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Václav Dupač

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STOCHASTIC APPROXIMATIONS IN THE PRESENCE OF TREND

VÁCLAV DUPAČ, Praha

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1. Summary. Two basic stochastic approximation methods deal with solving an equation (the Robbins-Monro method) or with seeking the point of a maximum (the Kiefer-Wolfowitz method), when the function values are determined with an experimental error. In the present paper both methods are adapted to the case, when the root or the point of a maximum move in a specified manner during the approximation process. As compared to the author's previous paper on this theme [1], the conditions, under which the approximations converge, are generalized in several directions.

2. The Robbins-Monro case. Denote by N the set of all positive integers, and by R the real line. For $n \in N$ let $M_n(x)$, $x \in R$, be Borel-measurable functions, let Θ_n be the unique root of the equation $M_n(x) = 0$. Both M_n and Θ_n are unknown to the experimenter, but it is supposed, that he can choose two sequences $a_n > 0$ and q_n such that

(1) $$\sum_1^\infty a_n = +\infty\,, \quad \sum_1^\infty a_n^2 < +\infty\,;$$

(2) $$\Theta_{n+1} = q_n \Theta_n + o(a_n)\,;$$

(3) $$(|q_n| - 1)^+ = o(a_n)\,;$$

(z^+ denotes $(z + |z|)/2$).

Let x_1 be an arbitrary random variable; for $n \in N$ define

(4) $$x_{n+1} = x_n^* - a_n y_n^*$$

where $x_n^* = q_n x_n$, and y_n^* is a random variable such that

(5) $$\mathsf{E}(y_n^* \mid x_1, x_2, \ldots, x_n) = M_{n+1}(x_n^*)$$

(6) $$\mathsf{Var}(y_n^* \mid x_1, x_2, \ldots, x_n) \leqq \text{const (say } \sigma^2)\,.$$

Theorem 1. *Let $M_n(x)$ satisfy the conditions:*

(7) $$|M_n(x)| \leqq A|x - \Theta_n| + B\,, \quad \text{for every} \quad x \in R,\ n \in N \quad \text{and suitable } A, B\,;$$

(8) $$\inf_{n \in N} \inf_{|x-\Theta_n|>\delta} \frac{M_n(x)}{x - \Theta_n} > 0 \quad \text{for every} \quad \delta > 0\,.$$

Then $x_n - \Theta_n \to 0$ with probability one; if $\mathsf{E}(x_1^2) < +\infty$, then also $\mathsf{E}[(x_n - \Theta_n)^2] \to \to 0$.

Remark 1. In particular, (7) and (8) are satisfied if they are satisfied for $n = 1$ and if $M_n(x) = M_1(x - \Theta_n + \Theta_1)$, $n \in N$.

Remark 2. The condition (8) cannot be replaced by

(8′) $$\inf_{n \in N} \inf_{|x-\Theta_n|>\delta} |M_n(x)| > 0\,, \quad \frac{M_n(x)}{x - \Theta_n} > 0\,, \quad x \neq \Theta_n\,, \quad n \in N$$

as the following counterexample shows (it satisfies (1)–(7) and (8′), but not (8)):

$$M_n(x) = \varepsilon \operatorname{sgn} x \quad \text{for all} \quad x \in R,\ n \in N \quad \text{and for some} \quad 0 < \varepsilon < \tfrac{1}{3}$$

(hence $\Theta_n = 0$, $n \in N$);

$$x_1 = 1\,; \quad \sigma^2 = 0\,; \quad q_n = 1 + \frac{1}{n}\,; \quad a_n = \frac{1}{n^\alpha}\,, \quad \tfrac{1}{2} < \alpha < 1\,.$$

It follows

$$x_{n+1} = \left(1 + \frac{1}{n}\right) x_n - \varepsilon \frac{1}{n^\alpha} \operatorname{sgn} x_n\,,$$

hence

$$x_n = n\left(1 - \varepsilon \sum_{i=1}^{n-1} \frac{1}{i^\alpha(i+1)}\right) \to +\infty\,.$$

Remark 3. Let $\Theta_n = gn^\beta + h$, where $\beta > 0$ is known, g and h unknown; then the conditions (1), (2), (3) are satisfied by the choice

(9) $$q_n = 1 + \binom{\beta}{1}\frac{1}{n} + \binom{\beta}{2}\frac{1}{n^2} + \ldots + \binom{\beta}{r}\frac{1}{n^r}$$

and

(10) $$a_n = \frac{a}{n^\alpha}\,, \quad a > 0\,, \quad \tfrac{1}{2} < \alpha < 1$$

with $r = [\alpha + \beta]$.

So we can choose $q_n = 1 + 1/n$, $a_n = a/n^\alpha$, $\frac{1}{2} < \alpha < 1$ for the linear trend, i.e. for $\beta = 1$; or $q_n = 1$, $a_n = a/n^\alpha$, $\frac{1}{2} < \alpha < 1 - \beta$ for $\beta < \frac{1}{2}$.

