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CARTAN'S METHOD OF SPECIALIZATION OF FRAMES*)

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PREFACE

The general theory of spaces with connection is now well known. The theory of pseudogroups has been also thoroughly studied. These two theories are based on the papers of E. CARTAN. Nevertheless, a great part of Cartan's results has not been revised, namely the theory of submanifolds of spaces with connection. Cartan introduced as the main tool of his studies his method of the specialization of the frames leading to the complete solution of the equivalence problem in the following sense: Be given a Lie group G, its Lie subgroup H, the homogeneous space G/H, a manifold M with $\dim M < \dim G/H$ and two embeddings $V, W: M \to G/H$; we have to decide whether there is an element $g \in G$ such that $V = gW$. This problem may be formulated more generally replacing the space G/H by a principal fibre bundle with a connection. In the classical differential geometry, the equivalence problem is often solved by means of the specialization of the frames (Frenet formulas for a curve etc.), but the general description of this procedure is given only in E. Cartan's papers in a rather unsatisfactory manner. Many Cartan's results are devoted to the theory of deformations which has been substantially completed by his successors. Nevertheless, the general definition of the deformation remains unclear. In many papers on local differential geometry, the frames are specialized (roughly speaking) as follows. Geometrically, i.e. intuitively, we estimate the equations of the considered submanifold, and we differentiate them quite precisely. But then: instead of ω's, we write some e's, and, according to some customs, some functions are said to be equal to one or to zero. In existence questions, we calculate the rangs of some matrices, and we declare that the investigated manifolds depend on A functions of B variables. It is necessary to say that the dependence of a solution on these functions has been made precise by KURANISHI; nevertheless, the theory of systems of exterior equations on the principal fibre bundles of frames is not quite clear. The worst thing,

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however, is to say that some function may be made equal to some constant by a suitable change of the frames assuming certain global properties of solutions of differentiable equations given only locally.

In this paper, I am very far of solving of all mentioned problems. I merely present a theory of spaces with Cartan's connection, define generally the developments of the curves and the notion of the deformation, and, in the last chapter, I try to give a more clear description of the specialization of the frames. The first chapter is devoted to an example, namely the theory of surfaces in 3-dimensional affine spaces treated in a more general way.

1. SURFACES IN AFFINE SPACES

1.1. Let us consider the n-dimensional affine space A^n. The *frame*

$$F = (M, e_1, \ldots, e_n) \tag{1.1}$$

of this space is an ordered set consisting of a point M and n linearly independent vectors e_i; the set of all frames may be made into an $(n + n^2)$-dimensional manifold $\mathscr{F}$. Let us denote by $\pi' : \mathscr{F} \to A^n$ the map given by

$$\pi'(M, e_1, \ldots, e_n) = M\,. \tag{1.2}$$

Further, consider the group $GA(n, \mathbf{R}) \subset GL(n + 1, \mathbf{R})$, its elements being the matrices of the form

$$A = \begin{pmatrix} 1 & 0 \\ \alpha & a \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ \alpha^1 & a^1_1 & a^1_2 & \ldots & a^1_n \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \alpha^n & a^n_1 & a^n_2 & \ldots & a^n_n \end{pmatrix}, \quad a \in GL(n, \mathbf{R}),\ \alpha^i \in \mathbf{R}\,. \tag{1.3}$$

The group $GA(n, \mathbf{R})$ is called the *real affine group*. The group $GA(n, \mathbf{R})$ acts freely on $\mathscr{F}$ on the right according to the rule

$$F' = FA\,, \tag{1.4}$$

FA being the usual product of the matrices F and A.

In the group $GA(n, \mathbf{R})$, we have two important subgroups. One of them, $GA_0(n, \mathbf{R})$, consists of the elements of the from

$$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \quad a \in GL(n, \mathbf{R})\,; \tag{1.5}$$

it is isomorphic to the group $GL(n, \mathbf{R})$. The second one consists of the matrices

$$\begin{pmatrix} 1 & 0 \\ \alpha & e_0 \end{pmatrix}, \quad e_0 \in GL(n, \mathbf{R}) \text{ being the identity}\,. \tag{1.6}$$

This so-called group of translations $T(n, \mathbf{R})$ is isomorphic to the additive group $\mathbf{R}^n$. The letter $\mathbf{R}$ in $GL(n, \mathbf{R})$ etc. will be often omitted.

Consider the sequence

$$e \to T(n) \xrightarrow{\alpha} GA(n) \xrightarrow{\beta} GA_0(n) \to e\,, \tag{1.7}$$

where e is the identity of the group $GA(n)$, α is the injection and

$$\beta \begin{pmatrix} 1 & 0 \\ \alpha & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}. \tag{1.8}$$

The sequence (1.7) is exact. $\gamma : GA_0(n) \to GA(n)$ being the natural injection, $\beta\gamma$ is the identical automorphism of the group $GA_0(n)$, and the sequence (1.7) admits the splitting. The group $GA(n)$ is the semidirect product of the groups $T(n)$ and $GA_0(n)$ in the following sense: To each element $A \in GA(n)$, there is a uniquely determined couple of elements $t \in T(n)$, $A_0 \in GA_0(n)$ such that

$$A = tA_0\,. \tag{1.9}$$

1.2. Be given an affine space A^n. Let D be a domain of the space $\mathbf{R}^m$, $m < n$. A *manifold* of the space A^n is an imbedding $V: D \to A^n$. We could consider an m-dimensional manifold instead of D, but our definition is quite sufficient because we are interested in the local theory only. Be given another manifold $W: D \to A^n$. Our problem is to determine whether the manifolds V and W are *equivalent*, i.e. to determine if there is an affine collineation $\mathscr{A} : A^n \to A^n$ such that the diagram

$$\begin{array}{ccc} & \overset{V}{\nearrow} & A^n \\ D & & \downarrow \mathscr{A} \\ & \underset{W}{\searrow} & A^n \end{array} \tag{1.10}$$

is commutative. First of all, we need to know how to determine the manifolds V and W. In the differential geometry, we use the following procedure (we restrict ourselves to the manifold V).

Let us consider the principal fibre bundle $P = A^n \times \mathscr{F}$ with the structure group $GA(n)$ acting by the rule

$$(M, F)\,.\,A = (M, FA)\,; \quad M \in A^n,\ F \in \mathscr{F},\ A \in GA(n)\,. \tag{1.11}$$

Let us denote by $\pi : P \to A^n$ the projection. We are going to construct the usual map $\varphi : P \to G$ such that

$$\varphi(pA) = \varphi(p)\,.\,A \quad \text{for each} \quad p \in P,\ A \in GA(n)\,. \tag{1.12}$$

Let $F_0 \in \mathscr{F}$ be a fixed frame of the space A^n, and consider the point $p = (M, F) \in P$. Then there is one and only one element $t \in T(n)$ such that

$$\pi'(F_0 t) = M . \tag{1.13}$$

Further, there is one and only one element $A \in GA(n)$ such that

$$F = F_0 t A . \tag{1.14}$$

We set

$$\varphi(M, F) = A , \tag{1.15}$$

the condition (1.12) being satisfied. The map φ depends on the frame F_0. Let us choose another frame F_1 such that

$$F_0 = F_1 t_1 B_1 \, ; \quad t_1 \in T(n), \ B_1 \in GA_0(n) . \tag{1.16}$$

We have $\pi'(F_1 t_1) = \pi'(F_0)$ and $\pi'(F_1 t_1 t) = \pi'(F_0 t) = M$. We may write

$$F = F_1 t_1 t A' \quad \text{and} \quad \varphi'(M, F) = A' . \tag{1.17}$$

From (1.14), (1.16) and (1.17), we obtain

$$A' = t^{-1} B_1 t A . \tag{1.18}$$

Let us return to our fixed frame F_0. Consider the reduction $Q \subset P$ to the group $GA_0(n)$ constructed as follows:

$$(M, F) \in Q \quad \text{if and only if} \quad \pi'(F) = M . \tag{1.19}$$

Let $q = (M, F) \in Q$. Then there is one and only one element $A \in GA(n)$ such that

$$F = F_0 A . \tag{1.20}$$

We have the uniquely determined decomposition

$$A = t A_0 \, ; \quad t \in T(n) , \quad A_0 \in GA_0(n) \, ; \tag{1.21}$$

of course, $\varphi(q) = A_0$. The elements of the matrix A are global coordinates on Q, elements of the matrix t are global coordinates on the base space A^n and the elements of the matrix A_0 are global coordinates in the fibre of the bundle $Q(A^n, GA_0(n))$ over the point $\pi'(F_0 t)$. We have also global coordinates on P, see (1.14). The elements of the matrix $t \in T(n)$ are the coordinates of the point $\pi'(F_0 t)$, the elements of the matrix $A \in GA(n)$ being the coordinates of the frame F in the fibre over the point $\pi'(F_0 t)$.

The Lie algebra $\mathfrak{ga}(n)$ of the group $GA(n)$ is isomorphic to the additive group of the matrices of the form

$$(1.22) \qquad r = \begin{pmatrix} 0 & 0 & \dots & 0 \\ r^1 & r_1^1 & \dots & r_n^1 \\ \vdots & \vdots & \dots & \vdots \\ r^n & r_1^n & \vdots & r_n^n \end{pmatrix},$$

where $[r, s] = rs - sr$. The subalgebras $\mathfrak{t}(n)$ and $\mathfrak{ga}_0(n)$ consist of the matrices of the form

$$(1.23) \qquad \begin{pmatrix} 0 & 0 & \dots & 0 \\ r^1 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ r^n & 0 & \dots & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & r_1^1 & \dots & r_n^1 \\ \vdots & \vdots & \dots & \vdots \\ 0 & r_1^n & \dots & r_n^n \end{pmatrix} \quad \text{resp.;}$$

of course,

$$(1.24) \qquad \mathfrak{ga}(n) = \mathfrak{ga}_0(n) \oplus \mathfrak{t}(n) .$$

In the equation (1.14), let us decompose the element A into the product of the elements of $T(n)$ and $GA_0(n)$, i.e. let us write

$$(1.25) \qquad F = F_0 t t_1 A_1 ; \quad t, t_1 \in T(n) , \quad A_1 \in GA_0(n) .$$

The frame F over the point $\pi'(F_0 t)$ belongs to Q if and only if $t_1 = e$. Let us consider the Lie algebra

$$(1.26) \qquad \mathfrak{l} = \mathfrak{ga}_0(n) \oplus \mathfrak{t}(n) \oplus \mathfrak{t}(n) .$$

On P, let us construct the l-forms

$$(1.27) \qquad \omega_1 = A_1^{-1} \, dA_1 , \quad \omega_2 = A_1^{-1} t_1^{-1} t^{-1} \, dt \, t_1 A_1 , \quad \omega_3 = A_1^{-1} t_1^{-1} \, dt_1 \, A_1 ,$$

ω_1 being $\mathfrak{ga}_0(n)$-valued, ω_2 and ω_3 being $\mathfrak{t}(n)$-valued. The form

$$(1.28) \qquad \omega^* = \omega_1 + \omega_2 + \omega_3$$

is an $\mathfrak{l}$-valued l-form on P. It is easy to prove

Theorem 1.1. *The form ω^* is a connection on P satisfying the structure equation*

$$(1.29) \qquad d\omega^* = -\omega^* \wedge \omega^* .$$

The restriction of ω^* to Q is the l-form with values in $\mathfrak{ga}_0(n) \oplus \mathfrak{t}(n) = \mathfrak{ga}(n)$, it is a so-called *Cartan's connection.* We have

$$(1.31) \qquad d\omega = -\omega \wedge \omega .$$

In the global coordinates on Q determined by the equation (1.20), we have

$$\omega = A^{-1}\,\mathrm{d}A\,, \tag{1.32}$$

and we may write $\mathrm{d}F = F_0\,\mathrm{d}A$, i.e.

$$\mathrm{d}F = F\omega\,. \tag{1.33}$$

Now, we are able to describe how to determine a manifold $V : D \to A^n$, this procedure being usual in the classical differential geometry. Let us choose a lift

$$v : D \to Q \tag{1.34}$$

of the map V, i.e. a map (1.34) such that the diagram

$$\begin{array}{ccc} & \overset{v}{\nearrow} & Q \\ D & & \downarrow \pi \\ & \underset{V}{\searrow} & A^n \end{array} \tag{1.35}$$

commutes. Let $\omega|_v$ be the restriction of the form ω to $v(D)$. The form

$$\omega_v = V_* \pi_* \omega|_v \tag{1.36}$$

is a $\mathfrak{ga}(n)$-valued 1-form on D satisfying the equation

$$\mathrm{d}\omega_v = -\omega_v \wedge \omega_v\,. \tag{1.37}$$

We may write ω_v as

$$\omega_v = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \omega_v^{\ 1} & \omega_{v1}^{\ 1} & \dots & \omega_{vn}^{\ 1} \\ \vdots & \vdots & \dots & \vdots \\ \omega_v^{\ n} & \omega_{v1}^{\ n} & \dots & \omega_{vn}^{\ n} \end{pmatrix}, \tag{1.38}$$

$\omega_v^{\ i}$ and $\omega_{vi}^{\ j}$ being real-valued 1-forms on D; because of the regularity of the map $V : D \to A^n$, there are m linearly independent forms among the forms $\omega_v^{\ 1}, \dots, \omega_v^{\ n}$, i.e.

(1.39) there are integers $i_1 < i_2 < \dots < i_m$; $i_k = 1, \dots, n$;
such that $\omega_v^{\ i_1} \wedge \omega_v^{\ i_2} \wedge \dots \wedge \omega_v^{\ i_m} \neq 0$.

The following existence theorem (stated here without proof) is fundamental.

Theorem 1.2. *Be given a domain* $D \subset R^m$ *and a* $\mathfrak{gl}(n)$*-valued* 1-form τ *on* D *satisfying the relations of the type* (1.37) *and* (1.39). *Further, be given points*

$u \in D$ and $q \in Q$. Then there is a neighbourhood $D' \subset D$ of the point u and a map $v : D' \to Q$ such that

$$(1.40) \qquad \omega_v = \tau\,, \quad v(u) = q\,.$$

The conditions (1.40) determine the map v uniquely.

Let us replace the lift (1.34) by another lift $v' : D \to Q$, and let us determine its form $\omega_{v'}$. Consider the map $A : D \to GA(n)$ such that the lift v is given by the relation

$$(1.41) \qquad F(u) = F_0\, A(u) \quad \text{for} \quad u \in D\,;$$

see (1.20). We have

$$(1.42) \qquad \omega_v = A^{-1}(u)\, \mathrm{d}A(u)\,.$$

Further, be given the map $B : D \to GA_0(n)$ in such a way that the lift v' is given by the equation

$$(1.43) \qquad F'(u) = F_0\, A(u)\, B(u) \quad \text{for} \quad u \in D\,.$$

Obviously,

$$(1.44) \qquad \omega_{v'} = B^{-1}(u)\, \omega_v\, B(u) + B^{-1}(u)\, \mathrm{d}B(u)\,.$$

This result is not surprising, ω being the form of a connection.

Let us state the following

Definition. Be given two manifolds $V, W : D \to A^n$. These manifolds are called equivalent if there are lifts $v, \mu : D \to Q$ of these maps V and W resp. such that we have

$$(1.45) \qquad \omega_v = \omega_\mu \quad \text{on} \quad D\,.$$

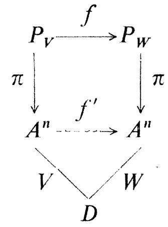
It is easy to prove the following two theorems.

Theorem 1.3. *Be given two manifolds $V, W : D \to A^n$ and arbitrary lifts $v, \mu : D \to Q$ of the maps V and W resp. The manifolds V, W are equivalent if and only if there is a map $B : D \to GA_0(n)$ such that*

$$\omega_\mu = B^{-1}(u)\, \omega_v\, B(u) + B^{-1}(u)\, \mathrm{d}B(u)\,, \quad u \in D\,.$$

Theorem 1.4. *Be given two manifolds $V, W : D \to A^n$. Let $P_V = V(D) \times \mathscr{F}$ be the restriction of the bundle P to the part $V(D)$ of the base space A^n; let P_W, Q_V, Q_W have analoguous significations. Let ω_V^* (ω_W^*) be the restriction of the form ω^* (1.28) to P_V (P_W). The manifolds V and W are equivalent if and only if there is* *a bundle isomorphism* $f : P_V \to P_W$ *with the following properties*: (1) *If the map* $f' : A^n \to A^n$ *is induced by the map* f, *the diagram*

(1.46)



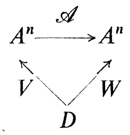
commutes. (2) $f(Q_V) = Q_W$. (3) *We have*

$$f_* \omega_W^* = \omega_W^* . \tag{1.47}$$

The notion of equivalence is geometrically obvious:

Theorem 1.5. *The manifolds* $V, W : D \to A^n$ *are equivalent if and only if there is an affine collineation* $\mathscr{A} : A^n \to A^n$ *such that the diagram*

(1.48)



is commutative.

1.3. In this section, let us speak more generally.

Definition. The space $\mathfrak{S} = \mathfrak{S}(P, M, G, Q, H, \omega)$ is a principal fibre bundle $P(M, G)$ with a given connection ω and a given reduction Q to the group $H \subset G$.

Definition. Be given two spaces

$$\mathfrak{S} = \mathfrak{S}(P, M, G, Q, H, \omega)\,, \quad \mathfrak{S}' = \mathfrak{S}'(P', M', G, Q', H, \omega')\,; \tag{1.49}$$

let M and M' be diffeomorphic. The diffeomorphism $f : M \to M'$ is called an equivalence if there is a lift $F : P \to P'$ of f such that (1) F is a fibre preserving isomorphism, (2) $F(Q) = Q'$, (3) $F_* \omega' = \omega$.

The main problem is to determine, for two given spaces $\mathfrak{S}$ and $\mathfrak{S}'$, all equivalences. Let us restrict ourselves to a less complicated problem: to determine if a given diffeomorphism is an equivalence. In the praxis (praxis = classical differential geometry), we have usually the following situation: Be given the spaces (1.49) and a diffeomorphism $f : M \to M'$. Further, be given the sections $\sigma : M \to Q$ and $\sigma' : M' \to Q'$. Let $\omega|_\sigma$ be the restriction of the form ω to $\sigma(M)$; let $\omega|_{\sigma'}$ have the analogous meaning. Knowing the $\mathfrak{g}$-valued 1-forms $\omega_\sigma = \sigma_* \omega\big|_\sigma$, $\omega_{\sigma'} = \sigma'_* \omega\big|_{\sigma'}$ on M and M' resp., we have to determine whether f is an equivalence.

