theorem is more simple than its statement. Let us keep the notation of the proof of Theorem 1. The curve γ be given by

$$t^\alpha = c^\alpha(t)\,, \quad t \in (-1, 1)\,; \quad c^\alpha(0) = 0\,, \tag{1.20}$$

i.e., in the linear coordinates in A^n, by

$$x^i = f^i(c^1(t), \ldots, c^r(t)) \equiv F^i(t)\,. \tag{1.21}$$

The curve γ' is given by

$$x^i = g^i(c^1(t), \ldots, c^r(t)) \equiv G^i(t) . \tag{1.22}$$

Because of (1.5) and (1.7), we have

$$(F^i)_0 = (G^i)_0 , \quad \left(\frac{\mathrm{d}F^i}{\mathrm{d}t}\right)_0 = \delta^i_\alpha \left(\frac{\mathrm{d}c^\alpha}{\mathrm{d}t}\right)_0 = \left(\frac{\mathrm{d}G^i}{\mathrm{d}t}\right)_0 ,$$

i.e. $j^1_0(\gamma) = j^1_0(\gamma')$. Of course,

$$v_p = \varrho \left(\frac{\mathrm{d}c^\alpha}{\mathrm{d}t}\right)_0 J_\alpha , \quad V = 2\varrho^2 \left(\frac{\mathrm{d}c^\alpha}{\mathrm{d}t}\right)_0 \left(\frac{\mathrm{d}c^\beta}{\mathrm{d}t}\right)_0 \left(\frac{\partial^2(g^i - f^i)}{\partial t^\alpha \, \partial t^\beta}\right)_0 J_i , \tag{1.23}$$

$\varrho \neq 0$ being a real number. From

$$\frac{\partial^2 F^i}{\partial t^2} = \frac{\partial^2 f^i}{\partial t^\alpha \, \partial t^\beta} \frac{\mathrm{d}c^\alpha}{\mathrm{d}t} \frac{\mathrm{d}c^\beta}{\mathrm{d}t} + \frac{\mathrm{d}f^i}{\mathrm{d}t^\alpha} \frac{\mathrm{d}^2 c^\alpha}{\mathrm{d}t^2}$$

and the similar equation for $\mathrm{d}^2 G^i/\mathrm{d}t^2$, we obtain

$$\left(\frac{\mathrm{d}^2(G^i - F^i)}{\mathrm{d}t^2}\right)_0 J_i = \frac{1}{2\varrho^2} V . \tag{1.24}$$

If $V = 0$, we have $(\mathrm{d}^2 G^i/\mathrm{d}t^2)_0 = (\mathrm{d}^2 F^i/\mathrm{d}t^2)_0$ for each i, and A is proved. Now, let us consider the case B, i.e.

$$V = \sigma \left(\frac{\mathrm{d}G^i}{\mathrm{d}t}\right)_0 J_i , \quad 0 \neq \sigma \in R . \tag{1.25}$$

Let $\delta = \delta(t)$ be an arbitrary function which is defined for $t \in (-\varepsilon_1, \varepsilon_1)$ and is such that $\delta\{(-\varepsilon_1, \varepsilon_1)\} \subset (-\varepsilon, \varepsilon)$ and

$$\delta(0) = 0 , \quad \left(\frac{\mathrm{d}\delta}{\mathrm{d}t}\right)_0 = 1 , \quad \left(\frac{\mathrm{d}^2\delta}{\mathrm{d}t^2}\right)_0 = -\frac{\sigma}{2\varrho^2} .$$

The curve $\gamma'' = \gamma'\delta$ is given by $H^i(t) = G^i(\delta(t))$, and we have

$$(H^i)_0 = (G^i)_0 , \quad \left(\frac{\mathrm{d}H^i}{\mathrm{d}t}\right)_0 = \left(\frac{\mathrm{d}G^i}{\mathrm{d}t}\right)_0 , \quad \left(\frac{\mathrm{d}^2 H^i}{\mathrm{d}t^2}\right)_0 = \left(\frac{\mathrm{d}^2 G^i}{\mathrm{d}t^2}\right)_0 - \frac{\sigma}{2\varrho^2} \left(\frac{\mathrm{d}G^i}{\mathrm{d}t}\right)_0 , \tag{1.26}$$

i.e.

$$\left(\frac{\mathrm{d}^2 H^i}{\mathrm{d}t^2}\right)_0 = \left(\frac{\mathrm{d}^2 F^i}{\mathrm{d}t^2}\right)_0 ,$$

substituting (1.25) and (1.26_3) into (1.24). The case C is obvious from (1.24).