Proof of Remark 3: We shall only verify (2), everything else is obvious; we have

$$\Theta_{n+1} = g(n+1)^\beta + h = gn^\beta \sum_{k=0}^{\infty} \binom{\beta}{k} \frac{1}{n^k} + h\,,$$

$$q_n\Theta_n = gn^\beta \sum_{k=0}^{r} \binom{\beta}{k} \frac{1}{n^k} + h \sum_{k=0}^{r} \binom{\beta}{k} \frac{1}{n^k}\,,$$

hence

$$\Theta_{n+1} - q_n\Theta_n = O\left(\frac{1}{n^{r+1-\beta}} + \frac{1}{n}\right) = o(a_n)\,,$$

since $[\alpha + \beta] + 1 - \beta > \alpha$.

Proof of Theorem 1: Rewrite the scheme (4) in the form

(11) $$x_{n+1} = x_{n]}^* - a_n M_{n+1}(x_n^*) + \varepsilon_n\,,$$

where $\varepsilon_n = -a_n(y_n^* - M_{n+1}(x_n^*))$. Subtract Θ_{n+1} on both sides of (11) and denote $\omega_n = \Theta_{n+1} - q_n\Theta_n$, $z_n = x_n - \Theta_n$; so that

$$x_n^* = q_n x_n = q_n z_n + q_n \Theta_n = q_n z_n - \omega_n + \Theta_{n+1}\,.$$

We get

(12) $$z_{n+1} = T_n(z_n) + \varepsilon_n$$

where

(13) $$T_n(r) = q_n r - \omega_n - a_n M_{n+1}(q_n r - \omega_n + \Theta_{n+1})\,.$$

We shall show, that the scheme (12) satisfies the conditions of Dvoretzky's theorem [2] for the convergence $z_n \to 0$ with probability one and in mean-square. The fulfilment of the conditions

(14) $$\mathsf{E}(\varepsilon_n \mid z_1, \ldots, z_n) = 0\,, \quad \sum_{n=1}^{\infty} \mathsf{E}(\varepsilon_n^2) < +\infty$$

is obvious from (5) and (6), so that it suffices to prove the inequality

(15) $$|T_n(r)| \leqq \max\{\alpha_n, |r| - \gamma_n\}\,, \quad n \in N\,, \quad r \in R\,,$$

for some positive α_n, γ_n such that $\alpha_n \to 0$, $\sum_1^{\infty} \gamma_n = +\infty$.

The next proposition follows easily from (8): For every sequence $\varrho_n > 0$, $\varrho_n \to 0$, bounded by a sufficiently small constant, there exists a sequence $\eta_n > 0$, $\eta_n \to 0$, such that

(16) $$|M_m(x)| > \varrho_n |x - \Theta_m| \quad \text{for all} \quad |x - \Theta_m| > \eta_n\,, \quad m \in N\,, \quad n \in N\,.$$

Let us choose ϱ_n such that

$$(17)\qquad \sum_1^\infty a_n\varrho_n^2 = +\infty\,, \quad \omega_n = o(a_n\varrho_n^2)\,, \quad (|q_n| - 1)^+ = o(a_n\varrho_n)\,;$$

let the corresponding η_n be such that

$$(18)\qquad a_n = o(\eta_n)\,;$$

this can always be done.

If $|q_n r - \omega_n| \leqq \eta_n$, then

$$(19)\qquad |T_n(r)| \leqq (1 + Aa_n)\,|q_n r - \omega_n| + Ba_n < 2\eta_n$$

for sufficiently large n, according to (7) and (18).

The case $|q_n r - \omega_n| > \eta_n$ is a little more complicated: The terms $q_n r - \omega_n$ and $a_n M_{n+1}(q_n r - \omega_n + \Theta_{n+1})$ are of the same sign, the first being larger than or equal to the second one in absolute value, for large n, so that we have

$$(20)\qquad |T_n(r)| = |q_n r - \omega_n| - a_n|M_{n+1}(q_n r - \omega_n + \Theta_{n+1})|\,.$$

Setting $m = n + 1$, $x = q_n r - \omega_n + \Theta_{n+1}$ in (16) and using it in (20), we get

$$(21)\qquad |T_n(r)| \leqq (1 - a_n\varrho_n)\,|q_n r - \omega_n| \leqq$$
$$\leqq (1 - a_n\varrho_n)\,(1 + (|q_n| - 1)^+)\,|r| + |\omega_n| = \tau_{n,r} \quad \text{(say)}\,.$$

Now, if $|r| \leqq \varrho_n$, then $\tau_{n,r} < 2\varrho_n$; if $|r| > \varrho_n$, then

$$(22)\qquad \tau_{n,r} < (1 - \tfrac{2}{3}a_n\varrho_n)\,|r| + |\omega_n| < |r| - \tfrac{1}{3}a_n\varrho_n^2\,,$$

both inequalities in (22) being consequencies of (17). So it is proved, that (15) holds with $\alpha_n = \max(2\eta_n, 2\varrho_n)$, $\gamma_n = a_n\varrho_n^2/3$.