It is quite clear that this problem may be reformulated analytically as follows: The diffeomorphism $f : M \to M'$ is an equivalence if and only if there is a map $h : M \to H$ such that

$$f_* \omega_{\sigma'} = \mathrm{ad}\,(h^{-1})\, \omega_\sigma + h^{-1}\, \mathrm{d}h\,. \tag{1.50}$$

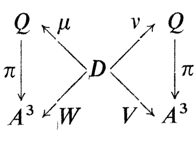
We have to solve the existence question for the map h.

The difficulties in solving this problem arise from the huge amount of possible maps $h : M \to H$. It would be very useful to restrict somewhat the possible candidates h. But this is exactly what is the subject of the Cartan's method of the specialization of the frames. This method may be described (very roughly speaking) as follows: Be given the spaces (1.49) and a diffeomorphism $f : M \to M'$. Succesively, we construct the reductions $Q \supset Q_1 \supset \ldots \supset Q_Z$, $Q' \supset Q'_1 \supset \ldots \supset Q'_Z$ of the bundles P and P' resp. to the groups $H \supset H_1 \supset \ldots \supset H_Z$ possessing the following property: f is an equivalence if and only if there is a lift $F_i : Q_i \to Q'_i$ of the map f such that (1) F_i is a fibre preserving isomorphism between the bundles $Q_i(M, H_i)$ and $Q'_i(M', H_i)$, (2) if ω_{Q_i} $(\omega_{Q'_i})$ is the restriction of the form ω (ω') to the manifold Q_i (Q'_i), we have $F_{i*}\omega_{Q'_i} = \omega_{Q_i}$. In the optimal case, $H_Z = e$ and Q_Z, Q'_Z are simply the sections of the bundles P and P' resp. In this case, we have just one bundle isomorphism $F_Z : Q_z \to Q'_Z$, and $F_{Z*}\omega_{Q_Z'} = \omega_{Q_Z}$ if and only if $f : M \to M'$ is an equivalence. To find out an effective solution of our problem, we have to solve the following one: Be given a section $\sigma : M \to Q$ of the bundle $P(M, G)$; we have to determine a map $h_1 : M \to H$ such that the section $\sigma(u)\, h_1(u)$; $u \in M$; is situated in Q_1. Later on, we shall see that we are able (speaking once more very roughly) to reduce this problem to a problem of the following type: Be given a Lie group G and its Lie subgroup H; $\mathfrak{g}$ and $\mathfrak{h}$ be the corresponding Lie algebras. In $\mathfrak{g}$, be given two linear subspaces K, L such that $\dim K = \dim L$ and $K, L \supset \mathfrak{h}$. We have to determine at least one solution of the equation $\mathrm{ad}\,(h)\, K = L$, $h \in H$, and all solutions of the equation $\mathrm{ad}\,(h)\, K = K$, $h \in H$.

1.4. Let us try to realize this program in a very simple situation: the local equivalence problem for surfaces in a 3-dimensional affine space.

First of all, let us formulate quite precisely our problem. Let A^3 be a 3-dimensional affine space; let us consider the previously introduced principal fibre bundle $P(A^3, GA(3))$ with the reduction Q (1.19) to the group $GA_0(3)$. Let $D \subset \mathbf{R}^2$ be a domain, and let $V, W : D \to A^3$ be two surfaces. Let us choose the lifts $\nu, \mu : D \to Q$ of these maps in such a way that the diagram

(1.51)



is commutative. Let ω^* be the connection form on P, let ω be its restriction to Q. We know the forms $\omega_\nu = \nu_* \omega|_{\nu(D)}$, $\omega_\mu = \mu_* \omega|_{\mu(D)}$; ω_ν and ω_μ are $\mathfrak{ga}(3)$-valued 1-forms on D satisfying the structure equations

$$\text{(1.52)} \qquad d\omega_\nu = -\omega_\nu \wedge \omega_\nu \,, \quad d\omega_\mu = -\omega_\mu \wedge \omega_\mu$$

and the conditions of the type (1.39). We have to decide whether the surfaces V and W are equivalent.

Let us consider, first of all, the surface $V: D \to A^3$ with the corresponding lift $\nu: D \to Q$. The form ω_ν is

$$\text{(1.53)} \qquad \omega_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega_\nu{}^1 & \omega_{\nu 1}^1 & \omega_{\nu 2}^1 & \omega_{\nu 3}^1 \\ \omega_\nu{}^2 & \omega_{\nu 1}^2 & \omega_{\nu 2}^2 & \omega_{\nu 3}^2 \\ \omega_\nu{}^3 & \omega_{\nu 1}^3 & \omega_{\nu 2}^3 & \omega_{\nu 3}^3 \end{pmatrix},$$

where $\omega_\nu{}^i$, $\omega_{\nu i}^j$ are $\mathbf{R}$-valued 1-forms on D. Without loss of generality, we may suppose that

$$\text{(1.54)} \qquad \omega_\nu{}^1 \wedge \omega_\nu{}^2 \neq 0 \,.$$

Let us denote by K the linear subspace of the Lie algebra $\mathfrak{ga}(3)$ spanned by the elements of the form

$$\text{(1.55)} \qquad r = \begin{pmatrix} 0 & 0 & 0 & 0 \\ r^1 & r_1^1 & r_2^1 & r_3^1 \\ r^2 & r_1^2 & r_2^2 & r_3^2 \\ 0 & r_1^3 & r_2^3 & r_3^3 \end{pmatrix}.$$

Let $H_1 \subset GA_0(3)$ be the group consisting of the elements of the form

$$\text{(1.56)} \qquad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_1^1 & a_2^1 & a_3^1 \\ 0 & a_1^2 & a_2^2 & a_3^2 \\ 0 & 0 & 0 & a_3^3 \end{pmatrix}, \quad a_3^3 \begin{vmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{vmatrix} \neq 0.$$

Let us denote by P_V (Q_V) the restriction of the bundle P (Q) to the base space $V(D)$. The lift $\varrho: D \to Q_V$ of the map $V: D \to A^3$ is called the *tangent lift* if the corresponding form ω_ϱ is K-valued. We have the following

Theorem 1.6. *There exists the reduction Q_1 of the bundle $Q_V(V(D), GA_0(3))$ to the group H_1 with this property: The lift $\varrho: D \to Q_V$ of the map $V: D \to A^3$ is tangent if and only if $\varrho(D) \subset Q_1$.*

Proof. First of all, let us produce a tangent lift. Consider the given lift $v : D \to Q_V$ with the corresponding form ω_v (1.53). A fixed frame F_0 of the space A^3 being given, there is a map $A : D \to GA(3)$ such that

$$v(u) = F_0 A(u) \quad \text{for} \quad u \in D . \tag{1.57}$$

Obviously,

$$\omega_v = A^{-1}(u)\, \mathrm{d}A(u) . \tag{1.58}$$

Be given a map $B : D \to GA_0(3)$. Then

$$\varrho(u) = v(u)\, B(u) = F_0\, A(u)\, B(u) \quad \text{for} \quad u \in D \tag{1.59}$$

is a lift $\varrho : D \to Q_V$; the corresponding form ω_ϱ is

$$\omega_\varrho = B^{-1}(u)\, \omega_v B(u) + B^{-1}(u)\, \mathrm{d}B(u) . \tag{1.60}$$

Of course, $B^{-1}(u)\, \mathrm{d}B(u) \in \mathfrak{ga}_0(3)$. If we use the notation

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b_1^1 & b_2^1 & b_3^1 \\ 0 & b_1^2 & b_2^2 & b_3^2 \\ 0 & b_1^3 & b_2^3 & b_3^3 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \tilde{b}_1^1 & \tilde{b}_2^1 & \tilde{b}_3^1 \\ 0 & \tilde{b}_1^2 & \tilde{b}_2^2 & \tilde{b}_3^2 \\ 0 & \tilde{b}_1^3 & \tilde{b}_2^3 & \tilde{b}_3^3 \end{pmatrix}, \tag{1.61}$$

we get

$$\omega_\varrho{}^3 = \tilde{b}_1^3 \omega_v{}^1 + \tilde{b}_2^3 \omega_v{}^2 + \tilde{b}_3^3 \omega_v{}^3 . \tag{1.62}$$

Because of (1.54), there are functions $\alpha, \beta : D \to \mathbf{R}$ such that

$$\omega_v{}^3 = \alpha \omega_v{}^1 + \beta \omega_v{}^2 . \tag{1.63}$$

Choosing $B(u)$ in such a way that the conditions

$$\tilde{b}_1^3 + \alpha \tilde{b}_3^3 = \tilde{b}_2^3 + \beta \tilde{b}_3^3 = 0 \tag{1.64}$$

are satisfied, ω_ϱ is K-valued, and this is the desired construction of a tangent lift.

Next, suppose that the lift ω_v is tangent, i.e. $\alpha, \beta : D \to 0 \in \mathbf{R}$. Then ω_ϱ is K-valued if and only if $\tilde{b}_1^3 = \tilde{b}_2^3 = 0$, i.e. $b_1^3 = b_2^3 = 0$, i.e. $B \in H_1$. Q.E.D.

Be given a surface $V : D \to A^3$ and its tangent lift $v : D \to Q_1$. Then we have $\omega_v^3 = 0$; the structure equation (1.52_1) yields

$$\mathrm{d}\omega_v{}^3 = {}^t\omega_v^{12} \wedge \omega_{v12} , \quad \text{where} \quad \omega_v^{12} = \begin{pmatrix} \omega_v{}^1 \\ \omega_v{}^2 \end{pmatrix}, \quad \omega_{v12} = \begin{pmatrix} \omega_{v1}^{\;3} \\ \omega_{v2}^{\;3} \end{pmatrix}. \tag{1.65}$$

Let us denote by $M(2)$ the set of symmetric 2×2 matrices. From the Cartan's lemma, it follows the existence of the map $S_v : D \to M(2)$ such that

$$\omega_{v12} = S_v \omega_v^{12} . \tag{1.66}$$

Let us determine the dependence of the function $s_v : D \to \mathbf{R}$,

$$s_v = \det S_v , \tag{1.67}$$

on the tangent lift v. Instead of the lift v, let us consider the lift ϱ (1.59) with

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b_1^1 & b_2^1 & b_3^1 \\ 0 & b_1^2 & b_2^2 & b_3^2 \\ 0 & 0 & 0 & b_3^3 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \tilde{b}_1^1 & \tilde{b}_2^1 & \tilde{b}_3^1 \\ 0 & \tilde{b}_1^2 & \tilde{b}_2^2 & \tilde{b}_3^2 \\ 0 & 0 & 0 & \tilde{b}_3^3 \end{pmatrix}, \tag{1.68}$$

$$\begin{pmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{pmatrix} = \beta \in GL(2), \quad \begin{pmatrix} \tilde{b}_1^1 & \tilde{b}_2^1 \\ \tilde{b}_1^2 & \tilde{b}_2^2 \end{pmatrix} = \beta^{-1}, \quad \tilde{b}_3^3 b_3^3 = 1 . \tag{1.69}$$

Using (1.60), we obtain

$$\omega_{\varrho 12} = \tilde{b}_3^3 \,.\, {}^t\beta \omega_{v12} , \quad \omega_v^{12} = \beta \omega_\varrho^{12} . \tag{1.70}$$

Substituting into the equation $\omega_{\varrho 12} = S_\varrho \omega_\varrho^{12}$, we get

$$S_\varrho = \tilde{b}_3^3 \,.\, {}^t\beta S_v \beta , \tag{1.71}$$

i.e.

$$s_\varrho = (\tilde{b}_3^3)^2 \,.\, (\det \beta)^2 \,.\, s_v . \tag{1.72}$$

Because of (1.71) and (1.72), we may formulate the following definitions: Be given a surface $V : D \to A^3$. The point $u \in D$ is called (1) *hyperbolic*, (2) *elliptic*, (3) *parabolic*, (4) *planar* if for an arbitrary lift $v : D \to Q_1$ (1) $s_v > 0$, (2) $s_v < 0$, (3) rang $S_v = 1$, (4) rang $S_v = 0$. In the following, we restrict ourselves to surfaces with the points of the same type; in this sense, we speak about (1) hyperbolic, (2) elliptic, (3) parabolic, (4) planar surfaces. If the surface has points of different types we could divide it into parts; however, this may lead to complications.

It is not very difficult to prove the following lemma: Let the surface $V : D \to A^3$ be (1) hyperbolic, (2) elliptic, (3) parabolic, (4) planar. Let us choose a tangent lift $v : D \to Q_1$. Then there are maps $\beta : D \to GL(2)$, $\tilde{b}_3^3 : D \to \mathbf{R}$ such that the matrix S_ϱ given by the equation (1.71) has the form

$$(1)\ S_\varrho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (2)\ S_\varrho = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.73}$$

$$(3)\ S_\varrho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (4)\ S_\varrho = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

All the solutions $\tilde{b}_3^3$, β of the equation $S_\varrho \approx \tilde{b}_3^3 \cdot {}^t\beta S_\varrho \beta$, S_ϱ being one of the matrices (1.73), are

(1.74) (1) $b_1^2 = b_2^1 = 0$, $(\tilde{b}_3^3)^{-1} = b_3^3 = b_1^1 b_2^2 \neq 0$; b_1^1, b_2^2 arbitrary

or

$b_1^1 = b_2^2 = 0$, $(b_1^2)^2 = (b_2^1)^2 = b_3^3$;

(2) $\beta = c\gamma$, where $0 \neq c \in \mathbf{R}$, $\gamma \in O(2)$; $b_3^3 = \pm c$;

(3) $b_2^1 = 0$, $b_3^3 = (b_1^1)^2$; b_1^1, b_1^2, b_2^2 arbitrary;

(4) b_3^3 and β arbitrary.

Let us consider the subgroups H_{2h}, H_{2l}, H_{2p} of the group H_1 formed by the elements B (1.68) satisfying (1), (2) or (3) of (1.74) resp. In the space $K \subset \mathfrak{ga}(3)$ (1.55), let us consider the subspace (1) K_{1h}, (2) K_{1l}, (3) K_{1p}, (4) K_{1pl} spanned by the elements of the form (1.55) where

(1.75) (1) $r_1^3 = r^2$, $r_2^3 = r^1$,

(2) $r_1^3 = r^1$, $r_2^3 = r^2$,

(3) $r_1^3 = r^1$, $r_2^3 = 0$,

(4) $r_1^3 = r_2^3 = 0$.

The tangent lift $v : D \to Q_1$ of a surface $V : D \to A^3$ is called asymptotic if the corresponding form ω_v takes values in the set $K_1 = K_{1h} \cup K_{1l} \cup K_{1p} \cup K_{1pl}$. We have (almost) proved

Theorem 1.7. *Be given a (1) hyperbolic, (2) elliptic, (3) parabolic surface $V : D \to A^3$. Then there is a reduction (1) Q_{2h}, (2) Q_{2l}, (3) Q_{2p} of the bundle $Q_1(V(D), H_1)$ to the group (1) H_{2h}, (2) H_{2l}, (3) H_{2p} with this property: The lift $\varrho : D \to Q_1$ of the map $V : D \to A^3$ is asymptotic if and only if $\varrho(D)$ is situated in (1) Q_{2h}, (2) Q_{2l}, (3) Q_{2p}. Each tangent lift of a planar surface is asymptotic.*

In what follows, let us restrict ourselves to hyperbolic surfaces $V : D \to A^3$. In this case, the asymptotic lifts $v : D \to Q$ are situated in the reduction Q_{2h} of the bundle Q to the group H_{2h}, this group being the set of elements of the form

$$(1.76) \qquad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b_1^1 & 0 & b_3^1 \\ 0 & 0 & b_2^2 & b_3^2 \\ 0 & 0 & 0 & b_1^1 b_2^2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & b_2^1 & b_3^1 \\ 0 & b_1^2 & 0 & b_3^2 \\ 0 & 0 & 0 & b_1^2 b_2^1 \end{pmatrix}.$$

The set of the elements of the form (1.76_1) is a subgroup, let us denote it by H_{2h}^+.