2. The goal of this paragraph is merely to show a utilization of our $*$-multiplication which may lead to natural non-trivial problems in areas which are considered to be "known".

Let S be the set of surfaces M in A^3 such that at each point $p \in M$ there are exactly two asymptotic tangents. Let $f : M \to M'$; $M, M' \in S$; be a diffeomorphism, and denote by $\tau(p)$ the tangent plane of M at $p \in M$. The map f is called the μ_i-*deformation* $(i = 1, 2, 3)$ if for each point $p \in M$ there is a linear transformation $C_p : V^3(A^3) \to V^3(A^3)$ such that $C_p\tau(p) = (\mathrm{d}f)_p$ and $*_{f,c_p}(\tau(p))$ (1) = trivial zero-vector space; (2) = one-dimensional tangent vector space at p; (3) = an asymptotic vector space at p. Here, $V^3(A^3)$ denotes the vector space associated to A^3, and $*(L)$ is the set of all vectors $l_1 * l_2$; $l_1, l_2 \in L$.

Theorem 4. (1) *If $f : M \to M'$ is a μ_1-deformation* (i.e. a deformation of second order), *the surfaces M, M' are equal up to an affine collineation of A^3.* (2) *Let $M \in S$ be given. The couples (f, M') such that $f : M \to M'$ is a μ_2-deformation exist and depend of five functions of one variable.* (3) *The triplets (f, M, M') such that $f : M \to M'$ is a μ_3-deformation exist and depend on seven functions of one variable.*

In (2) and (3), we suppose that M and M' are not equal up to an affine collineation. The generality is to be understood in the terms of Cartan-Kuranishi's theory of systems in involution.

Proof. Associating to each point $p \in M$ the frame A, J_1, J_2, J_3 such that $A = p$ and J_1, J_2 are tangent vectors, we may write (at least locally)

$$(2.1)\qquad \mathrm{d}A = \omega^1 J_1 + \omega^2 J_2\,,\qquad \mathrm{d}J_2 = \omega_2^1 J_1 + \omega_2^2 J_2 + \omega_2^3 J_3\,,$$
$$\mathrm{d}J_1 = \omega_1^1 J_1 + \omega_1^2 J_2 + \omega_1^3 J_3\,,\qquad \mathrm{d}J_3 = \omega_3^1 J_1 + \omega_3^2 J_2 + \omega_3^3 J_3$$

with the integrability conditions

$$(2.2)\qquad \mathrm{d}\omega^i = \omega^j \wedge \omega_j^i\,,\quad \mathrm{d}\omega_i^j = \omega_i^k \wedge \omega_k^j\,;\quad i, j = 1, \ldots, 3\,.$$

Our surface is given by the equation

$$(2.3)\qquad \omega^3 = 0$$

with the integrability conditions

$$(2.4)\qquad \omega_1^3 = \alpha\omega^1 + \beta\omega^2\,,\quad \omega_2^3 = \beta\omega^1 + \gamma\omega^2\,.$$

The vectors J_1, J_2 being asymptotic, we may choose the frames in such a way that

$$(2.5)\qquad \omega_1^3 = \omega^2\,,\quad \omega_2^3 = \omega^1\,,$$

the integrability conditions being

$$(2.6)\qquad 2\omega_1^2 \wedge \omega^1 + (\omega_1^1 + \omega_2^2 - \omega_3^3) \wedge \omega^2 = 0\,,$$
$$(\omega_1^1 + \omega_2^2 - \omega_3^3) \wedge \omega^1 + 2\omega_2^1 \wedge \omega^2 = 0\,.$$

The surface M' be given by the equations (23'), (2.5') and the diffeomorphism f by

$$\tau^1 = 0\,, \quad \tau^2 = 0\,; \tag{2.7}$$

we use the notation

$$\tau^i = \omega^i - \omega'^i\,, \quad \tau_i^j = \omega_i^j - \omega_i'^j\,. \tag{2.8}$$

The differential df being now given by

$$(df)(x^1 J_1 + x^2 J_2) = x^1 J_1' + x^2 J_2'\,,$$

let $C : V^3 \to V^3$ be given by $C(x^i J_i) = x^i J_i'$. We have

$$\begin{aligned} 2\,.\,(\omega^1 J_1 + \omega^2 J_2) *_{f,c} (\omega^1 J_1 + \omega^2 J_2) = \\ = (\tau_1^1 \omega^1 + \tau_2^1 \omega^2)\, J_1 + (\tau_1^2 \omega^1 + \tau_2^2 \omega^2)\, J_2 + (\tau_1^3 \omega^1 + \tau_2^3 \omega^2)\, J_3\,, \end{aligned} \tag{2.9}$$

this equation being deduced from the expression $C\,d^2 A - d^2 A'$ following the proof of Theorem 3. Finally, from (2.7) and the obvious equation $\tau^3 = 0$ we get

$$\tau_1^i = a^i \omega^1 + b^i \omega^2\,, \quad \tau_2^i = b^i \omega^1 + c^i \omega^2\,; \quad i = 1, 2, 3\,. \tag{2.10}$$

(1) The triplets (f, M, M') such that $f : M \to M'$ is a μ_1-deformation are given by the equations (2.3), (2.5), (2.3'), (2.7) and

$$\tau_1^1 = \tau_2^1 = \tau_1^2 = \tau_2^2 = \tau_1^3 = \tau_2^3 = 0 \tag{2.11}$$

with the integrability conditions (2.6) and

$$\begin{aligned} \omega^1 \wedge \tau_3^1 = \omega^2 \wedge \tau_3^1 = 0\,, \quad \omega^1 \wedge \tau_3^2 = \omega^2 \wedge \tau_3^2 = 0\,, \\ \omega^1 \wedge \tau_3^3 = \omega^2 \wedge \tau_3^3 = 0\,. \end{aligned} \tag{2.12}$$

From (2.12), we obtain $\tau_3^1 = \tau_3^2 = \tau_3^3 = 0$, and the surfaces M, M' are equal up to an affine collineation $A^3 \to A^3$, the systems (2.1) and (2.1') being equal.

(2) Let $M \in S$ be given, i.e. the left-hand side forms in (2.3) and (2.4) are known. The couples (f, M') such that $f : M \to M'$ is a μ_2-deformation and $*_{f,c}(\tau) = (.)\, J_1$ are given by the system (2.3'), (2.7) and

$$\tau_1^2 = \tau_2^2 = \tau_1^3 = \tau_2^3 = 0 \tag{2.13}$$

with the integrability conditions

$$\begin{aligned} \omega^1 \wedge \tau_1^1 + \omega^2 \wedge \tau_2^1 = 0\,, \quad \omega_1^2 \wedge \tau_1^1 - (\alpha\omega^1 + \beta\omega^2) \wedge \tau_3^2 = 0\,, \\ \tau_2^1 \wedge \omega_1^2 + (\beta\omega^1 + \gamma\omega^2) \wedge \tau_3^2 = 0\,, \quad (\alpha\omega^1 + \beta\omega^2) \wedge (\tau_3^3 - \tau_1^1) = 0\,, \\ (\beta\omega^1 + \gamma\omega^2) \wedge \tau_3^3 + \tau_2^1 \wedge (\alpha\omega^1 + \beta\omega^2) = 0\,. \end{aligned} \tag{2.14}$$

This system is in involution, the determinant of the polar matrix being equal to

$$\omega_1^3(\omega^1\omega_1^3 + \omega^2\omega_2^3)(\tau_1^1\omega_2^3 - \omega_1^3\tau_2^1).$$