The restriction of the form ω to Q_{2h} is K_{2h}-valued, K_{2h} being spanned by the elements of the type

(1.77) $$r = \begin{pmatrix} 0 & 0 & 0 & 0 \\ r^1 & r_1^1 & r_2^1 & r_3^1 \\ r^2 & r_1^2 & r_2^2 & r_3^2 \\ 0 & r^2 & r^1 & r_3^3 \end{pmatrix}.$$

Now, let $V : D \to A^3$ be a hyperbolic surface and $v : D \to Q_{2h}$ be an asymptotic lift of V; let us consider it in the form (1.57). Be given a map $B : D \to H_{2h}^+$. According to (1.60), we get

(1.78) $$\omega_{\varrho 1}^{\ 1} + \omega_{\varrho 2}^{\ 2} - \omega_{\varrho 3}^{\ 3} = \omega_{v1}^{\ 1} + \omega_{v2}^{\ 2} - \omega_{v3}^{\ 3} - 2b_3^2(b_3^3)^{-1}\,\omega_v^{\ 1} - 2b_3^1(b_3^3)^{-1}\,\omega_v^{\ 2}.$$

Let us denote by $K_3 \subset K_{2h}$ the linear subspace spanned by the elements (1.77) satisfying the condition

(1.79) $$r_1^1 + r_2^2 = r_3^3.$$

Further, denote by $H_3 \subset H_{2h}$ the group of elements (1.76) such that

(1.80) $$b_3^1 = b_3^2 = 0;$$

let $H_3^+ = H_3 \cap H_{2h}^+$. The lift $v : D \to Q_{2h}$ is called the *Darboux's lift* if the corresponding form ω_v is K_3-valued. The equation (1.78) shows how to get a Darboux's lift from an asymptotic one. It is merely a matter of computation to prove

Theorem 1.8. *To each hyperbolic surface $V : D \to A^3$, there exists the reduction Q_3 of the bundle $Q_2(V(D), H_{2h})$ to the group H_3 with the following property: The lift $\varrho : D \to Q_2$ of the map V is a Darboux's lift if and only if $\varrho(D)$ is situated in Q_3.*

The bundle Q_3 is sufficiently small, nevertheless, let us try to reduce it once more. Let $V : D \to A^3$ be a hyperbolic surface and $v : D \to Q_3$ be a Darboux's lift of the map V. We have

(1.81) $$\omega_v^{\ 3} = 0; \quad \omega_{v1}^{\ 3} = \omega_v^{\ 2},\ \omega_{v2}^{\ 3} = \omega_v^{\ 1}; \quad \omega_{v1}^{\ 1} + \omega_{v2}^{\ 2} - \omega_{v3}^{\ 3} = 0.$$

The exterior differentiation of the equations $(1.81_{2,3})$ yields

(1.82) $$\omega_{v1}^{\ 2} \wedge \omega_v^{\ 1} = \omega_{v2}^{\ 1} \wedge \omega_v^{\ 2} = 0,$$

see (1.52_1). This shows the existence of the functions $k_v, l_v : D \to \mathbf{R}$ such that

(1.83) $$\omega_{v1}^{\ 2} = k_v\omega_v^{\ 1}, \quad \omega_{v2}^{\ 1} = l_v\omega_v^{\ 2}.$$

Let $\varrho : D \to Q_3$ (1.59) be another Darboux's lift; at this moment, let us suppose $B(D) \subset H_3^+$. From (1.60), we get

(1.84) $$\omega_\varrho^{\ 1} = \tilde{b}_1^1\omega_v^{\ 1}, \quad \omega_\varrho^{\ 2} = \tilde{b}_2^2\omega_v^{\ 2}; \quad \omega_{\varrho 2}^{\ 1} = \tilde{b}_1^1 b_2^2\omega_{v2}^{\ 1}, \ \omega_{\varrho 1}^{\ 2} = \tilde{b}_2^2 b_1^1\omega_{v1}^{\ 2}$$

and

$$k_\varrho = (b_1^1)^2\, \tilde{b}_2^2 k_v\,, \quad l_\varrho = (b_2^2)^2\, \tilde{b}_1^1 l_v\,. \tag{1.85}$$

Now, we have to consider an arbitrary map $B : D \to H_3$. Let us denote

$$e_{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in H_3\,. \tag{1.86}$$

It is easy to see that $H_3^+ \,.\, e_{-1} = H_3 - H_3^+$. Therefore, it is sufficient to consider the Darboux's lift $v_{-1} : D \to Q_3$ (1.59), where $B(D) = e_{-1}$. Because of $(e_{-1})^{-1} = e_{-1}$, we get

$$k_{v_{-1}} = l_v\,, \quad l_{v_{-1}} = k_v\,. \tag{1.87}$$

We have just proved that the following definition has sense: The surface $V : D \to A^3$ s called *D-general* if it is hyperbolic and we have

$$k_v l_v \neq 0 \tag{1.88}$$

for a Darboux's lift $v : D \to Q_3$ of the map V.

Let $K_4 \subset K_3$ be the linear space consisting of the elements of the form

$$r = \begin{pmatrix} 0 & 0 & 0 & 0 \\ r^1 & r_1^1 & r^2 & r_3^1 \\ r^2 & r^1 & r_2^2 & r_3^2 \\ 0 & r^2 & r^1 & r_1^1 + r_2^2 \end{pmatrix}; \tag{1.89}$$

further, let $H_4 \subset H_3$ be the group consisting of the elements $e \in GA(3)$ and e_{-1} (1.86). The Darboux's lift v is called *canonical* if the corresponding form ω_v is K_4-valued. We have (almost) shown how to construct a canonical lift to a given Darboux's lift of a D-general surface. Now we have

Theorem 1.9. *To each D-general surface $V : D \to A^3$, there exists the reduction Q_4 fo the bundle $Q_3(V(D), H_3)$ to the group H_4 with the following property: The lift $D \to Q_3$ of the map $V : D \to A^3$ is canonical if and only if $\varrho(D) \subset Q_4$.*

Of course, the reduction Q_4 consists of two sections of the bundle Q_3. We *orient* the surface by declaring one of them for the positive one (the other being negative).

Be given a D-general surface $V : D \to A^3$; let $v : D \to Q_4$ be a canonical iift of the map V and ω_v be its associated form. Let us denote by $\mu : D \to Q_4$ the other canonical lift. It is easy to see that

$$\omega_\mu{}^1 = \omega_v{}^2,\ \omega_\mu{}^2 = \omega_v{}^1,\ \omega_{\mu 1}^{\ 1} = \omega_{v 2}^{\ 2},\ \omega_{\mu 2}^{\ 2} = \omega_{v 1}^{\ 1},\ \omega_{\mu 3}^{\ 1} = \omega_{v 3}^{\ 2},\ \omega_{\mu 3}^{\ 2} = \omega_{v 3}^{\ 1}\,. \tag{1.90}$$

Taking in regard the structure equations (1.52_1), the equations

$$(1.91) \qquad \omega_{\nu 1}^{2} = \omega_\nu{}^1\,, \quad \omega_{\nu 2}^{1} = \omega_\nu{}^2\,, \quad \omega_{\nu 3}^{3} = \omega_{\nu 1}^{1} + \omega_{\nu 2}^{2}$$

yield

$$(1.92) \qquad \begin{aligned} \omega_\nu{}^1 \wedge (\omega_{\nu 2}^{2} - 2\omega_{\nu 1}^{1}) + \omega_\nu{}^2 \wedge \omega_{\nu 3}^{2} &= 0\,, \\ \omega_\nu{}^1 \wedge \omega_{\nu 3}^{2} + \omega_\nu{}^2 \wedge \omega_{\nu 3}^{1} &= 0\,, \\ \omega_\nu{}^1 \wedge \omega_{\nu 3}^{1} + \omega_\nu{}^2 \wedge (\omega_{\nu 1}^{1} - 2\omega_{\nu 2}^{2}) &= 0\,, \end{aligned}$$

and we have analoguous equations for the lift μ. It follows the existence of the functions $A_\nu, B_\nu, C_\nu, D_\nu, E_\nu : D \to \mathbf{R}$ such that

$$(1.93) \qquad \begin{aligned} \omega_{\nu 1}^{1} &= A_\nu \omega_\nu{}^1 + B_\nu \omega_\nu{}^2\,, \\ \omega_{\nu 2}^{2} &= C_\nu \omega_\nu{}^1 + D_\nu \omega_\nu{}^2\,, \\ \omega_{\nu 3}^{2} &= (D_\nu - 2B_\nu)\,\omega_\nu{}^1 + E_\nu \omega_\nu{}^2\,, \\ \omega_{\nu 3}^{1} &= E_\nu \omega_\nu{}^1 + (A_\nu - 2C_\nu)\,\omega_\nu{}^2\,. \end{aligned}$$

The passage to the lift μ yields

$$(1.94) \qquad A_\mu = D_\nu,\ B_\mu = C_\nu,\ C_\mu = B_\nu,\ D_\mu = A_\nu,\ E_\mu = E_\nu\,.$$

From the structure equations, we get

$$(1.95) \qquad d\omega_\nu{}^1 = B_\nu \omega_\nu{}^1 \wedge \omega_\nu{}^2\,, \quad d\omega_\nu{}^2 = -C_\nu \omega_\nu{}^1 \wedge \omega_\nu{}^2$$

and analoguous equations for μ. This solves completely the equivalence problem for D-general surfaces as may be seen from the following

Theorem 1.10. *Be given two surfaces $V, W : D \to A^3$; the surface V be D-general. If the surfaces V and W are equivalent, the surface W is D-general, too. Let now V and W be D-general surfaces. Let us construct the canonical lifts ν, μ of the surfaces V and W resp. (for each surface, we choose one of the lifts) and the associated $\mathbf{R}$-valued 1-forms $\omega_\nu{}^1, \omega_\nu{}^2, \omega_\mu{}^1, \omega_\mu{}^2$ on D and the functions $A_\nu, D_\nu, E_\nu, A_\mu, D_\mu, E_\mu$: : $D \to \mathbf{R}$. The surfaces V and W are equivalent if and only if*

$$(1.96) \qquad \omega_\nu{}^1 = \omega_\mu{}^1,\ \omega_\nu{}^2 = \omega_\mu{}^2\,;\quad A_\nu = A_\mu,\ D_\nu = D_\mu,\ E_\nu = E_\mu$$

or

$$\omega_\nu{}^1 = \omega_\mu{}^2,\ \omega_\nu{}^2 = \omega_\mu{}^1\,;\quad A_\nu = D_\mu,\ D_\nu = A_\mu,\ E_\nu = E_\mu\,.$$

1.5. In this section, let us prove an existence theorem. The vector space K_4 is spanned by all elements of the form (1.89), and we have $\dim K = 6$; the numbers $r^1, r^2, r_1^1, r_2^2, r_3^1, r_3^2$ are the coordinates of the element r. Be given a D-general surface

$V : D \to A^3$ and its canonical lifts $\nu, \mu : D \to Q_4$. Let τ_u be the tangent vector space of the domain D at the point u, and let us introduce the notation

$$(1.97) \qquad \alpha_\nu(u) = \omega\,(\mathrm{d}\nu(\tau_u))\,, \quad \alpha_\mu(u) = \omega(\mathrm{d}\mu(\tau_u))\,;$$

$\alpha_\nu(u)$ and $\alpha_\mu(u)$ are planes in K_4. The equations of these planes are

$$(1.98) \qquad \begin{aligned} r_1^1 &= A_\nu(u)\,r^1 + B_\nu(u)\,r^2\,, \quad r_2^2 = C_\nu(u)\,r^1 + D_\nu(u)\,r^2\,,\\ r_3^2 &= \{D_\nu(u) - 2B_\nu(u)\}\,r^1 + E_\nu(u)\,r^2\,,\\ r_3^1 &= E_\nu(u)\,r^1 + \{A_\nu(u) - 2C_\nu(u)\}\,r^2 \end{aligned}$$

and

$$(1.99) \qquad \begin{aligned} r_1^1 &= D_\nu(u)\,r^1 + C_\nu(u)\,r^2\,, \quad r_2^2 = B_\nu(u)\,r^1 + A_\nu(u)\,r^2\,,\\ r_3^2 &= \{A_\nu(u) - 2C_\nu(u)\}\,r^1 + E_\nu(u)\,r^2\,,\\ r_3^1 &= E_\nu(u)\,r^1 + \{D_\nu(u) - 2B_\nu(u)\}\,r^2 \end{aligned}$$

resp. Let us denote by $\mathscr{P}$ the set of all planes α of the space K_4 such that the intersection of α with the space $r^1 = r^2 = 0$ consists of the zero vector only. Each plane $\alpha \in \mathscr{P}$ is given by the equations

$$(1.100) \qquad \begin{aligned} r_1^1 &= Kr^1 + Lr^2\,, \quad r_2^2 = Mr^1 + Nr^2\,,\\ r_3^2 &= Pr^1 + Qr^2\,, \quad r_3^1 = Rr^1 + Sr^2\,, \end{aligned}$$

and we have a 1-l-correspondence $\mathscr{P} \to \mathbf{R}^8$. Let $\mathscr{R} \subset \mathbf{R}^8$ be the 5-dimensional vector subspace determined by the equations

$$(1.101) \qquad P - N + 2L = 0\,, \quad Q = R\,, \quad S - K + 2M = 0\,.$$

Introduce the 1-l-correspondence $\iota : \mathscr{R} \to \mathscr{R}$ associating to the plane

$$(1.102) \qquad \begin{aligned} r_1^1 &= Kr^1 + Lr^2\,, \quad r_2^2 = Mr^1 + Nr^2\,,\\ r_3^2 &= (N - 2L)\,r^1 + Qr^2\,, \quad r_3^1 = Qr^1 + (K - 2M)\,r^2 \end{aligned}$$

the plane

$$(1.103) \qquad \begin{aligned} r_1^1 &= Nr^1 + Mr^2\,, \quad r_2^2 = Lr^1 + Kr^2\,,\\ r_3^2 &= (K - 2M)\,r^1 + Qr^2\,, \quad r_3^1 = Qr^1 + (N - 2L)\,r^2\,; \end{aligned}$$

obviously, $\iota^2\alpha = \alpha$. Let $\mathscr{R}\iota$ be the set of all couples (α, β); $\alpha, \beta \in \mathscr{R}$; such that $\iota\alpha = \beta$. From the equations (1.98) and (1.99), we get the following

Theorem 1.11. *A D-general surface $V : D \to A^3$ determines uniquely a map $V^0 : D \to \mathscr{R}_\iota$.*

The function $\varphi : \mathcal{R}_\iota \to \mathbf{R}$ is called *admissible* if there is a non-constant function $\Phi_\varphi : \mathbf{R}^5 \to \mathbf{R}$ such that

$$(1.104)\qquad \Phi_\varphi(x_1, x_2, x_3, x_4, x_5) = \Phi_\varphi(x_4, x_3, x_2, x_1, x_5) \quad \text{for} \quad x_i \in \mathbf{R}$$

and

$$(1.105)\qquad \varphi(\alpha, \iota\alpha) = \Phi_\varphi(K, L, M, N, Q)\,,$$

the plane α being given by (1.102). We have the following

Theorem 1.12. *Be given an admissible function $\varphi : \mathcal{R}_\iota \to \mathbf{R}$. Then there are D-general surfaces $V: D \to A^3$ such that $V^0(D) \subset \varphi^{-1}(0)$.*

Proof. Let the surface $V: D \to A^3$ be D-general with $V^0(D) \subset \varphi^{-1}(0)$; let $v : D \to Q_4$ be its canonical lift. There are functions $A, \ldots, E$ such that

$$(1.106)\qquad \begin{aligned} &\omega_1^1 = A\omega^1 + B\omega^2\,, \quad \omega_2^2 = C\omega^1 + D\omega^2\,,\\ &\omega_3^2 = (D - 2B)\,\omega^1 + E\omega^2\,, \quad \omega_3^1 = E\omega^1 + (A - 2C)\,\omega^2\,;\\ &\Phi(A, B, C, D, E) = 0\,. \end{aligned}$$

The exterior differentiation of the equations (1.106) yields

$$(1.107)\qquad \begin{aligned} &\omega^1 \wedge \mathrm{d}A + \omega^2 \wedge \mathrm{d}B + (1 - E - AB + BC)\,\omega^1 \wedge \omega^2 = 0\,,\\ &\omega^1 \wedge \mathrm{d}C + \omega^2 \wedge \mathrm{d}D + (E - 1 - BC + CD)\,\omega^1 \wedge \omega^2 = 0\,,\\ &\omega^1 \wedge (\mathrm{d}D - 2\mathrm{d}B) + \omega^2 \wedge \mathrm{d}E +\\ &\quad + (2C - A - 2BD + 4B^2 + AE + CE)\,\omega^1 \wedge \omega^2 = 0\,,\\ &\omega^1 \wedge \mathrm{d}E + \omega^2 \wedge (\mathrm{d}A - 2\mathrm{d}C) +\\ &\quad + (D - 2B + 2AC - 4C^2 - DE - BE)\,\omega^1 \wedge \omega^2 = 0\,; \end{aligned}$$

$$(1.108)\qquad \Phi_1\,\mathrm{d}A + \Phi_2\,\mathrm{d}B + \Phi_3\,\mathrm{d}C + \Phi_4\,\mathrm{d}D + \Phi_5\,\mathrm{d}E = 0\,,$$

where $\Phi_i = \partial\Phi(A, \ldots, E)/\partial x_i$. The polar determinant of the system (1.106) + (1.107) is

$$(1.109)\qquad \Delta = \begin{vmatrix} \omega^1 & \omega^2 & 0 & 0 & 0\\ 0 & 0 & \omega^1 & \omega^2 & 0\\ 0 & -2\omega^1 & 0 & \omega^1 & \omega^2\\ \omega^2 & 0 & -2\omega^2 & 0 & \omega^1\\ \Phi_1 & \Phi_2 & \Phi_3 & \Phi_4 & \Phi_5 \end{vmatrix} =$$

$$\begin{aligned} &= -(\Phi_2 + 2\Phi_4)\,(\omega^1)^4 + (\Phi_1 + 2\Phi_3)\,(\omega^1)^3\,\omega^2 + 3\Phi_5(\omega^1)^2\,(\omega^2)^2 +\\ &\quad + (2\Phi_2 + \Phi_4)\,\omega^1(\omega^2)^3 - (2\Phi_1 + \Phi_3)\,(\omega^2)^4\,. \end{aligned}$$

The function Φ being non-constant, we have $\Delta \neq 0$. The system (1.106) + (1.107) is in involution and its solutions depend on five functions of one variable in the usual sense. Q.E.D.

2. GENERAL THEORY OF SPACES WITH CONNECTION

2.1. We recall here some fundamental definitions. The *principal fibre bundle* $P(M, G)$ consists of manifolds P, M and a Lie group G such that (1) G acts freely on P on the right, and we write $R_g p = pg$ for $p \in P$, $g \in G$; (2) $M = P/G$, and the canonical projection $\pi : P \to M$ is differentiable; (3) P is locally trivial. We may choose an open covering $\{U_\alpha\}$ of the base space M and the corresponding set of diffeomorphisms $\varphi_\alpha : \pi^{-1}(U_\alpha) \to G$ such that

$$\varphi_\alpha(pg) = \varphi_\alpha(p)\, g \quad \text{for} \quad p \in \pi^{-1}(U_\alpha),\ g \in G\,. \tag{2.1}$$

Define the maps $\psi_{\beta\alpha} : U_\alpha \cap U_\beta \to G$ by the relations

$$\psi_{\beta\alpha}(u) = \varphi_\beta(p) \cdot (\varphi_\alpha(p))^{-1}\,; \quad u \in U_\alpha \cap U_\beta\,; \tag{2.2}$$
$$p \in \pi^{-1}(u) \quad \text{being an arbitrary point}\,;$$

$\psi_{\beta\alpha}(u)$ does not depend on the choice of the point $p \in \pi^{-1}(u)$. The functions $\psi_{\beta\alpha}$ are the transition functions, and we have

$$\psi_{\gamma\alpha}(u) = \psi_{\gamma\beta}(u)\, \psi_{\beta\alpha}(u) \quad \text{for} \quad u \in U_\alpha \cap U_\beta \cap U_\gamma\,. \tag{2.3}$$

Let $A \in \mathfrak{g}$ be an element of the Lie algebra $\mathfrak{g}$ of the Lie group G. The associated *fundamental vector field* A^* on P is defined as follows: Let $p \in P$, and let us consider the map $\mu_p : G \to P$ defined by $\mu_p(g) = pg$; we have

$$A^*_p = (\mu_p)_* \mathrm{A} \equiv (\mathrm{d}\mu_p)_e\, A\,, \tag{2.4}$$

$e \in G$ being the identity.

Let $P(M, G)$ be a principal fibre bundle and ϱ a representation of G on a finite dimensional vector space V. A *pseudotensorial r-form on P of type* (ϱ, V) is a V-valued r-form φ on P such that

$$R_{g*}\varphi = \varrho(g^{-1})\, \varphi\,, \tag{2.5}$$

i.e.

$$\varphi(R_{g*}X_1, \ldots, R_{g*}X_r) = \varrho(g^{-1})\, \varphi(X_1, \ldots, X_r) \quad \text{for each} \quad g \in G\,. \tag{2.6}$$

Such a form is called *tensorial* if

$$\varphi(X_1, \ldots, X_r) = 0 \quad \text{whenever at least one of the vectors } X_i \text{ is vertical.} \tag{2.7}$$

If $V = \mathfrak{g}$ and ϱ is the adjoint representation ad : $G \to$ Aut $(\mathfrak{g} \to \mathfrak{g})$, the pseudotensorial form of type (ϱ, V) is called of *type* ad G.

In this terminology, the *connection* on $P(M, G)$ is a pseudotensorial $\mathfrak{g}$-valued 1-form on P of type ad G such that $\omega(A^*) = A$ for each vector $A \in \mathfrak{g}$ (A^* being the fundamental field associated to A). Obviously, we get $X = 0$ from $\omega(X) = 0$. Let us denote by $hX(vX)$ the horizontal (vertical) part of the vector X tangent to P.

On the principal bundle $P(M, G)$ with a connection ω, the pseudotensorial forms have following properties (see [1], p. 76, Proposition 5.1): If φ is a pseudotensorial r-form on P of type (ϱ, V), then (a) the form φh defined by

$$(2.8) \qquad (\varphi h)(X_1, \ldots, X_r) = \varphi(hX_1, \ldots, hX_r)$$

is a tensorial form of type (ϱ, V); (b) $\mathrm{d}\varphi$ is a pseudotensorial $(r + 1)$-form of type (ϱ, V); (c) the form $\mathrm{D}\varphi = (\mathrm{d}\varphi)\, h$, the so-called *exterior covariant derivative* of φ, is a tensorial $(r + 1)$-form of type (ϱ, V).

The form $\mathrm{D}\omega = \Omega$ is a tensorial $\mathfrak{g}$-valued 2-form of type ad G, and it is called the *curvature form* of the connection ω. We have the following structure equation

$$(2.9) \qquad \mathrm{d}\omega(X, Y) = -\tfrac{1}{2}[\omega(X), \omega(Y)] + \Omega(X, Y)\,;$$

see [1], p. 77, Theorem 5.2.

The fundamental subject of our investigations is given by the following

Definition. The space with a connection $\mathfrak{S} = \mathfrak{S}(P, M, G, Q, H, \omega)$ consists of (1) a principal fibre bundle $P(M, G)$, (2) a reduction Q of the bundle $P(M, G)$ to a Lie subgroup $H \subset G$, (3) a connection ω on $P(M, G)$. Be given two spaces with connection

$$(2.10) \qquad \mathfrak{S} = \mathfrak{S}(P, M, G, Q, H, \omega)\,, \quad \mathfrak{S}' = \mathfrak{S}'(P', M', G, Q', H, \omega')\,.$$

The map $f : P \to P'$ is called the equivalence between $\mathfrak{S}$ and $\mathfrak{S}'$ if (1) f is a bundle isomorphism, (2) $f(Q) = Q'$, (3) $f_* \omega' = \omega$.

Our main task is to solve the following problem: Be given the spaces (2.10) and a diffeomorphism $f_0 : M \to M'$. We have to decide whether there is a lift $f : P \to P'$ of the map f_0, f being an equivalence.

2.2. Be given a space with a connection $\mathfrak{S} = \mathfrak{S}(P, M, G, Q, H, \omega)$. Let us denote by

$$(2.11) \qquad \iota_H : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$$

the natural homomorphism. For $h \in H$, define

$$(2.12) \qquad \mathrm{ad}(h) : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$$

in the natural manner: Let $v \in \mathfrak{g}/\mathfrak{h}$, and let $v' \in \iota_H^{-1}(v)$ be an arbitrary vector; then

$$\text{(2.13)} \qquad \mathrm{ad}(h)\, v = \iota_H(\mathrm{ad}(h)\, v')\,, \quad \text{i.e.} \quad \mathrm{ad}(h)\,(\iota_H v') = \iota_H(\mathrm{ad}(h)\, v')\,.$$

This is obviously a good definition; ad : $H \to \mathrm{Aut}\,(\mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h})$ is a representation.

On Q, let us define the forms

$$\text{(2.14)} \qquad \varphi(X) = \iota_H(\omega(X))\,,$$

$$\text{(2.15)} \qquad \Phi(X, Y) = \iota_H(\Omega(X, Y))\,.$$

It is easy to prove

Theorem 2.1. *The form* $\varphi(X)$ $[\Phi(X, Y)]$ *is a* $\mathfrak{g}/\mathfrak{h}$*-valued tensorial* 1*-form* (2*-form*) *of type* ad H *defined on the bundle* $Q(M, H)$.

The form $\Phi(X, Y)$ is the *torsion form* of the space $\mathfrak{S}$. From the structure equation (2.9), we get

Theorem 2.2. *On* $Q(M, H)$, *we have*

$$\text{(2.16)} \qquad \mathrm{d}\varphi(X, Y) = -\tfrac{1}{2}\iota_H[\omega(X), \omega(Y)] + \Phi(X, Y)\,.$$

We get a more interesting situation in the case of a reductive algebra $\mathfrak{g}$ with the decomposition

$$\text{(2.17)} \qquad \mathfrak{g} = \mathfrak{h} + N; \quad \mathrm{ad}(H)N = N, \quad [\mathfrak{h}, N] \subset N\,.$$

In this case, we identify $\mathfrak{g}/\mathfrak{h}$ with N. Let us denote by

$$\text{(2.18)} \qquad \iota_H : \mathfrak{g} \to N\,, \quad \iota_N : \mathfrak{g} \to \mathfrak{h}$$

the natural projections. On Q, we may write

$$\text{(2.19)} \qquad \omega(X) = \omega'(X) + \varphi(X)\,, \quad \text{where} \quad \omega'(X) = \iota_N\, \omega(X) \in \mathfrak{h}\,,$$

and $\varphi(X)$ is defined by the equation (2.14). It is well known that $\omega'(X)$ is a connection on $Q(M, H)$; see [1], p. 83, Proposition 6.4. Hence a connection ω on $P(M, G)$ induces a connection ω' on $Q(M, H)$ and a tensorial N-valued 1-form $\varphi(X)$ of type ad H on the same bundle Q. Conversely, we have

Theorem 2.3. *Be given a principal fibre bundle* $P(M, G)$ *and its reduction* $Q(M, H)$. *Let* G *be reductive with the decomposition* (2.17). *On* $Q(M, H)$, *be given a connection* ω' *and a tenosiral* N*-valued* 1*-form* φ *of type* ad H. *Then there is a unique connection* ω *on* $P(M, G)$ *such that*

$$\text{(2.20)} \qquad \omega(X) = \omega'(X) + \varphi(X) \quad \text{on} \quad Q\,.$$

Proof. Let $p \in P$ and $X \in T_p(P)$. Let us choose $q \in Q$ in such a way that $\pi(p) = \pi(q)$; let $p = qg$, $g \in G$. Further, let us choose $Y \in T_q(Q)$ in such a way that $X = R_{g*}Y + A^*$, A^* being vertical. Let $A \in \mathfrak{g}$ be the vector such that the value of the associated fundamental field at the point p is just A^*. Then we set

$$\omega(X) = \operatorname{ad}(g^{-1})(\omega'(Y) + \varphi(Y)) + A . \tag{2.21}$$

It is easy to show that ω is uniquely determined, and that it is a connection. Q.E.D.

The preceding theorem is a direct generalization of Proposition 3.1 in [1], p. 127.

Theorem 2.4. *Let $\mathfrak{S} = \mathfrak{S}(P, M, G, Q, H, \omega)$ be a space, and let G/H be reductive with the decomposition* (2.17). *Let ω' be the induced connection on $Q(M, H)$, let φ be the form* (2.14). *Further, let Ω (Ω') be the curvature form of the connection ω (ω'). Then*

$$\Omega(X, Y) = \Omega'(X, Y) + (\mathrm{D}_{\omega'}\varphi)(X, Y) + \tfrac{1}{2}[\varphi(X), \varphi(Y)] \quad \text{on} \quad Q . \tag{2.22}$$

Here, $\mathrm{D}_{\omega'}$ denotes the operator of the covariant exterior differentiation with respect to ω'.

Proof. It is sufficient to consider two cases. (1) X is vertical. The curvature form being tensorial, we have $\Omega(X, Y) = \Omega'(X, Y) = 0$ on Q. Further, $(\mathrm{D}_{\omega'}\varphi)(X, Y) = \mathrm{d}\varphi(h_{\omega'}X, h_{\omega'}Y) = 0$, $h_{\omega'}X$ being the horizontal part of the vector X in the connection ω'; in our case, $h_{\omega'}(X) = 0$. Finally, $\omega(X) = \omega'(X) + \varphi(X) \in \mathfrak{h}$, i.e. $\varphi(X) = 0$. (2) X and Y are horizontal with respect to the connection ω'. Because $h_\omega(h_{\omega'}Z) = h_\omega Z$ for each Z, we have $\Omega(h_{\omega'}X, h_{\omega'}Y) = \Omega(X, Y)$. The structure equation (2.9) yields

$$\mathrm{d}\omega'(h_{\omega'}X, h_{\omega'}Y) + \mathrm{d}\varphi(h_{\omega'}X, h_{\omega'}Y) = -\tfrac{1}{2}[\varphi(X), \varphi(Y)] + \Omega(X, Y),$$

the left hand side being equal to $\Omega'(X, Y) + (\mathrm{D}_{\omega'}\varphi)(X, Y)$. Q.E.D.

This theorem is a generalisation of Proposition 3.2 in [1] p. 128. Comparing the $\mathfrak{h}$- and N-components in the structure equation (2.9) and using (1.19), we get

Theorem 2.5. *Let the situation be the same as in Theorem* 2.4. *On Q, we have*

$$\mathrm{d}\varphi(X, Y) = -\tfrac{1}{2}[\omega'(X), \varphi(Y)] - \tfrac{1}{2}[\varphi(X), \omega'(Y)] + (\mathrm{D}_{\omega'}\varphi)(X, Y), \tag{2.23}$$

$$\Phi(X, Y) = (\mathrm{D}_{\omega'}\varphi)(X, Y) + \tfrac{1}{2}\iota_H[\varphi(X), \varphi(Y)] . \tag{2.24}$$

2.3. Let us try to express our results in the local coordinate systems. Be given a space $\mathfrak{S} = \mathfrak{S}(P, M, G, Q, H, \omega)$. Consider the coordinate neighbourhood $U \subset M$ with the local coordinates $(u^A) = (u^1, \ldots, u^n)$. Let $e_1, \ldots, e_{r+s}$ be a basis in $\mathfrak{g}$ such that $e_1, \ldots, e_r$ is a basis in $\mathfrak{h}$. Use the following indices:

$$i, j, \ldots = 1, \ldots, r + s\,; \quad a, b, \ldots = 1, \ldots, r\,; \tag{2.25}$$

$$\alpha, \beta, \ldots = r + 1, \ldots, r + s\,.$$

We have

$$(2.26) \qquad [e_i, e_j] = \sum_{k=1}^{r+s} c_{ij}^k e_k$$

and

$$(2.27) \qquad c_{ab}^{\alpha} = 0\,.$$

The connection on P is given by the form

$$(2.28) \qquad \omega = \sum_{i=1}^{r+s} \omega^i e_i\,,$$

and the structure equation is

$$(2.29) \qquad \mathrm{d}\omega^i = -\tfrac{1}{2} \sum_{j,k=1}^{r+s} c_{jk}^i \omega^j \wedge \omega^k + \Omega^i\,,$$

$\Omega = \sum_{i=1}^{r+s} \Omega^i e_i$ being the curvature form; see [1], p. 78. Let $I_{r+1}, \ldots, I_{r+s}$ be a basis in $\mathfrak{g}/\mathfrak{h}$ such that the homomorphism (2.11) is expressed by

$$(2.30) \qquad \iota_H\Big(\sum_{a=1}^{r} x^a e_a + \sum_{\alpha=r+1}^{r+s} y^\alpha e_\alpha\Big) = \sum_{\alpha=r+1}^{r+s} y^\alpha I_\alpha\,.$$

This means that we have

$$(2.31) \qquad \varphi = \sum_{\alpha=r+1}^{r+s} \omega^\alpha I_\alpha\,, \quad \Phi = \sum_{\alpha=r+1}^{r+s} \Omega^\alpha I_\alpha$$

on Q. The forms φ and Φ being tensorial on Q, the forms ω^α, Ω^α are tensorial, too. In the coordinate neighbourhood $Q \cap \pi^{-1}(U)$, we may write

$$(2.32) \qquad \omega^\alpha = \sum_{A=1}^{n} a_A^\alpha(u^1, \ldots, u^n; q)\, \mathrm{d}u^A\,,$$

$$(2.33) \qquad \Omega^\alpha = \sum_{A,B=1}^{n} a_{AB}^\alpha(u^1, \ldots, u^n; q)\, \mathrm{d}u^A \wedge \mathrm{d}u^B\,, \quad a_{BA}^\alpha + a_{BA}^\alpha = 0$$

in the following sense: Let

$$q \in Q,\ \pi(q) = (u^1, \ldots, u^n),\ X \in T_q(Q),\ \pi_* X = \sum_{A=1}^{\alpha} x^A \cdot \frac{\partial}{\partial u^A}\Big|_u\,,$$

$$(2.34) \qquad \omega^\alpha(X) = \sum_{A=1}^{n} a_A^\alpha(u^1, \ldots, u^n; q)\, x^A\,,$$

and analoguously for Ω^α.

From (2.16), we get

$$(2.35) \qquad d\omega^\alpha = -\tfrac{1}{2}\sum_{i,j=1}^{r+s} c^\alpha_{ij}\omega^i \wedge \omega^j + \Omega^\alpha\,; \quad \alpha = r+1, \ldots, r+s\,;$$

on Q. We may explain these equations as follows: Let us consider a coordinate neigbourhood D on Q such that $\pi(D) \subset U$ and each point $q \in D$ has the coordinates $(u^1, \ldots, u^n, q^1, \ldots, q^t)$, the point $\pi(q) \in U$ having the coordinates $(u^1, \ldots, u^n)$. In the domain D, we may write

$$(2.36) \qquad \omega^a = \sum_{A=1}^{n} b^a_A(u, q)\, du^A + \sum_{K=1}^{t} f^a_K(o, q)\, dq^K\,; \quad a = 1, \ldots, r\,;$$

$$\omega^\alpha = \sum_{A=1}^{n} a^\alpha_A(u, q)\, du^A\,; \quad \alpha = r+1, \ldots, r+s\,;$$

$$\Omega^\alpha = \sum_{A=1}^{n} a^\alpha_{AB}(u, q)\, du^A \wedge du^B\,; \quad \alpha = r+1, \ldots, r+s\,.$$

On Q, the 2-forms

$$(2.37) \qquad d\omega^\alpha + \tfrac{1}{2}\sum_{i,j=1}^{r+s} c^\alpha_{ij}\omega^i \wedge \omega^j\,; \quad \alpha = r+1, \ldots, r+s\,;$$

are (at least formally) linear combinations of the forms $du^A \wedge du^B$, $du^A \wedge dq^K$, $dq^L \wedge dq^L$; the equation (2.35) shows that (2.37) are linear combinations of the forms $du^A \wedge du^B$ only. In other words, we have

$$(2.38) \qquad \frac{\partial a^\alpha_A(u, q)}{\partial q^K} + \sum_{a=1}^{r}\sum_{\beta=r+1}^{r+s} c^\alpha_{a\beta} f^a_K(u, q)\, a^\beta_A(u, q) = 0 \quad \text{for } \alpha = r+1, \ldots, r+s\,;$$

$$A = 1, \ldots, n\,; \quad K = 1, \ldots, t\,.$$

Thus we have explained the (somewhat confused) considerations due to E. Cartan, Oeuvres, III. 1, pp. 701 and 757.

2.4. Let $GL(n) = GL(n, \mathbf{R})$ be the group of non-singular matrices $A = (a^j_i)$; $i, j = 1, \ldots, n$; the element a^j_i standing in the i-th column and the j-th row. The multiplication is given by

$$(2.39) \qquad (a^j_i)(b^j_i) = (c^j_i)\,, \quad \text{where} \quad c^j_i = \sum_{k=1}^{n} a^j_k b^k_i\,.$$

The Lie algebra $\mathfrak{gl}(n)$ of the group $GL(n)$ is the vector space of all matrices of the type $n \times n$ with

$$(2.40) \qquad [R, S] = RS - SR \quad \text{for} \quad R, S \in \mathfrak{gl}(n)\,.$$

It is well known that

$$(2.41) \qquad \mathrm{ad}(A)\, R = ARA^{-1} \quad \text{for} \quad A \in GL(n),\ R \in \mathfrak{gl}(n)\,.$$

(1) Spaces with affine connection. Be given a manifold M; $\dim M = n$. Let $m \in M$ be a point. The *affine frame* F at m is a set

(2.42) $$F = (e_0, e_1, \ldots, e_n)$$

of the vectors of the vector space $T_m(M)$ such that the vectors $e_1, \ldots, e_n$ are linearly independent. Introducing in the obvious way the differentiable structure into the set of all affine frames of the manifold M, we get a manifold denoted by P. F being formed by the vectors of the space $T_m(M)$, we set $\pi(F) = m$, this defining the map $\pi : P \to M$.