(3) The triplets (f, M, M') such that $f : M \to M'$ is a μ_3-deformation and $*_{f,c}(\tau) = (.)\, J_1$, J_1 being an asymptotic vector, are given by (2.3), (2.5), (2.3'), (2.7) and (2.13) with the integrability conditions (2.6) and

$$\begin{gathered}\omega^1 \wedge \tau_1^1 + \omega^2 \wedge \tau_2^1 = 0, \quad \omega_1^2 \wedge \tau_1^1 + \omega^2 \wedge \tau_3^2 = 0, \\ \tau_2^1 \wedge \omega_1^2 + \omega^1 \wedge \tau_3^2 = 0, \quad \omega^1 \wedge \tau_3^2 - \omega^2 \wedge \tau_2^1 = 0, \\ \omega^2 \wedge (\tau_3^3 - \tau_1^1) = 0.\end{gathered} \tag{2.15}$$

The determinant of the polar matrix is

$$2\omega^1(\omega^2)^4(\omega^1\tau_1^1 - \omega^2\tau_2^1),$$

and the system is in involution. Q.E.D.

3. In this paragraph, we shall describe the set of (so-called *special*) diffeomorphisms $f : \Omega \to A'^3$, $\Omega \subset A^3$, with this property: there is a vector field V on Ω such that $v *_{f,\mathrm{d}f} w = (.)\, V$ for any vector fields v, w on Ω. The $*$-multiplication being commutative, it is sufficient to replace the considered property with a weaker one: $v *_{f,\mathrm{d}f} v = (.)\, V$ for each vector field v on Ω. The only diffeomorphisms $f : \Omega \to A^3$ with $v *_{f,\mathrm{d}f} v = 0$ for each v on Ω being the affine collineations, we exclude them from further consideration.

To each point $p \in \Omega$, let us associate a frame A, J_1, J_2, J_3 in A^3 and A', J'_1, J'_2, J'_3 in A'^3 such that $A = p$, $A' = f(p)$ and $(\mathrm{d}f)_p(x^iJ_i) = x^iJ'_i$. Then we have the equations

$$\mathrm{d}A = \omega^i J_i, \quad \mathrm{d}J_i = \omega_i^j J_j; \quad \mathrm{d}A' = \omega'^i J'_i, \quad \mathrm{d}J'_i = \omega_i'^j J'_j \tag{3.1}$$

with the integrability conditions

$$\begin{gathered}\mathrm{d}\omega^i = \omega^j \wedge \omega_j^i, \quad \mathrm{d}\omega_i^j = \omega_i^k \wedge \omega_k^j; \\ \mathrm{d}\omega'^i = \omega'^j \wedge \omega_j'^i, \quad \mathrm{d}\omega_i'^j = \omega_i'^k \wedge \omega_k'^j,\end{gathered} \tag{3.2}$$

the map f being expressed – see (2.8) for notation – by

$$\tau^1 = \tau^2 = \tau^3 = 0. \tag{3.3}$$

Of course, we suppose

$$\omega^1 \wedge \omega^2 \wedge \omega^3 \neq 0. \tag{3.4}$$

It is easy to obtain – see (2.9) – the formula

$$(\omega^i J_i) *_{f,\mathrm{d}f} (\omega^i J_i) = \tfrac{1}{2}\tau_j^i \omega^j J_i. \tag{3.5}$$

Let f be special, and suppose that the vector field V coincides with J_3. The special diffeomorphisms f satisfy the system (3.3) and

$$\tau_1^1 = \tau_1^2 = \tau_2^1 = \tau_2^2 = \tau_3^1 = \tau_3^2 = 0 . \tag{3.6}$$

The integrability conditions of (3.3) and (3.6) are

$$\omega^1 \wedge \tau_1^3 + \omega^2 \wedge \tau_2^3 + \omega^3 \wedge \tau_3^3 = 0 , \tag{3.7}$$

$$\omega_3^1 \wedge \tau_1^3 = \omega_3^2 \wedge \tau_1^3 = \omega_3^1 \wedge \tau_2^3 = \omega_3^2 \wedge \tau_2^3 = \omega_3^1 \wedge \tau_3^3 = \omega_3^2 \wedge \tau_3^3 = 0 . \tag{3.8}$$