Let $GA(n)$ be the affine group, i.e. the subgroup of $GL(n+1)$ consisting of the elements

(2.43) $$A = \begin{pmatrix} 1 & 0 \\ \alpha & a \end{pmatrix} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ \alpha^1 & a_1^1 & \ldots & a_n^1 \\ \vdots & \vdots & \ldots & \vdots \\ \alpha^n & a_1^n & \ldots & a_n^n \end{pmatrix}, \quad a \in GL(n).$$

The group $GA(n)$ operates freely on P on the right according to the rule

(2.44) $$R_A F = FA,$$

FA being the usual product of the matrices F and A; thus we get the principal fibre bundle $P(M, GA(n))$. Let Q be the manifold of all frames (2.42) such that $e_0 = 0$. Let us denote by $GA_0(n) \subset GA(n)$ the group consisting of all elements (2.43) such that $\alpha^1 = \ldots = \alpha^n = 0$, this group is isomorphic to $GL(n)$. It is obvious that Q is a reduction of the bundle P to the group $GA_0(n)$. ω being a connection on P, the space

$$\mathfrak{S} = \mathfrak{S}(P, M, GA(n), Q, GA_0(n), \omega)$$

is called the *space with affine connection*.

Let $T(n)$ be the group of translations, i.e. the group of elements (2.43) where $a \in GL(n)$ is the identity. The Lie algebra $\mathfrak{ga}(n)$ consists of all matrices of the form

(2.45) $$R = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ r^1 & r_1^1 & \ldots & r_n^1 \\ \vdots & \vdots & \ldots & \vdots \\ r^n & r_1^n & \ldots & r_n^n \end{pmatrix}.$$

Obviously,

(2.46) $$\mathfrak{ga}(n) = \mathfrak{ga}_0(n) \oplus \mathfrak{t}(n),$$

$\mathfrak{ga}(n)$ being reductive with the decomposition (2.46); we use the identification $\mathfrak{ga}(n)/\mathfrak{ga}_0(n) = \mathfrak{t}(n)$. The connection ω be given by

(2.47) $$\omega = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ \omega^1 & \omega_1^1 & \ldots & \omega_n^1 \\ \vdots & \vdots & \ldots & \vdots \\ \omega^n & \omega_1^n & \ldots & \omega_n^n \end{pmatrix},$$

ω^i, ω^j_i being **R**-valued 1-forms on P. Let φ be the form on Q defined by (2.14); obviously,

$$\varphi = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \omega^1 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ \omega^n & 0 & \dots & 0 \end{pmatrix}. \tag{2.48}$$

On $Q(M, GA_0(n))$, we get the so-called *linear connection* given by the form

$$\omega' = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & \omega^1_1 & \dots & \omega^1_n \\ \vdots & \vdots & \dots & \vdots \\ 0 & \omega^n_1 & \dots & \omega^n_n \end{pmatrix}. \tag{2.49}$$

The curvature form on P is

$$\Omega = \mathrm{d}\omega + \omega \wedge \omega . \tag{2.50}$$

The curvature form of the linear connection ω' is

$$\Omega' = \mathrm{d}\omega' + \omega' \wedge \omega' . \tag{2.51}$$

Using Theorem 2.5, let us calculate the torsion form. Obviously,

$$[\varphi(X), \varphi(Y)] = \varphi(X)\,\varphi(Y) - \varphi(Y)\,\varphi(X) = 0 .$$

Further,

$$[\omega'(X), \varphi(Y)] + [\varphi(X), \omega'(Y)] =$$
$$= \omega'(X)\,\varphi(Y) - \varphi(Y)\,\omega'(X) + \varphi(X)\,\omega'(Y) - \omega'(Y)\,\varphi(X) =$$
$$= 2(\omega' \wedge \varphi)(X, Y) + 2(\varphi \wedge \omega')(X, Y),$$

but we have $\varphi \wedge \omega' = 0$. The equation (2.23) reduces to

$$\mathrm{d}\varphi + \omega' \wedge \varphi = \Phi . \tag{2.52}$$

The forms Ω' and Φ are tensorial 2-forms on Q, the first one being $\mathfrak{ga}_0(n)$-valued, the second one $\mathfrak{t}(n)$-valued. The forms $\omega^1, \dots, \omega^n$ generating the $\mathscr{F}(Q)$-module of tensorial 1-forms of type ad H on Q ($\mathscr{F}(Q)$ is the ring of functions on Q), the elements of the matrices Ω' and Φ are 2-forms from the $\mathscr{F}(Q)$-module generated by the 2-forms $\omega^i \wedge \omega^j$.

(2) Spaces with projective connection. Be given a manifold M, $\dim M = n$. Let $m \in M$ be a fixed point and $T_m = T_m(M)$ the tangent space at m. The *analytic point* of the space T_m is a couple (ε, e); $\varepsilon \in \mathbf{R}$, $e \in T_m$; the couple $(0, 0)$ being excluded.

The analytic points $(\varepsilon_1, e_1), \ldots, (\varepsilon_r, e_r)$ are linearly dependent if there are numbers $k_1, \ldots, k_r \in \mathbf{R}$ such that

$$(2.53) \qquad k_1\varepsilon_1 + \ldots + k_r\varepsilon_r = 0 \in \mathbf{R}\,, \quad k_1 e_1 + \ldots + k_r e_r = 0 \in T_m\,.$$

In the space T_m, there are $n + 1$ linearly independent analytic points: it is sufficient to consider any linearly independent vectors $e'_1, \ldots, e'_n \in T_n$ and the analytic points $(0, e'_1), \ldots, (0, e'_n), (1, e'_n)$. If $(\varepsilon_0, e_0), \ldots, (\varepsilon_n, e_n)$ are linearly independent and (ε, e) is an analytic point, there is a unique set of numbers $k_0, \ldots, k_n \in \mathbf{R}$ such that

$$(2.54) \qquad (\varepsilon, e) = k_0(\varepsilon_0, e_0) + \ldots + k_n(\varepsilon_n, e_n)$$

The *analytic frame* E at the point $m \in M$ is any ordered set $(\varepsilon_0, e_0), \ldots, (\varepsilon_n, e_n)$ of linearly independent analytic points. Two analytic frames E, E' at the point m are called equivalent if there is a $k \in \mathbf{R}$ such that $(\varepsilon_i, e_i) = k(\varepsilon'_i, e'_i)$ for $i = 0, \ldots, n$. The *geometric frame* is the class of equivalent analytic frames.

Let P be the manifold (with the obvious differentiable structure) of all geometric frames of the manifold M; we have the natural projection $\pi : P \to M$. Let us denote by $SL(n + 1) \subset GL(n + 1)$ the group of all matrices $A \in GL(n + 1)$ with $\det A = \pm 1$. The Lie algebra $\mathfrak{sl}(n + 1)$ is formed by all $(n + 1) \times (n + 1)$ matrices R satisfying the condition trace $R = 0$. The group $SL(n + 1)$ operates freely on P on the right as follows: Let $F \in P$, $\pi(F) = m$. Let us choose an analytic frame $E \in F$,

$$(2.55) \qquad E = ((\varepsilon_0, e_0), \ldots, (\varepsilon_n, e_n))\,,$$

and set $R_A E = EA$, where EA is the obviously defined matrix product. The analytic frame EA is the element of some geometric frame, this frame being denoted by $R_A F = FA$. Let Q be the submanifold of the manifold P consisting of all geometric frames containing an analytic frame of the type (2.55) with $e_0 = 0 \in T_m$. The manifold Q is a reduction of the bundle $P(M, SL(n + 1))$ to the group $SL_0(n + 1)$ consisting of the elements of the form

$$(2.56) \qquad A = \begin{pmatrix} a_0^0 & a_1^0 & \ldots & a_n^0 \\ 0 & a_1^1 & \ldots & a_1^1 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & a_1^n & \ldots & a_n^n \end{pmatrix} \in SL(n + 1)\,.$$

The space

$$\mathfrak{S} = \mathfrak{S}(P, M, SL(n + 1), Q, SL_0(n + 1), \omega)$$

is the *space with projective connection*. Let us write

$$(2.57) \qquad \omega = \begin{pmatrix} \omega_0^0 & \omega_1^0 & \ldots & \omega_n^0 \\ \omega_0^1 & \omega_1^1 & \ldots & \omega_n^1 \\ \vdots & \vdots & \ldots & \vdots \\ \omega_0^n & \omega_1^n & \ldots & \omega_n^n \end{pmatrix}, \quad \text{trace}\, \omega = 0\,.$$

The group $SL_0(n+1)$ is not reductive in $SL(n+1)$. The vector space $W = \mathfrak{sl}(n+1)/\mathfrak{sl}_0(n+1)$ may be identified with the space of all matrices of the form

$$R = \begin{pmatrix} 0 & 0 & \dots & 0 \\ r_0^1 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ r_0^n & 0 & \dots & 0 \end{pmatrix}, \tag{2.58}$$

where

$$\iota_W \begin{pmatrix} r_0^0 & r_1^0 & \dots & r_n^0 \\ r_0^0 & r_1^1 & \dots & r_n^1 \\ \vdots & \vdots & \dots & \vdots \\ r_0^n & r_1^n & \dots & r_n^n \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ r_0^1 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ r_0^n & 0 & \dots & 0 \end{pmatrix}. \tag{2.59}$$

Writing the elements (2.56) and (2.58) in the form

$$A = \begin{pmatrix} a_0^0 & \alpha \\ 0 & a \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix}, \tag{2.60}$$

we get

$$\operatorname{ad}(A)\, R = \iota_W(ARA^{-1}) = \begin{pmatrix} 0 & 0 \\ (a_0^0)^{-1}\, ar & 0 \end{pmatrix}. \tag{2.61}$$

Let us apply Theorem 2.2 to our situation. We have

$$[\omega(X), \omega(Y)] = \omega(X)\,\omega(Y) - \omega(Y)\,\omega(X) = 2(\omega \wedge \omega)(X, Y). \tag{2.62}$$

If we write the form ω (2.57) as

$$\omega = \begin{pmatrix} \omega_0^0 & \omega^0 \\ \omega_0 & \psi \end{pmatrix}, \tag{2.63}$$

we have

$$\omega \wedge \omega = \begin{pmatrix} \omega^0 \wedge \omega_0 & \omega_0^0 \wedge \omega^0 + \omega^0 \wedge \psi \\ \omega_0 \wedge \omega_0^0 + \psi \wedge \omega_0 & \omega_0 \wedge \omega^0 + \psi \wedge \psi \end{pmatrix},$$

and the equation (2.16) reduces to

$$d\omega_0 + \omega_0 \wedge \omega_0^0 + \psi \wedge \omega_0 = \Phi_0, \tag{2.64}$$

i.e.

$$d\omega_0^i + \omega_0^i \wedge \omega_0^0 + \sum_{k=1}^{n} \omega_k^i \wedge \omega_0^k = \Phi_0^i; \quad i = 1, \dots, n. \tag{2.65}$$

The torsion form Φ_0 belongs to the $\mathscr{F}(Q)$-module generated by the 2-forms $\omega_0^i \wedge \omega_0^j$; $i, j = 1, \dots, n$.

2.5. Be given a space $\mathfrak{S} = \mathfrak{S}(P, M, G, Q, H, \omega)$. Let $\{U_\alpha\}$ be an open covering of the base space M; let $\varphi_\alpha : \pi^{-1}(U_\alpha) \to G$ be maps such that $\varphi_\alpha(pg) = \varphi_\alpha(p)\, g$ for each $p \in \pi^{-1}(U_\alpha)$, $g \in G$. Let $\psi_{\beta\alpha} : U_\alpha \cap U_\beta \to G$ be the associated transition functions. The *H-set* over $u \in M$ is each set $S \subset \pi^{-1}(u)$ such that $SH = S$, SH denoting the set of all points sh, $s \in S$, $h \in H$. Let $S \subset \pi^{-1}(u)$, $u \in U_\alpha$, be a given H-set; let us choose a point $s \in S$ and define the maps

$$\Phi_\alpha\{S\} = \varphi_\alpha(s)\, H(\varphi_\alpha(s))^{-1} \subset G\,, \tag{2.66}$$

$$\boldsymbol{\Phi}_\alpha\{S\} = \operatorname{ad}(\varphi_\alpha(s))\, \mathfrak{h} \subset \mathfrak{g}\,. \tag{2.67}$$

The use of $\{.\}$ instead of $(.)$ indicates that there is *no* given point map $S \to \Phi_\alpha\{S\}$ or $S \to \boldsymbol{\Phi}_\alpha\{S\}$ resp. $\Phi_\alpha\{S\}$ is obviously a subgroup of the group G, and its Lie algebra is $\boldsymbol{\Phi}_\alpha\{S\}$. It is easy to see that the maps (2.66) and (2.67) do not depend on the choice of the point $s \in S$.

Theorem 2.6. *If $u \in U_\alpha \cap U_\beta$ and $S \subset \pi^{-1}(u)$ is an H-set, we have*

$$\Phi_\beta\{S\} = \psi_{\beta\alpha}(u)\,.\,\Phi_\alpha\{S\}\,.\,(\psi_{\beta\alpha}(u))^{-1}\,, \tag{2.68}$$

$$\boldsymbol{\Phi}_\beta\{S\} = \operatorname{ad}(\psi_{\beta\alpha}(u))\,.\,\boldsymbol{\Phi}_\alpha\{S\}\,. \tag{2.69}$$

An arc in the space $\mathfrak{S}$ is simply a map $\mu : (-1, 1) \to M$. Let us define the development of the arc μ at the point $\mu(0)$. Let $x : (-1, 1) \to Q$ be the lift of the map μ, i.e. $\pi(x(t)) = \mu(t)$ for each $t \in (-1, 1)$. The following is well known: The horizontal lift $y(t)$ of the arc $\mu(t)$ passing through the point $x(0)$ is given by

$$y(t) = x(t)\, g(t)\,, \quad t \in (1-, 1)\,, \tag{2.70}$$

$g(t)$ being the solution of the differential equation

$$\frac{\mathrm{d}g(t)}{\mathrm{d}t}\,.\,g^{-1}(t) = -\omega\left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right) \tag{2.71}$$

determined by the initial condition

$$g(0) = e\,. \tag{2.72}$$

Let $\mu(0) \in U_\alpha$. Then the development of the arc μ is defined by

$$\mu^*_\alpha(t) = \varphi_\alpha(x(0))\,.\,g^{-1}(t)\,.\,H\,.\,g(t)\,.\,\varphi_\alpha^{-1}(x(0)) \tag{2.73}$$

or by

$$\boldsymbol{\mu}^*_\alpha(t) = \operatorname{ad}(\varphi_\alpha(x(0)))\,.\,\operatorname{ad}(g^{-1}(t))\,.\,\mathfrak{h} \tag{2.74}$$

resp.

Theorem 2.7. *The developments μ_α^*, $\boldsymbol{\mu}_\alpha^*$ are well defined, i.e., they do not depend on the special choice of the lift $x : (-1, 1) \to Q$. If $\mu(0) \in U_\alpha \cap U_\beta$, we have*

$$\mu_\beta^*(t) = \psi_{\beta\alpha}(\mu(0)) \,.\, \mu_\alpha^*(t) \,.\, \psi_{\beta\alpha}^{-1}(\mu(0)) \,, \tag{2.75}$$

$$\boldsymbol{\mu}_\beta^*(t) = \mathrm{ad}\,(\psi_{\beta\alpha}(\mu(0))) \,.\, \boldsymbol{\mu}_\alpha^*(t) \,. \tag{2.76}$$

Proof. The relations (2.75) and (2.76) are obvious. Let $x' : (-1, 1) \to Q$ be another lift of the arc μ; we have the map $h : (-1, 1) \to H$ such that $x'(t) = x(t)\, h(t)$ for each $t \in (-1, 1)$. The lift (2.70) of the arc μ being horizontal, any other horizontal lift is given by $y(t) \,.\, g$, where $g \in G$ is any fixed element. Thus the horizontal lift $y' : (-1, 1) \to Q$ passing through the point $x'(0)$ is given by

$$y'(t) = x'(t) \,.\, g'(t) \,, \quad \text{where} \quad g'(t) = h^{-1}(t)\, g(t)\, h(0) \,. \tag{2.77}$$

But this means $\mu_\alpha^*(t) = \mu_\alpha'^*(t)$, $\boldsymbol{\mu}_\alpha^*(t) = \boldsymbol{\mu}_\alpha'^*(t)$. Q.E.D.

The following theorem has only an analytic signification, but it is of great importance for what follows.

Theorem 2.8. *Be given a Lie group G and a map $A : (-1, 1) \to \mathfrak{g}$. Let the map $g : (-1, 1) \to G$ be the solution of the differential equation*

$$\frac{\mathrm{d}g(t)}{\mathrm{d}t} \,.\, g^{-1}(t) = -A(t) \tag{2.78}$$

determined by the initial condition

$$g(0) = e \in G \,. \tag{2.79}$$

Let $V_0 \in \mathfrak{g}$ be a fixed vector. Define the map $V : (-1, 1) \to \mathfrak{g}$ by

$$V(t) = \mathrm{ad}\,(g^{-1}(t))\, V_0 \,. \tag{2.80}$$

Then $V(0) = V_0$ and

$$\frac{\mathrm{d}V(0)}{\mathrm{d}t} = [A(0), V_0] \,, \tag{2.81}$$

$$\frac{\mathrm{d}^2 V(0)}{\mathrm{d}t^2} = \left[\frac{\mathrm{d}A(0)}{\mathrm{d}t}, V_0\right] + [A(0), [A(0), V_0]] \,. \tag{2.82}$$

Proof. During the proof, we shall often use the Leibniz's formula; see [1], p. 11. Let us choose an arbitrary map $y : (-1, 1) \to G$ satisfying the conditions $y(0) = e$ and

$$\frac{\mathrm{d}y(0)}{\mathrm{d}s} = V_0 \,. \tag{2.83}$$

The function $z : (-1, 1) \times (-1, 1) \to G$ be defined by

$$z(t, s) = g^{-1}(t) \,.\, y(s) \,.\, g(t) \,. \tag{2.84}$$

Obviously, $z(t, 0) = e$ and

$$V(t) = \frac{\partial z(t, 0)}{\partial s} \,. \tag{2.85}$$

From (2.84), we get $g(t)\, z(t, s) = y(s)\, g(t)$; the differentiation with respect to s yields

$$g(t)\, V(t) = V_0\, g(t) \tag{2.86}$$

at the point $s = 0$. Differentiating with respect to t and using the supposition (2.78), i.e.

$$\frac{\mathrm{d}g(t)}{\mathrm{d}t} = -A(t)\, g(t) \,, \tag{2.87}$$

we get

$$-A(t)\, g(t)\, V(t) + g(t)\, \frac{\mathrm{d}V(t)}{\mathrm{d}t} = -V_0\, A(t)\, g(t) \,; \tag{2.88}$$

for $t = 0$, we get

$$\frac{\mathrm{d}V(0)}{\mathrm{d}t} = A(0)\, V_0 - V_0\, A(0) \,,$$

i.e. (2.81). A further differentiation of the equation (2.88) and the substitution $t = 0$ gives

$$\frac{\mathrm{d}^2 V(0)}{\mathrm{d}t^2} = \frac{\mathrm{d}A(0)}{\mathrm{d}t}\, V_0 - V_0\, \frac{\mathrm{d}A(0)}{\mathrm{d}t} + V_0\, A(0)\, A(0) - 2\, A(0)\, V_0\, A(0) +$$

$$+ A(0)\, A(0)\, V_0 = \left[\frac{\mathrm{d}A(0)}{\mathrm{d}t}, V_0\right] + A(0)[A(0), V_0] - [A(0), V_0]\, A(0) \,,$$

i.e. (2.82). Q.E.D.

Let V be a finite dimensional vector space over reals, let $\dim V = n$. Denote by $V^{[d]}$ the *Stiefel manifold* of all p-frames in V, a p-frame being an ordered set of p linearly independent vectors of V. Further, denote by $V^{(p)}$ the *Grassmann manifold* of all p-dimensional subspaces in V. The linear group $GL(p)$ acts freely on $V^{[p]}$ on the right as follows: if $e = [e_1, \ldots, e_p] \in V^{[p]}$ and $A \in GL(p)$, then we get $eA \in V^{[p]}$ as the obvious matrix product of e and A. Of course, $V^{(p)} = V^{[p]}/GL(p)$, and there is the natural map $V^{[p]} \to V^{(p)}$ associating to each p-frame e the p-space determined by the vectors $e_1, \ldots, e_p$.

Definition. Let $O \subset \mathbf{R}^m$ be a domain and $o \in O$ be its fixed point. (1) Be given maps $f, g : O \to V^{[p]}$. We say that f and g belong to the same t-jet at the point o

if $j^t_o(\iota f) = j^t_o(\iota g)$ where the natural map $\iota : V^{[p]} \to V \times V \times \ldots \times V$ (p-times) is defined by $\iota e = (e_1, \ldots, e_p)$. (2) Be given maps $F, F' : O \to V^{(p)}$. The maps F and F' are said to belong to the same t-jet at the point o if there are sections $f, f' : D \to V^{[p]}$; $D \subset V^{(p)}$ being some neighbourhood of the points $F(o)$ and $G(o)$, such that $j^t_o(fF) = j^t_o(f'F')$. (3) Let $\mathfrak{S} = \mathfrak{S}(P, M, G, Q, H, \omega)$ be a given space. Be given maps $f, g : O \to M$. We say that f and g belong to the same $\mathfrak{S}$-jet of order t at the point o, and we write $\mathfrak{S}j^t_o(f) = \mathfrak{S}j^t_o(g)$, if the following is true: Let $\varrho : (-1, 1) \to \mathbf{R}^m$ be an arbitrary map such that $\varrho(-1, 1) \subset O$ and $\varrho(0) = o$. By means of ϱ, we get the maps $\mu = f\varrho$, $v = g\varrho : (-1, 1) \to M$. Let us denote by $\boldsymbol{\mu}^*, \mathbf{v}^* : (-1, 1) \to \mathfrak{g}^{(\dim H)}$ the developments of the arcs μ, v at the point 0. Then $j^t_0(\boldsymbol{\mu}^*) = j^t_0(v^*)$.

Be given a space $\mathfrak{S}$ and maps $f, g : (-1, 1) \to M$. If $j^t_0(f) = j^t_0(g)$, we have $\mathfrak{S}j^t_0(f) = \mathfrak{S}j^t_0(g)$. However, the converse is not true: the most simple example is that of a space $\mathfrak{S}$ with $H = G$; for any two maps $f, g : (-1, 1) \to M$ satisfying $f(0) = g(0)$, we have $\mathfrak{S}j^t_0(f) = \mathfrak{S}j^t_0(g)$ for each t.

Theorem 2.9. *Be given a space $\mathfrak{S} = \mathfrak{S}(P, M, G, Q, H, \omega)$ and maps $f, g : (-1, 1) \to M$ such that $f(0) = g(0)$. Let $x, y : (-1, 1) \to Q$ be the lifts of the maps f and g resp. such that $z_0 = x(0) = y(0)$. Define the maps $A, B : (-1, 1) \to \mathfrak{g}$ by the equations*

$$A(t) = \omega\left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right), \quad B(t) = \omega\left(\frac{\mathrm{d}y(t)}{\mathrm{d}t}\right). \tag{2.89}$$

Then (1) $\mathfrak{S}j^1_0(f) = \mathfrak{S}j^1_0(g)$, (2) $\mathfrak{S}j^2_0(f) = \mathfrak{S}j^2_0(g)$ *if and only if* (1)

$$[A(0) - B(0), \mathfrak{h}] \subset \mathfrak{h}\,, \tag{2.90}$$

(2) *we have* (2.90) *and*

$$\left[\frac{\mathrm{d}A(0)}{\mathrm{d}t} - \frac{\mathrm{d}B(0)}{\mathrm{d}t}, v\right] + [A(0) - B(0), [A(0), v]] - \\ - [B(0), [A(0) - B(0), v]] \in \mathfrak{h} \tag{2.91}$$

for each $v \in \mathfrak{h}$.

Proof. Let $\{U_\alpha\}$ be the usual open covering of the base space M with the homeomorphisms $\varphi_\alpha : \pi^{-1}(U_\alpha) \to G$. The developments $\mathbf{f}^*_\alpha, \mathbf{g}^*_\alpha : (-1, 1) \to \mathfrak{g}^{(\dim H)}$ are given by

$$\mathbf{f}^*_\alpha(t) = \mathrm{ad}\,(\varphi_\alpha(z_0))\,\mathrm{ad}\,(g^{-1}(t))\,\mathfrak{h}\,, \quad \mathbf{g}^*_\alpha(t) = \mathrm{ad}\,(\varphi_\alpha(z_0))\,\mathrm{ad}\,(h^{-1}(t))\,\mathfrak{h}\,, \tag{2.92}$$

where $g(t)$ is the solution of the equations (2.78), (2.79) and $h(t)$ is the solution of analogous equations

$$\frac{\mathrm{d}h(t)}{\mathrm{d}t}\,.\,h^{-1}(t) = -B(t)\,, \quad h(0) = e \in G\,. \tag{2.93}$$

Looking for the contact of the maps $\mathbf{f}_\alpha^*$, $\mathbf{g}_\alpha^*$, we may as well restrict ourselves to the investigation of the contact of the maps $F, G : (-1, 1) \to \mathfrak{g}^{(\dim H)}$ given by the equations

$$(2.94) \qquad F(t) = K(t)\,\mathfrak{h}\,, \quad G(t) = L(t)\,\mathfrak{h}\,,$$

where

$$(2.95) \qquad K(t) = \operatorname{ad}(g^{-1}(t))\,, \quad L(t) = \operatorname{ad}(h^{-1}(t))\,.$$

Let $E_0 = (e_1, \ldots, e_{\dim H})$ be a fixed basis in $\mathfrak{h}$. Then

$$(2.96) \qquad V(t) = K(t)\,E_0\,, \quad W(t) = L(t)\,E_0$$

are basis of the spaces $F(t)$ and $G(t)$ resp. The general basis of the space $G(t)$ is

$$(2.97) \qquad W'(t) = W(t)\,S(t) = L(t)\,E_0\,S(t)\,,$$

where $S : (-1,1) \to GL(\dim H)$ is a map. Of course,

$$(2.98) \qquad \frac{\mathrm{d}V(t)}{\mathrm{d}t} = \frac{\mathrm{d}K(t)}{\mathrm{d}t}\,E_0\,, \quad \frac{\mathrm{d}^2V(t)}{\mathrm{d}t^2} = \frac{\mathrm{d}^2K(t)}{\mathrm{d}t^2}\,E_0\,,$$

$$\frac{\mathrm{d}W'(t)}{\mathrm{d}t} = \frac{\mathrm{d}L(t)}{\mathrm{d}t}\,E_0\,S(t) + L(t)\,E_0\,\frac{\mathrm{d}S(t)}{\mathrm{d}t}\,,$$

$$\frac{\mathrm{d}^2W'(t)}{\mathrm{d}t^2} = \frac{\mathrm{d}^2L(t)}{\mathrm{d}t^2}\,E_0\,S(t) + 2\frac{\mathrm{d}L(t)}{\mathrm{d}t}\,E_0\,\frac{\mathrm{d}S(t)}{\mathrm{d}t} + L(t)\,E_0\,\frac{\mathrm{d}^2S(t)}{\mathrm{d}t^2}\,.$$

Let us use the following notation: if $\varepsilon = (e_1, \ldots, e_z)$, $e_\xi \in \mathfrak{g}$, and $v \in \mathfrak{g}$, then $[v, \varepsilon] = ([v, e_1], \ldots, [v, e_z])$. From Theorem 2.8 and (2.96)–(2.98), we get

$$(2.99) \qquad V(0) = E_0\,, \quad \frac{\mathrm{d}V(0)}{\mathrm{d}t} = [A(0), E_0]\,,$$

$$\frac{\mathrm{d}^2V(0)}{\mathrm{d}t^2} = \left[\frac{\mathrm{d}A(0)}{\mathrm{d}t}, E_0\right] + [A(0), [A(0), E_0]]\,,$$

$$(2.100) \qquad W'(0) = E_0S_0\,, \quad \frac{\mathrm{d}W'(0)}{\mathrm{d}t} = [B(0), E_0]\,S_0 + E_0S_1\,,$$

$$\frac{\mathrm{d}^2W'(0)}{\mathrm{d}t^2} = \left[\frac{\mathrm{d}B(0)}{\mathrm{d}t}, E_0\right]S_0 + [B(0), [B(0), E_0]]\,S_0 + 2[B(0), E_0]\,S_1 + E_0S_2\,,$$

where

$$(2.101) \qquad S_0 = S(0) \in GL(\dim H)\,, \quad S_1 = \frac{\mathrm{d}S(0)}{\mathrm{d}t}\,, \quad S_2 = \frac{\mathrm{d}^2S(0)}{\mathrm{d}t^2}\,.$$

The condition $\mathfrak{S}j_0^2(f) = \mathfrak{S}j_0^2(g)$ is equivalent to the existence of matrices (2.101) such that

$$(2.102) \qquad V(0) = W'(0), \quad \frac{dV(0)}{dt} = \frac{dW'(0)}{dt}, \quad \frac{d^2V(0)}{dt^2} = \frac{d^2W'(0)}{dt^2}.$$

From (2.102_1), we get that S_0 is the identity of the group $GL(\dim H)$, this group acting freely on the Stiefel manifold $\mathfrak{h}^{[\dim H]}$. If (2.102_2) is satisfied, there is a matrix S_1 of the type $\dim H \times \dim H$ such that

$$(2.103) \qquad [A(0) - B(0), E_0] = E_0 S_1 .$$

This means that we have $[A(0) - B(0), e_i] \in \mathfrak{h}$ for each vector $e_i \in E_0$, i.e. (2.90). Conversely, let us suppose (2.90). $[A(0) - B(0), E_0]$ is the set of $\dim H$ vectors in $\mathfrak{h}$, and there is a matrix S such that (2.103) is satisfied. The condition (2.102_3) is – see (2.103) – equivalent to the existence of a matrix S_2 such that

$$\left[\frac{dA(0)}{dt}, E_0\right] + [A(0), [A(0), E_0]] = \left[\frac{dB(0)}{dt}, E_0\right] +$$
$$+ [B(0), [B(0), E_0]] + 2[A(0) - B(0), [B(0), E_0]] + E_0 S_2 ,$$

and it is quite easy to see that this is equivalent to (2.91). Q.E.D.

It is natural to present the following

Definition. Be given spaces $\mathfrak{S} = \mathfrak{S}(P, M, G, Q, H, \omega)$ and $\mathfrak{S}' = \mathfrak{S}'(P', M', G, Q', H, \omega')$. A diffeomorphism $f : M \to M'$ is called the deformation of order r if there is a lift $F : P \to P'$ of f, this lift being a bundle isomorphism and satisfying the following conditions: (1) $F(Q) = Q'$. (2) Denote by $\omega^* = F_* \omega'$ the induced connection on P, and let us write $\mathfrak{S}^* = \mathfrak{S}^*(P, M, G, Q, H, \omega^*)$. Let $u \in M$ be an arbitrary point and $v : (-1, 1) \to M$ an arbitrary map such that $v(0) = u$. Let $\boldsymbol{\varphi} : (-1, 1) \to \mathfrak{g}^{(\dim H)}$ be the development of the arc v with respect to the connection ω; analoguously, let $\boldsymbol{\varphi}^*$ be the development of the arc v with respect to the connection ω^*. Then $j_0^r(\boldsymbol{\varphi}) = j_0^r(\boldsymbol{\varphi}^*)$.

The proof of the following theorem is very similar to that of Theorem 2.9.

Theorem 2.10. *Be given spaces $\mathfrak{S} = \mathfrak{S}(P, M, G, Q, H, \omega)$, $\mathfrak{S}' = \mathfrak{S}'(P', M', G, Q', H, \omega')$ and a diffeomorphism $f : M \to M'$. The map is a deformation of the* (1) *first*, (2) *second order if and only if there is a bundle isomorphism $F : P \to P'$ such that F is a lift of f, $F(Q) = Q'$ and* (1)

$$(2.104) \qquad [\omega(X) - \omega^*(X), \mathfrak{h}] \subset \mathfrak{h} \quad on \quad Q ,$$

(2) *we have* (2.104) *and, on* Q,

$$(2.105) \qquad [\omega(X) - \omega^*(X), [\omega(X), v]] - [\omega^*(X), [\omega(X) - \omega^*(X), v]] \in \mathfrak{h}$$

for each vector $v \in \mathfrak{h}$.

Here, $\omega^* = F_*\omega'$. *In the case of* $\mathfrak{h}$ *reductive in* g, *the condition* (2.104) *is equivalent to the condition*

$$(2.106) \qquad \varphi(X) = \varphi^*(X) \quad on \quad Q\,.$$

3. SPECIALIZATION OF FRAMES

3.1. Our fundamental problem may be formulated as follows:

Problem I. *Be given spaces*

$$(3.1) \qquad \mathfrak{S} = \mathfrak{S}(P, M, G, Q, H, \omega)\,, \quad \mathfrak{S}' = \mathfrak{S}'(P, M, G, Q, H, \omega')\,;$$

we have to decide whether there is a bundle isomorphism

$$(3.2) \qquad F : P \to P$$

with the following properties: (1) *the diagram*

$$(3.3) \qquad \begin{array}{ccc} P & \xrightarrow{\ F\ } & P \\ \pi\big\downarrow & & \big\downarrow\pi \\ M & \xrightarrow{\ \text{id}\ } & M \end{array}$$

is commutative; (2) $F(Q) = Q$; (3) $F_*\omega' = \omega$.

Very often, this problem has the following formulation:

Problem II. *Be given spaces* (3.1) *and local sections*

$$(3.4) \qquad v : M \to Q\,, \quad v' : M \to Q\,.$$

Thus we get $\mathfrak{g}$*-valued 1-forms*

$$(3.5) \qquad \omega_v = v_*\omega\,, \quad \omega_{v'} = v'_x\omega'$$

on M. *We have to decide whether there is a* (*local*) *map*

$$(3.6) \qquad h : M \to H$$

such that

$$(3.7) \qquad \omega_{v'} = \text{ad}\,(h^{-1})\,\omega_v + h^{-1}\,\mathrm{d}h$$

on M.

If the map (3.6) satisfies (3.7), let us define the map $F : P \to P$ as follows: Let $p \in P$, $\pi(p) = m$. Then there is a uniquely determined element $g \in G$ such that $p = v(m)\, hg$. We set

$$F(p) = v'(m)\, g\,. \tag{3.8}$$

The map F is a bundle isomorphism satisfying the conditions (1)–(3) of our Problem I. Thus both the problems are equivalent. In what follows, we shall try to present an algorism leading, in the general case (what is general is to be explained later on), to the solution of Problem II.

Be given spaces (3.1). *From now on, let us suppose that*

$$\dim M < \dim \mathfrak{g}/\mathfrak{h}\,. \tag{3.9}$$

First of all, let us study the space $\mathfrak{S}$. Let $m_0 \in M$ be a fixed point and $U \subset M$ its neighbourhood such that the local sections (3.4) are defined over it. Let $T_{v(m_0)}(Q)$ be the tangent vector space of the manifold Q at the point $v(m_0)$. Introduce the notation

$$K_{v(m_0)} = \omega(T_{v(m_0)}(Q))\,. \tag{3.10}$$

Obviously,

$$K_{v(m_0)} \supset \mathfrak{h}\,, \quad \dim K_{v(m_0)} = \dim \mathfrak{h} + \dim M\,. \tag{3.11}$$

Further, introduce the notation

$$L_{v(m_0)} = \iota_H(K_{v(m_0)}) \subset \mathfrak{g}/\mathfrak{h}\,, \tag{3.12}$$

$\iota_H : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ being the natural homomorphism (2.11). From (3.11_2) and (3.9), we get

$$\dim L_{v(m_0)} = \dim M < \dim \mathfrak{g}/\mathfrak{h}\,. \tag{3.13}$$

Let $h_0 \in H$ be a fixed element. Then

$$K_{v(m_0)h_0} = \mathrm{ad}\,(h_0^{-1})\, K_{v(m_0)}\,, \tag{3.14}$$

and we have

$$L_{v(m_0)h_0} = \mathrm{ad}\,(h_0^{-1})\, L_{v(m_0)}\,, \tag{3.15}$$

ad being the representation (2.12).