The map f being not an affine collineation, we have

$$\tau_i^3 \neq 0 \quad \text{for at least one} \quad i = 1, 2, 3 . \tag{3.9}$$

This assumption and (3.8) lead to

$$\omega_3^1 \wedge \omega_3^2 = 0 , \tag{3.10}$$

and there is a 1-form θ such that $\omega_3^1 = a_1\theta$, $\omega_3^2 = a_2\theta$. We have

$$\mathrm{d}J_3 = \theta(a_1 J_1 + a_2 J_2) + \omega_3^3 J_3 ,$$

and we may specialize the frames in such a way that

$$\omega_3^2 = 0 . \tag{3.11}$$

Further, we have $\mathrm{d}\omega_3^1 = \omega_3^1 \wedge (\omega_1^1 - \omega_3^3)$, and the case $\omega_3^1 = 0$ is geometrically significant. Let us introduce the following types of correspondences:

Type I is given by the system (3.3), (3.6) and

$$\omega_3^1 = \omega_3^2 = 0 ; \tag{3.12}$$

Type II is given by the system (3.3), (3.6) and

$$\omega_3^2 = 0 , \quad \omega_3^1 \neq 0 . \tag{3.13}$$

First of all, let us determine the correspondences of type I. For $\omega^1 = \omega^2 = 0$, we have

$$\mathrm{d}A = \omega^3 J_3 , \quad \mathrm{d}J_3 = (\omega_3^3)_{\omega^1 = \omega^2 = 0} \, J_3 ,$$

$$\mathrm{d}A' = \omega^3 J'_3 , \quad \mathrm{d}J'_3 = (\omega_3^3 + \tau_3^3)_{\omega^1 = \omega^2 = 0} \, J'_3 ,$$

i.e., both the points A, A' run along a line. It is easy to see

Theorem 5. *The special diffeomorphisms $f : \Omega \to A'^3$ of type* I *are given, in suitable coordinate systems in A^3 and A'^3, by*

$$x' = ax + by , \quad y' = cx + ey , \quad z' = \varphi(x, y, z) . \tag{3.14}$$

Let us consider type II. From (3.8), we obtain

$$\tau_1^3 = \alpha_1 \omega_3^1 , \quad \tau_2^3 = \alpha_2 \omega_3^1 , \quad \tau_3^3 = \alpha_3 \omega_3^1 , \tag{3.15}$$

and we have $\alpha_1 = \alpha_2 = \alpha_3 = 0$ only for affine colineations. Further, $\omega_3^1 \wedge \mathrm{d}\omega_3^1 = 0$, and such a function t exists (at least locally) that $\omega_3^1 \wedge \mathrm{d}t = 0$. $\mathrm{d}f$ being the differential of our map, let us introduce the affine collineations $C = C(p) : A^3 \to A'^3$, $p \in \Omega$, by $CA = A'$, $CJ_i = J'_i = (\mathrm{d}f)(J_i)$. We get $\mathrm{d}C \,.\, A = 0, \mathrm{d}C \,.\, J_1 = \alpha_1 \omega_3^1 J'_3$, $\mathrm{d}C \,.\, J_2 = \alpha_2 \omega_3^1 J'_3$, $\mathrm{d}C \,.\, J_3 = \alpha_3 \omega_3^1 J'_3$. The affine collineations C depend on t only, and if the point A runs through the plane given by $\alpha_1 \xi^1 + \alpha_2 \xi^2 + \alpha_3 \xi^3 = 0$ in the local coordinates $A + \xi^1 J_1 + \xi^2 J_2 + \xi^3 J_3$, C is fixed. Roughly speaking, we get two one-parametric systems of planes $\alpha(t)$, $\alpha'(t)$ in a 1-1-correspondence in A^3 and A'^3 resp., and our diffeomorphism is the union of ∞^1 affine collineations between the couples of corresponding planes. This being done, we have only to consider the possibilities for the structure of the families $\alpha(t)$, $\alpha'(t)$ and the structure of the family $C = C(t) : \alpha(t) \to \alpha'(t)$. To obtain the precise statement, let us introduce the following sets of homeomorphisms $f : \tilde{\Omega} \to A^3$, $\tilde{\Omega} \subset \tilde{A}^3$ (we present only the types of these maps; in each case, we must establish the conditions for the functions in the formulas in order to obtain really a diffeomorphism – not a map only; this is left to the reader):