Let $H_{v(m_0)}$ be the set of all $h \in H$ such that

$$K_{v(m_0)h} = K_{v(m_0)}\,. \tag{3.16}$$

The set $H_{v(m_0)}$ is obviously a Lie subgroup of the group H. Let $q \in Q$ be an arbitrary point such that $\pi(q) = m_0$; let us suppose that $q = v(m_0)\,\varkappa$, $\varkappa \in H$. If $K_q = \omega(T_q(Q))$, we have

$$K_q = \mathrm{ad}\,(\varkappa^{-1})\,K_{v(m_0)}\,. \tag{3.17}$$

Defining H_q analoguously as the Lie group of all elements $h \in H$ such that $K_{qh} = K_q$, we have

$$H_q = \varkappa^{-1} H_{v(m_0)}\,\varkappa\,. \tag{3.18}$$

Recalling the known formulas

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{ad}\,(\exp t\,Y)\,X|_{t=0} = [Y, X]\,, \quad \mathrm{ad}\,(\exp Y)\,X = \exp\,(\mathrm{ad}\,Y)\,X\,; \tag{3.19}$$

see [2], pp. 227–228; we get

Theorem 3.1. *Be given Lie groups $H \subset G$ with the Lie algebras $\mathfrak{h} \subset \mathfrak{g}$. Let K be a linear space such that $\mathfrak{h} \subset K \subset \mathfrak{g}$. Let*

$$H_K = \{h \in H \mid \mathrm{ad}\,(h)\,K = K\}\,, \quad \mathfrak{h}_K = \{v \in \mathfrak{h} \mid [v, K] \subset K\}\,. \tag{3.20}$$

Then H_K is a Lie group and $\mathfrak{h}_K$ its Lie algebra.

It follows that the Lie algebra $\mathfrak{h}_{v(m_0)}$ of the group $H_{v(m_0)}$ is the Lie algebra of all vectors $v \in \mathfrak{h}$ such that

$$[v, K_{v(m_0)}] \subset K_{v(m_0)}\,. \tag{3.21}$$

From (3.18), we get

$$\mathfrak{h}_q = \mathrm{ad}\,(\varkappa^{-1})\,\mathfrak{h}_{v(m_0)}\,. \tag{3.22}$$

Let us summarize the preceding results in

Theorem 3.2. *Be given a space $\mathfrak{S} = \mathfrak{S}(P, M, G, Q, H, \omega)$ and a fixed point $m_0 \in M$. To each point $q \in Q$, $\pi(q) = m_0$, let us associate the space*

$$K_q = \omega(T_q(Q))\,, \quad \mathfrak{h} \subset K_q \subset \mathfrak{g}\,; \tag{3.23}$$

the space

$$K_{qh} = \mathrm{ad}\,(h^{-1})\,K_q \tag{3.24}$$

being associated to the point $qh \in Q$, $h \in H$. Further, to the point q, we associate the Lie group $H_q \subset H$ consisting of the elements $k \in H$ such that $\mathrm{ad}\,(k)\,K_q = K_q$; *the*

Lie algebra $\mathfrak{h}_q$ *of* H_q *is the set of the vectors* $v \in \mathfrak{h}$ *such that* $[v, K_q] \subset K_q$. *We have*

$$H_{qh} = h^{-1} H_q h\,, \quad \mathfrak{h}_{qh} = \operatorname{ad}(h^{-1})\,\mathfrak{h}_q\,. \tag{3.25}$$

Let us introduce the notation $L_q = \iota_H K_q$, $\iota_H : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ *being the natural homomorphism. According to* (2.14), *we have*

$$L_q = \varphi(T_q(Q))\,. \tag{3.26}$$

Further,

$$L_{qh} = \operatorname{ad}(h^{-1})\,L_q\,, \tag{3.27}$$

$\operatorname{ad}(h^{-1}) : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$ *being the map* (2.12). *Of course,* $\dim L_q = \dim M$.

3.2. Denote by Z the manifold of all spaces K such that $\mathfrak{h} \subset K \subset \mathfrak{g}$, $\dim K = \dim \mathfrak{h} + \dim M$. Analoguously, denote by Z' the manifold of all subspaces $L \subset \mathfrak{g}/\mathfrak{h}$ such that $\dim L = \dim M$. The manifolds Z and Z' are clearly diffeomorphic, the natural identification being given by the map $K \to \iota_H(K)$.

Let $K_0 \in Z$ be a fixed space. In the space $\mathfrak{g}$, let us choose a vector basis

$$e_1, \ldots, e_r, e_{r+1}, \ldots, e_{r+s}, e_{r+s+1}, \ldots, e_{r+s+t} \tag{3.28}$$

such that $e_1, \ldots, e_r$ is a basis of the space $\mathfrak{h}$ and $e_1, \ldots, e_{r+s}$ a basis of the space K_0. We have $\dim \mathfrak{h} = r$, $\dim M = s$. Introducing in $\mathfrak{g}$ the coordinates $x^1, \ldots, x^{r+s+t}$ by the relation

$$e = \sum_{i=1}^{r+s+t} x^i e_i\,, \tag{3.29}$$

$$x^{r+s+1} = \ldots = x^{r+s+t} = 0 \tag{3.30}$$

are the equations of the space K_0. Any system of t linearly independent linear equations in x^i determines a subspace K of the dimension $r + s$; $\mathfrak{h} \subset K$ if and only if the system consists of linear equations in $x^{r+1}, \ldots, x^{r+s+t}$ only. Each space $K \in Z$ is thus given by the equations

$$\sum_{\alpha=1}^{s} b^\nu_\alpha x^{r+\alpha} + \sum_{\mu=1}^{t} b^\nu_\mu x^{r+s+\mu} = 0\,; \quad \nu = 1, \ldots, t\,. \tag{3.31}$$

Clearly, there is a neighbourhood O of the space K_0 in Z such that each space $K \in O$ is given by the system (3.31) with rang $(b^\nu_\mu) = t$, i.e. by a set of the form

$$x^{r+s+\mu} = \sum_{\alpha=1}^{s} a^\mu_\alpha x^{r+\alpha}\,; \quad \mu = 1, \ldots, t\,. \tag{3.32}$$

This system is determined uniquely by the spaces K_0, K and the basis (3.28). The numbers a_α^μ are thus the coordinates in the neighbourhood O, and we get

$$\dim Z = \dim Z' = st = \dim M \,.\, (\dim \mathfrak{g}/\mathfrak{h} - \dim M)\,. \tag{3.33}$$

Let $K_0 \in Z$ be our fixed space; let us determine $\dim \mathfrak{h}_{K_0}$, the space $\mathfrak{h}_{K_0}$ being defined by the equation (3.20_2). The multiplication in the Lie algebra $\mathfrak{g}$ be given by

$$[e_i, e_j] = \sum_{k=1}^{r+s+t} c_{ij}^k e_k, \quad i, j = , \ldots, r + s + t\,. \tag{3.34}$$

$\mathfrak{h}$ being a Lie subalgebra, we have

$$[e_\varrho, e_\sigma] = \sum_{\tau=1}^{r} c_{\varrho\sigma}^\tau e_\tau\,; \quad \varrho, \sigma = 1, \ldots, r\,; \tag{3.35}$$

i.e.

$$c_{\varrho\sigma}^{r+\alpha} = c_{\varrho\sigma}^{r+s+\mu} = 0\,; \quad \varrho, \sigma = 1, \ldots, r\,; \quad \alpha = 1, \ldots, s\,; \quad \mu = 1, \ldots, t\,. \tag{3.36}$$

Let $v \in \mathfrak{h}$, $w \in K_0$, i.e.

$$v = \sum_{\varrho=1}^{r} v^\varrho e_\varrho\,, \quad w = \sum_{\sigma=1}^{r} w^\sigma e_\sigma + \sum_{\alpha=1}^{s} w^{r+\alpha} e_{r+\alpha}\,. \tag{3.37}$$

Then

$$[v, w] = \sum_{\tau=1}^{r} \sum_{\varrho=1}^{r} v^\varrho \Big(\sum_{\sigma=1}^{r} c_{\varrho\sigma}^\tau w^\sigma + \sum_{\alpha=1}^{s} c_{\varrho,r+\alpha}^\tau w^{r+\alpha} \Big) e_\tau +$$

$$+ \sum_{\varrho=1}^{r} \sum_{\alpha,\beta=1}^{s} c_{\varrho,r+\alpha}^{r+\beta} v^\varrho w^{r+\alpha} e_{r+\beta} + \sum_{\varrho=1}^{r} \sum_{\alpha=1}^{s} \sum_{\mu=1}^{t} c_{\varrho,r+\alpha}^{r+s+\mu} v^\varrho w^{r+\alpha} e_{r+s+\mu}\,.$$

If $[v, w] \in K_0$ for each vector $w \in K_0$, we have

$$\sum_{\varrho=1}^{r} c_{\varrho,r+\alpha}^{r+s+\mu} v^\varrho = 0\,; \quad \alpha = 1, \ldots, s\,; \quad \mu = 1, \ldots, t\,. \tag{3.38}$$

The system (3.38) is the system of equations of the space $\mathfrak{h}_{K_0} \subset \mathfrak{h}$. According to (3.35), we have

Theorem 3.3. *Be given Lie algebras $\mathfrak{h} \subset \mathfrak{g}$ and a space K_0 such that $\mathfrak{h} \subset K_0 \subset \mathfrak{g}$. Let Z be the manifold of all spaces K such that $\mathfrak{h} \subset K \subset \mathfrak{g}$, $\dim K = \dim K_0$. Let $\mathfrak{h}_{K_0}$ be the Lie algebra consisting of the vectors $v \in \mathfrak{h}$ such that $[v, K_0] \subset K_0$. Then*

$$\dim \mathfrak{h}_{K_0} \geqq \dim \mathfrak{h} - \dim Z\,. \tag{3.39}$$

Let us introduce the notation

$$\mathscr{D}(K) \equiv \mathscr{D}(\mathfrak{h} \subset K \subset \mathfrak{g}) = \dim \mathfrak{h} - \dim \mathfrak{h}_K ; \tag{3.40}$$

$$\mathscr{D} = \max \mathscr{D}(K) , \quad K \in Z . \tag{3.41}$$

The space $K \in Z$ is called *regular* if $\mathscr{D}(K) = \mathscr{D}$.

Theorem 3.4. *The set of regular spaces $K \in Z$ is open.*

Proof. Let $K_0 \in Z$ be a regular space. In $\mathfrak{g}$, let us choose a basis (3.28) with the described properties. The space K (3.32) is determined by the vectors

$$e_\varrho ; \quad \varrho = 1, \ldots, r ; \tag{3.42}$$

$$f_{r+\alpha} = e_{r+\alpha} + \sum_{\mu=1}^{t} a_\alpha^\mu e_{r+s+\mu} ; \quad \alpha = 1, \ldots, s .$$

Let $w \in K$, i.e.

$$w = \sum_{\sigma=1}^{r} w^\sigma e_\sigma + \sum_{\alpha=1}^{s} w^{r+\alpha} f_{r+\alpha} . \tag{3.43}$$

If $v \in \mathfrak{h}$ is given by (3.37_1), we have

$$[v, w] = \sum_{\varrho=1}^{r} \sum_{\alpha=1}^{s} v^\varrho w^{r+\alpha} \Big(\sum_{\beta=1}^{s} d_{\varrho,r+\alpha}^{r+\beta} e_{r+\beta} + \sum_{\mu=1}^{t} d_{\varrho,r+\alpha}^{r+s+\mu} e_{r+s+\mu} \Big) \bmod \mathfrak{h} , \tag{3.44}$$

where

$$d_{\varrho,r+\alpha}^{r+\beta} = c_{\varrho,r+\alpha}^{r+\beta} + \sum_{\mu=1}^{t} a_\alpha^\mu c_{r+s+\mu}^{r+\beta} , \tag{3.45}$$

$$d_{\varrho,r+\alpha}^{r+s+\mu} = c_{\varrho,r+\alpha}^{r+s+\mu} + \sum_{\nu=1}^{t} a_\alpha^\nu c_{\varrho,r+s+\nu}^{r+s+\mu} .$$

Now,

$$[v, w] - \sum_{\varrho=1}^{r} \sum_{\alpha=1}^{s} v^\varrho w^{r+\alpha} \sum_{\beta=1}^{s} d_{\varrho,r+\alpha}^{r+\beta} f_{r+\beta} \equiv$$

$$\equiv \sum_{\varrho=1}^{r} \sum_{\alpha=1}^{s} v^\varrho w^{r+\alpha} \sum_{\mu=1}^{t} \Big(d_{\varrho,r+\alpha}^{r+s+\mu} - \sum_{\beta=1}^{s} d_{\varrho,r+\alpha}^{r+\beta} a_\beta^\mu \Big) e_{r+s+\mu} \bmod \mathfrak{h} .$$

The equations of the space $\mathfrak{h}_K$ are

$$\sum_{\varrho=1}^{r} \Big(d_{\varrho,r+\alpha}^{r+s+\mu} - \sum_{\beta=1}^{s} d_{\varrho,r+\alpha}^{r+\beta} a_\beta^\mu \Big) v^\varrho = 0 ; \quad \alpha = 1, \ldots, s ; \quad \mu = 1, \ldots, t ;$$

i.e.

$$(3.46)\quad \sum_{\varrho=1}^{r}\Big(c_{\varrho,r+\alpha}^{r+s+\mu} + \sum_{\nu=1}^{t} a_\alpha^\nu c_{\varrho,r+s+\nu}^{r+s+\mu} - \sum_{\beta=1}^{s} a_\beta^\mu c_{\varrho,r+\alpha}^{r+\beta} - \sum_{\beta=1}^{s}\sum_{\nu=1}^{t} a_\beta^\mu a_\alpha^\nu c_{\varrho,r+s+\nu}^{r+\beta}\Big)\, v^\varrho = 0 :$$

$$\alpha = 1, \ldots, s\,; \quad \mu = 1, \ldots, t\,.$$

For $K_0 = K$, i.e. $a_\alpha^\mu = 0$, we get (3.40). Choosing a sufficiently small $\varepsilon > 0$, the rang of the system (3.46) with $|a_\alpha^\nu| < \varepsilon$ is not less than that of the system (3.38). On the other hand, it is not greater, the space K_0 being regular. Q.E.D.

3.3. We shall say that the spaces $K, K' \in Z$ are situated in the same *orbit* if there is an $h \in H$ such that

$$(3.47)\qquad K' = \operatorname{ad}(h)\, K\,.$$

Let us denote by $\{K\}$ the orbit determined by the space $K \in Z$; in the case (3.47), we have $\{K\} = \{K'\}$. Quite analoguously, the spaces $L, L' \in Z'$ are situated in the same orbit if there is an $h \in H$ such that

$$(3.48)\qquad L' = \operatorname{ad}(h)\, L\,;$$

introduce again the notation $\{L\}$ for the orbit determined by the space L. It is easy to see that $\{K\} = \{K'\}$ if and only if $\{\iota_H(K)\} = \{\iota_H(K')\}$.

Investigating a given space $\mathfrak{S}$, we have to determine the system of all orbits in the manifold Z. Because the dimensions of the orbits may differ, this system is often very complicated, and it is difficult to formulate general theorems.

Let us start with the Lie algebra $\mathfrak{g}$, its subalgebra $\mathfrak{h}$ and a space K_0, $\mathfrak{h} \subset K_0 \subset \mathfrak{g}$, and let us consider the basis (3.28). Each space $K \in Z$ situated in some neighbourhood O of the space $K_0 \in Z$ is given by the equations (3.32), the numbers a_α^μ being the coordinates in O; the space K is given by the vectors (3.42).

Let $K \in Z$ and let $v \in \mathfrak{h}$ be a fixed non-zero vector. Consider the system of spaces

$$(3.49)\qquad K(v, t) = \operatorname{ad}(\exp tv)\, K\,,$$

where $t \in (-\delta, \delta)$ and $\delta > 0$ is small. Obviously, $K(v, 0) = K$ for each vector $v \in \mathfrak{h}$. The space (3.49) is determined by the vectors e_ϱ; $\varrho = 1, \ldots, r$; and

$$(3.50)\qquad g_{r+\alpha}(v, t) = \operatorname{ad}(\exp tv) f_{r+\alpha}\,; \quad \alpha = 1, \ldots, s\,;$$

$f_{r+\alpha}$ being the vector (3.42_2). Again, we have $g_{r+\alpha}(v, 0) = f_{r+\alpha}$. We may write

$$(3.51)\qquad g_{r+\alpha}(v, t) = f_{r+\alpha} + t h_{r+\alpha}(v) + O(t^2)\,,$$

where

$$(3.52)\qquad h_{r+\alpha}(v) = \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{ad}(\exp tv)\big|_{t=0} \cdot f_{r+\alpha} = [v, f_{r+\alpha}]\,;$$

the last equation being the consequence of (3.19). Let us determine the equations of the space $K(v, t)$. Restricting δ in such a way that $K(v\ t) \in O$ for each $|t| < \delta$, these equations are of the form

$$(3.53) \qquad x^{r+s+\mu} = \sum_{\beta=1}^{s} (a^{\mu}_{\beta} + tb^{\mu}_{\beta}(v) + O(t^2))\, x^{r+\beta} \quad \mu = 1, \ldots, t\,;$$

our task is to determine $b^{\mu}_{\beta}(v)$. Because of (3.44), we have

$$(3.54) \qquad h_{r+\alpha}(v) = \sum_{\varrho=1}^{r} v^{\varrho}\left(\sum_{\beta=1}^{s} d^{r+\beta}_{\varrho,r+\alpha} e_{r+\beta} + \sum_{\mu=1}^{t} d^{r+s+\mu}_{\varrho,r+\alpha} e_{r+s+\mu}\right) \bmod \mathfrak{h}\,,$$

$d^{r+\beta}_{\varrho,r+\alpha}$ and $d^{r+s+\mu}_{\varrho,r+\alpha}$ being determined by the relations (3.45). The vector (3.51) has the coordinates

$$(3.55) \qquad x^{\varrho}\,; \quad \varrho = 1, \ldots, r\,;$$

$$x^{r+\beta} = \delta^{r+\beta}_{r+\alpha} + t\sum_{\varrho=1}^{r} v^{\varrho} d^{r+\beta}_{\varrho,r+\alpha} + O(t^2)\,; \quad \beta = 1, \ldots, s\,;$$

$$x^{r+s+\mu} = a^{\mu}_{\alpha} + t\sum_{\varrho=1}^{r} v^{\varrho} d^{r+s+\mu}_{\varrho,r+\alpha} + O(t^2)\,; \quad \mu = 1, \ldots, t\,;$$

where $\delta^{r+\beta}_{r+\alpha}$ is the Kronecker's delta. Substituting into (3.53), we get

$$(3.56) \qquad b^{\mu}_{\alpha}(v) = \sum_{\varrho=1}^{r} v^{\varrho}\left(d^{r+s+\mu}_{\varrho,r+\alpha} - \sum_{\beta=1}^{s} a^{\mu}_{\beta} d^{r+\beta}_{\varrho,r+\alpha}\right),$$

$b^{\mu}_{\alpha}(v)$ being just the left hand side of the equation (3.46).

On O, the vector fields $\partial/\partial a^{\mu}_{\alpha}$; $\alpha = 1, \ldots, s$; $\mu = 1, \ldots, t$; are the basis of the vector fields. Let us consider the vector fields

$$(3.57) \qquad V_{\varrho} = \sum_{\alpha=1}^{s} \sum_{\mu=1}^{t} \left(d^{r+s+\mu}_{\varrho,r+\alpha} - \sum_{\beta=1}^{s} a^{\mu}_{\beta} d^{r+\beta}_{\varrho,r+\alpha}\right) \frac{\partial}{\partial a^{\mu}_{\alpha}}\,; \quad \varrho = 1, \ldots, r\,;$$

on O. Let $\Theta_K \subset T_K(O)$ be the linear subspace spanned by the vectors (3.57).

Theorem 3.5. *We have*

$$(3.58) \qquad \dim \Theta_K = \mathcal{D}(K)\,,$$

the number $\mathcal{D}(K)$ *being defined by* (3.40).

Using the known results on a Lie group acting on a manifold — see [1], Proposition 4.1, p. 42 — we get

Theorem 3.6. *Let $K_0 \in Z$ be a regular space. Then there exists its neighbourhood $O \subset Z$ with the following properties:* (1)

$$\dim \Theta_K = \mathscr{D} \quad for \quad K \in O . \tag{3.59}$$

(2) *The subspaces Θ_K are an involutive distribution on O.* (3) *Let $K_1 \in Z$ be a fixed space and $V \subset O$ be the integral manifold of dimension $\mathscr{D}$ of the distribution Θ_K passing through the space K_1, then $V = O \cap \{K_1\}$, $\{K_1\}$ being the orbit of the space K_1.*

The following theorem is a simple consequence of the preceding one.

Theorem 3.7. *Let $K_0 \in Z$ be a regular space. Then there exist its neighbourhood $O \subset Z$ and a manifold $W \subset O$ with the following properties:* (1) $K_0 \in W$, $\dim W = \dim Z - \mathscr{D}$ (2) *If $K \in O$, there is one and only one point $K_W \in W$ such that $\{K\} = \{K_W\}$.*

The manifold W may be any manifold through the space K_0 satisfying $T_{K_0}(W) \cap \Theta_{K_0} = 0$; of course, we might be pressed to restrict our manifold O. If O/Θ_K has the usual meaning, the manifolds W and O/Θ_K are diffeomorphic.

Let us consider the coordinates in $\mathfrak{g}$ introduced above. The space Θ_{K_0} at a regular point K_0 is determined by the vectors

$$V_\varrho = \sum_{\alpha=1}^{s} \sum_{\mu=1}^{t} c_{\varrho,r+\alpha}^{r+s+\mu} \left.\frac{\partial}{\partial a_\alpha^\mu}\right|_{K_0} ; \quad \varrho = 1, \ldots, r ; \tag{3.60}$$

$\mathscr{D}$ of them are linearly independent; let us choose the numeration in such a way that

$$V_A = \sum_{\alpha=1}^{s} \sum_{\mu=1}^{t} c_{A,r+\alpha}^{r+s+\mu} \left.\frac{\partial}{\partial a_\alpha^\mu}\right|_{K_0} ; \quad A = 1, \ldots, \mathscr{D} \tag{3.61}$$

are linearly independent. Choose the numbers

$$r_{B\mu}^\alpha ; \quad \alpha = 1, \ldots, s ; \quad \mu = 1, \ldots, t ; \quad B = 1, \ldots, \mathscr{D} ; \tag{3.62}$$

such that (1) the rang of the matrix of type $ts \times \mathscr{D}$ formed, in the obvious sense, by the elements (3.62) is equal to $\mathscr{D}$, (2) we have

$$\sum_{\alpha=1}^{s} \sum_{\mu=1}^{t} r_{B\mu}^\alpha c_{A,r+\alpha}^{r+s+\mu} \neq 0 ; \quad A, B = 1, \ldots, \mathscr{D} . \tag{3.63}$$

Then the space determined by the equations

$$\sum_{\alpha=1}^{s} \sum_{\mu=1}^{t} r_{B\mu}^\alpha a_\alpha^\mu = 0 ; \quad B = 1, \ldots, \mathscr{D} ; \tag{3.64}$$

is an example of the manifold W of Theorem 3.7. In other words: There is a neighbourhood $O \subset Z$ of the space K_0 such that to each $K \in O$ there exists a unique space K_W (3.32) such that its coordinates a_α^μ satisfy (3.64) and we have $\{K\} = \{K_W\}$.

3.4. Be given a space $\mathfrak{S} = \mathfrak{S}(M, P, G, Q, H, \omega)$. Let $m_0 \in M$ be a fixed point and $U \subset M$ a neighbourhood of the point m_0. Further, be given a section $v : U \to Q$. Let us write

$$K_{v(m)} = \omega(T_{v(m)}(Q)) . \tag{3.65}$$

Obviously, $K_m \in Z$. Analoguously, we get the space

$$K_{v'(m)} = \omega(T_{v'(m)}(Q)) \tag{3.66}$$

for each other section $v' : U \to Q$. If

$$v'(m) = v(m)\, h(m) , \tag{3.67}$$

$h : U \to H$ being a given map, we have

$$K_{v'(m)} = \operatorname{ad}(h^{-1}(m))\, K_{v(m)} . \tag{3.68}$$

Thus $\{K_{v\prime(m)}\} = \{K_{v(m)}\}$ for each $m \in U$.

Suppose that space K_{m_0} is regular. According to Theorem 3.4, there exists a neighbourhood $U' \subset U$ of the point m_0 such that, for each point $m \in U'$, the space K_m is regular, too. In what follows, *we restrict ourselves to the case*

$$\mathscr{D} = \dim Z , \tag{3.69}$$

the case $\mathscr{D} < \dim Z$ being not too much complicated. According to Theorem 3,7, there exists a neighbourhood $U'' \subset U'$ of the point m_0 such that

$$\{K_{v(m)}\} = \{K_{v(m_0)}\} \quad \text{for each point} \quad m \in U'' . \tag{3.70}$$

In other words, there is a neighbourhood U'' of the point m_0 and a map $h : U'' \to H$ such that

$$K_{v(m_0)} = \operatorname{ad}(h^{-1}(m))\, K_{v(m)} \quad \text{for} \quad m \in U'' . \tag{3.71}$$

The section $\mu : U'' \to Q$ given by the equation $\mu(m) = v(m)\, h(m)$ has the property that

$$K_{\mu(m_0)} = K_{\mu(m)} \quad \text{for} \quad m \in U'' . \tag{3.72}$$

Theorem 3.8. *Be given a space* $\mathfrak{S} = \mathfrak{S}(P, M, G, Q, H, \omega)$, *a fixed point* $m_0 \in M$, *its neighbourhood* $U \subset M$ *and a section* $v : U \to Q$. *Suppose that we have*

$$\dim \mathfrak{h} - \dim \mathfrak{h}_{v(m_0)} = \dim Z \tag{3.73}$$

for the space $K_{v(m_0)} = \omega(T_{v(m_0)}(Q))$; *here, Z is the manifold of spaces K such that that* $\mathfrak{h} \subset K \subset \mathfrak{g}$, $\dim K = \mathfrak{h} + \dim M$; $H_{v(m_0)}$ *is the Lie group of all solutions* $h \in H$

of the equation $K_{\nu(m_0)} = \mathrm{ad}\,(h)\,K_{\nu(m_0)}$, $\mathfrak{h}_{\nu(m_0)}$ *is its Lie algebra consisting of all vectors* $v \in \mathfrak{h}$ *satisfying* $[v, K_{\nu(m_0)}] \subset K_{\nu(m_0)}$. *Then there is a neighbourhood* $U' \subset U$ *of the point* m_0 *and a map* $k : U' \to H$ *such that, for the section*

$$\mu(m) = \nu(m)\,k(m)\,, \quad m \in U'\,, \tag{3.74}$$

we have

$$K_{\nu(m_0)} = K_{\mu(m_0)} = K_{\mu(m)} \quad \textit{for} \quad m \in U'\,. \tag{3.75}$$

Let $Q(U', H)$ *be the restriction of the bundle* $Q(M, H)$ *to the base space* $U' \subset M$. *Then there is one and only one reduction* $Q_{\nu(m_0)}(U', H_{\nu(m_0)})$ *of the bundle* $Q(U', H)$ *to the group* $H_{\nu(m_0)}$ *with the following property: For the section* $\sigma : U' \to Q$, *we have* $\sigma(U') \subset Q_{\nu(m_0)}$ *if and only if* $K_{\nu(m_0)} = K_{\sigma(m)}$ *for each* $m \in U'$.

Be given spaces

$$\mathfrak{S} = \mathfrak{S}(M, P, G, Q, H, \omega)\,, \quad \mathfrak{S}' = \mathfrak{S}'(M, P, G, Q, H, \omega')\,, \tag{3.76}$$

a fixed point $m_0 \in M$, its neighbourhood $U \subset M$ and the local sections $\nu, \nu' : U \to Q$. We are going to study Problem II. All considerations being local, we shall often diminish our neighbourhood U without mentioning it; let $M = U$. Let us repeat Problem II: The forms

$$\omega_\nu = \nu_*\omega\,, \quad \omega_{\nu'} = \nu'_*\omega' \tag{3.77}$$

being given, we have to decide whether there exists a map

$$h : M \to H \tag{3.78}$$

such that

$$\omega_{\nu'} = \mathrm{ad}\,(h^{-1})\,\omega_\nu + h^{-1}\,\mathrm{d}h\,. \tag{3.79}$$

Let us suppose that the spaces

$$K_{m_0} = \omega(T_{\nu(m_0)}(Q))\,, \quad K'_{m_0} = \omega'(T_{\nu'(m_0)}(Q)) \tag{3.80}$$

are regular and satisfy the equation (3.73). Applying Theorem 3.8, we see the existence of the sections $\mu, \mu' : U \to Q$ such that

$$\omega(T_{\mu(m)}(Q)) = K_{m_0}\,, \quad \omega'(T_{\mu'(m)}(Q)) = K'_{m_0} \quad \text{for} \quad m \in U\,. \tag{3.81}$$

If there is a map (3.78) satisfying (3.79), we have

$$K'_{m_0} = \mathrm{ad}\,(h^{-1}(m))\,K_{m_0} \quad \text{for each} \quad m \in U\,. \tag{3.82}$$

Especially, there is an $h_0 = h(m_0)$ such that

$$K'_{m_0} = \mathrm{ad}\,(h_0^{-1})\,K_{m_0}\,. \tag{3.83}$$

Let us consider an h_0 satisfying (3.83), and let us write

$$(3.84) \qquad h(m) = k(m)\, h_0 .$$

From (3.82), we get

$$(3.85) \qquad K'_{m_0} = \mathrm{ad}\,(h_0^{-1})\,\mathrm{ad}\,(k^{-1}(m))\,K_{m_0}, \quad \text{i.e.} \quad K_{m_0} = \mathrm{ad}\,(k^{-1}(m))\,K_{m_0} .$$

But this means $k(m) \in H_K$, where $K = K_{m_0}$.

Theorem 3.9. *Be given spaces* (3.76), *a fixed point* $m_0 \in M$ *and sections* $v, v' : M \to Q$. *On* M, *consider the* $\mathfrak{g}$*-valued 1-forms* (3.77). *Suppose the existence of a map* (3.78) *satisfying* (3.79). *Finally, suppose that the spaces* (3.80) *are regular and that they satisfy equations of the type* (3.73). *Then there is a neighbourhood* $U \subset M$ *of the point* m_0 *and sections* $\mu, \mu' : U \to Q$ *satisfying* (3.81). *Let* $l : U \to H$ *be a map such that*

$$(3.86) \qquad \omega_{\mu'} = \mathrm{ad}\,(l^{-1}(m))\,\omega_\mu + l^{-1}(m)\,\mathrm{d}l(m) ;$$

here, $\omega_\mu = \mu_* \omega$, $\omega_{\mu'} = \mu'_* \omega'$. *Let* h_0 *be an arbitrary solution of the equation* (3.83). *Then there is a map* $k : U \to H_K$, $K = K_{m_0}$, *such that*

$$(3.87) \qquad l(m) = k(m)\, h_0 \quad \textit{for} \quad m \in U .$$

This theorem makes the equivalence problem less difficult: Instead of the study of all the maps $U \to H$, we have to produce one solution of the equation (3.83), the group H_K of the solutions of the equation $K_{m_0} = \mathrm{ad}\,(h)\,K_{m_0}$, and afterwards we have to study the maps $U \to H_K$ only. In all concrete cases, we are led by this general procedure; Theorem 3.9 makes it more precise.

3.5. Let us consider once more our example, i.e. the equivalence problem for surfaces in affine 3-spaces. The Lie algebra $\mathfrak{g} = \mathfrak{ga}(3)$ of the affine group $GA(3)$ is isomorphic to the additive group of matrices of the form

$$(3.88) \qquad r = \begin{pmatrix} 0 & 0 & 0 & 0 \\ r^1 & r_1^1 & r_2^1 & r_3^1 \\ r^2 & r_1^2 & r_2^2 & r_3^2 \\ r^3 & r_1^3 & r_2^3 & r_3^3 \end{pmatrix},$$

where $[r, s] = rs - sr$. A general element of the group $GA(3)$ is, of course,

$$(3.89) \qquad a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a^1 & a_1^1 & a_2^1 & a_3^1 \\ a^2 & a_1^2 & a_2^2 & a_3^2 \\ a^3 & a_1^3 & a_2^3 & a_3^3 \end{pmatrix};$$

further, we have

$$\text{(3.90)} \qquad \operatorname{ad}(a)\, r = ara^{-1}\,.$$

The Lie algebra $\mathfrak{h} = \mathfrak{ga}_0(3) \subset \mathfrak{ga}(3)$ is formed by the elements (3.88) satisfying $r^1 = r^2 = r^3 = 0$. Considering a surface in the space A^3, we have $\dim \mathfrak{g} = 12$, $\dim \mathfrak{h} = 9$, $\dim M = 2$. Thus the manifold Z consists of all spaces K satisfying $\mathfrak{h} \subset K \subset \mathfrak{g}$ and $\dim K = 11$. Each space $K \in Z$ is given by one equation of the type $\alpha_1 r^1 + \alpha_2 r^2 + \alpha_3 r^3 = 0$; without loss of generality, we may suppose

$$\text{(3.91)} \qquad r^3 = \alpha_1 r^1 + \alpha_2 r^2\,.$$

We have to produce all vectors

$$\text{(3.92)} \qquad v = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & v_1^1 & v_2^1 & v_3^1 \\ 0 & v_1^2 & v_2^2 & v_3^2 \\ 0 & v_1^3 & v_2^3 & v_3^3 \end{pmatrix} \in \mathfrak{h}$$

such that $[v, K] \subset K$. We have

$$\text{(3.93)} \qquad [v, r] = vr - rv = \begin{pmatrix} 0 & 0 & 0 & 0 \\ s^1 & s_1^1 & s_2^1 & s_3^1 \\ s^2 & s_1^2 & s_2^2 & s_3^2 \\ s^3 & s_1^3 & s_2^3 & s_3^3 \end{pmatrix},$$

$$s^i = v_1^i r^1 + v_2^i r^2 + v_3^i r^3\,; \quad i = 1, 2, 3\,.$$

(3.91) being satisfied and $s^3 = \alpha_1 s^1 + \alpha_2 s^2$ for each r^1, r^2, we get the equations

$$\text{(3.94)} \qquad \begin{aligned} &v_1^3 + \alpha_1(v_3^3 - v_1^1) - \alpha_2 v_2^2 - (\alpha_1)^2 v_3^1 - \alpha_1\alpha_2 v_3^2 = 0\,, \\ &v_2^3 - \alpha_1 v_2^1 + \alpha_2(v_3^3 - v_2^2) - \alpha_1\alpha_2 v_3^1 - (\alpha_2)^2 v_3^2 = 0\,; \end{aligned}$$

(3.94) are the equations of the space $\mathfrak{h}_K \subset \mathfrak{h}$. This equations are always linearly independent, and we have $\dim \mathfrak{h}_K = 7$. Because of $\dim Z = 2 = \dim \mathfrak{h} - \dim \mathfrak{h}_K$, each space $K \in Z$ is regular, and we may apply Theorem 3.8. But we may say even more. The manifold Z is connected and, according to Theorem 3.4, there is just one orbit equal to Z. We may therefore choose an arbitrary fixed space $K_0 \in Z$, and, according to Theorem 3.8, concetrate ourselves to the reduction of the considered bundle to the group H_{K_0}. As usual, K_0 is given by the equation $r^3 = 0$. Then $\mathfrak{h}_K$ is given by the equations $v_1^3 = v_2^3 = 0$; the space K_0 becomes the set of elements (1.55) and the group H_K is just the group H_1 consisting of the elements (1.56). Theorem 1.6 is a special case of Theorem 3.8.

Next, we investigate the manifold Z_1 of spaces L such that $\mathfrak{h}_K \subset L \subset \mathfrak{g}$, $\dim L = \dim \mathfrak{h}_K + \dim M = 9$. There are only some technical difficulties in doing this. Applying successively Theorems 3.4, 3.7 and 3.8, we get the full classification of surfaces and Theorems 1.7, 1.8 and 1.9. It would be instructive to accomplish this.

References

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Резюме

МЕТОД КАРТАНА СПЕЦИАЛИЗАЦИИ РЕПЕРОВ

АЛОИС ШВЕЦ (Alois Švec), Прага

Пространство со связностью $S = S(P, M, G, Q, H, \omega)$ – (1) главное расслоенное пространство $P(M, G)$, (2) приведение Q пространства $P(M, G)$ к подгруппе $H \subset G$, (3) связность ω на $P(M, G)$. Определяется развитие кривых бази и деформация двух пространств со связностью. Главной темой работы – решение проблемы эквивалентности с помощью обобщенного метода специализации реперов.