(a) $f \in \Phi_1$ is given by

$$x = \varphi_1(w) + u \frac{\mathrm{d}\varphi_1(w)}{\mathrm{d}w} + v \frac{\mathrm{d}^2\varphi_1(w)}{\mathrm{d}w^2} , \tag{3.16}$$

$$y = \varphi_2(w) + u \frac{\mathrm{d}\varphi_2(w)}{\mathrm{d}w} + v \frac{\mathrm{d}^2\varphi_2(w)}{\mathrm{d}w^2} , \quad z = \varphi_3(w) + u \frac{\mathrm{d}\varphi_3(w)}{\mathrm{d}w} + v \frac{\mathrm{d}^2\varphi_3(w)}{\mathrm{d}w^2} ;$$

(b) $f \in \Phi_2$ is given by

(3.17)

$$x = u\, \varphi_1(w) + v \frac{\mathrm{d}\varphi_1(w)}{\mathrm{d}w} , \quad y = u\, \varphi_2(w) + v \frac{\mathrm{d}\varphi_2(w)}{\mathrm{d}w} , \quad z = u\, \varphi_3(w) + v \frac{\mathrm{d}\varphi_3(w)}{\mathrm{d}w} ;$$

(c) $f \in \Phi_3$ is given by

$$x = \varphi_1(w) + u + v \frac{\mathrm{d}\varphi_1(w)}{\mathrm{d}w} , \tag{3.18}$$

$$y = \varphi_2(w) + v \frac{\mathrm{d}\varphi_2(w)}{\mathrm{d}w} , \quad z = \varphi_3(w) + v \frac{\mathrm{d}\varphi_3(w)}{\mathrm{d}w} ;$$

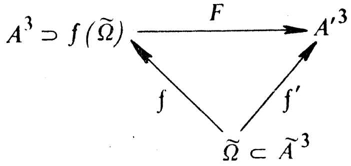
(d) $f \in \Phi_4$ is given by

$$x = u + v\, \varphi_1(w) , \quad y = v\, \varphi_2(w) , \quad z = v\, \varphi_3(w) ; \tag{3.19}$$

(e) $f \in \Phi_5$ is given by

$$x = u + \varphi_1(w) , \quad y = v + \varphi_2(w) , \quad z = \varphi_3(w) . \tag{3.20}$$

Theorem 6. *The special diffeomorphisms* $F : \Omega \to A'^3$, $\Omega \subset A^3$, *of type* II *are constructed as follows: Take two diffeomorphisms* $f : \tilde{\Omega} \to A^3$, $f' : \tilde{\Omega} \to A'^3$; $\tilde{\Omega} \subset \tilde{A}^3$; $f \in \Phi_i$, $f' \in \Phi_j$ (the values of i, j being specified below) *such that the map* $F : f(\tilde{\Omega}) \to A'^3$ *given by the commutative diagram*



be a non-linear diffeomorphism. The diffeomorphism F is special of type II *in the following cases*: (1) $f \in \Phi_1, f' \in \Phi_1$; (2) $f \in \Phi_2, f' \in \Phi_2$; (3) $f \in \Phi_3, f' \in \Phi_3$; (4) $f \in \Phi_3$, $f' \in \Phi_2$; (5) $f \in \Phi_4, f' \in \Phi_4$; (6) $f \in \Phi_5, f' \in \Phi_5$; (7) $f \in \Phi_5, f' \in \Phi_4$.

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Резюме

ОБОБЩЕНИЯ ИНФИНИТЕЗИМАЛЬНЫХ РОСТКОВ ЭРЕСМАНА

АЛОИС ШВЕЦ (Alois Švec), Прага

Для отображения $f : M \to N$; M, N – многообразия в A^n; дается обобщение линеаризирующего соответствия Вилла-Чеха. Решаются некоторие проблемы существования специальных отображений $f : M^2 \to N^2$ в случае $n = 3$ и отображений $f : A^3 \to A^3$.