3. The multidimensional Kiefer-Wolfcwitz case. Denote by R^p the p-dimensional. Euclidean space. For $n \in N$ let $M_n(x)$, $x \in R^p$, be Borel-measurable functions, let $\Theta_n \in R^p$ be the point at which $M_n(x)$ has the unique maximum. It is supposed, that the experimenter can choose two sequences of positive constants a_n, c_n and a sequence of real matrices $Q_n(p \times p)$ such that

$$(23)\qquad c_n \to 0\,, \quad \sum_1^\infty a_n = +\infty\,, \quad \sum_1^\infty (a_n^2/c_n^2) < +\infty\,, \quad a_n/c_n^2 \to 0\,;$$

$$(24)\qquad \|\Theta_{n+1} - Q_n\Theta_n\| = o(a_n)\,;$$

$$(25)\qquad (\|Q_n\| - 1)^+ = o(a_n)\,.$$

Let x_1 be an arbitrary p-dimensional random vector; for $n \in N$ define

(26) $$x_{n+1} = x_n^* + a_n \frac{y_{2n}^* - y_{2n-1}^*}{c_n}$$

where $x_n^* = Q_n x_n$ and y_{2n}^*, y_{2n-1}^* are random vectors such that their coordinates $y_{2n,i}^*, y_{2n-1,i}^*$, $i = 1, 2, \ldots, p$, are all conditionally independent given $x_1, x_2, \ldots, x_n$ and satisfy

(27) $$\mathsf{E}(y_{2n,i}^* \mid x_1, \ldots, x_n) = M_{n+1}(x_n^* + c_n e_i) ,$$
$$\mathsf{E}(y_{2n-1,i}^* \mid x_1, \ldots, x_n) = M_{n+1}(x_n^* - c_n e_i) , \quad i = 1, 2, \ldots, p ,$$

(28) $$\mathsf{Var}(y_{v,i}^* \mid x_1, \ldots, x_n) \leqq \text{const.} , \quad v = 2n,\ 2n - 1 , \quad i = 1, 2, \ldots, p$$

(e_i, $i = 1, 2, \ldots, p$ are elements of the usual orthonormal set in R^p; the constant in (28) is independent of n). Denote by $D_\varepsilon M_n(x)$ the vector with coordinates

$$\frac{M_n(x + \varepsilon e_i) - M_n(x - \varepsilon e_i)}{\varepsilon} , \quad i = 1, 2, \ldots, p .$$

Theorem 2. *Let $M_n(x)$ satisfy the conditions*

(29) $$|M_n(x + \varepsilon e_i) - M_n(x)| \leqq A\|x - \Theta_n\| + B$$

for all $0 < \varepsilon < 1$, $i = 1, 2, \ldots, p$, $x \in R^p$, $n \in N$ *and suitable* A, B;

(30) $$\sup_{n \in N} \sup_{\|x - \Theta_n\| > \delta} \sup_{0 < \varepsilon < \delta} \frac{(D_\varepsilon M_n(x), x - \Theta_n)}{\|x - \Theta_n\|^2} < 0 \quad \textit{for each} \quad 0 < \delta < \delta_0 .$$

Then $x_n - \Theta_n \to 0$ *with probability one.*

Remark 4. Let $\Theta_n = n^B g + h$, where B is a known matrix, g and h unknown vectors; let

(31) $$\|n^B\| = O(n^\beta) \quad \text{for some} \quad \beta > 0 .$$

Then the conditions (23), (24), (25) are satisfied by the choice

(32) $$Q_n = E + \binom{B}{1}\frac{1}{n} + \binom{B}{2}\frac{1}{n^2} + \ldots + \binom{B}{r}\frac{1}{n^r} ,$$

(33) $$a_n = \frac{a}{n^\alpha} , \quad a > 0 , \quad \tfrac{1}{2} < \alpha < 1 ,$$

(34) $$c_n = \frac{c}{n^\gamma} , \quad c > 0 , \quad 0 < \gamma < \alpha - \tfrac{1}{2}$$

where $\binom{B}{k}$ denotes $[B(B-E)(B-2E)\dots(B-(k-1)E)]/k!$, and $r = [\alpha + \beta]$. So we can choose $Q_n = (1 + 1/n) E$ for the case of a linear trend in each coordinate, i.e. for $\Theta_{n,i} = g_i n + h_i$, $i = 1, 2, \dots, p$. If $\beta < \frac{1}{2}$, B may be unknown; (23)–(25) are then satisfied by $Q_n = E$, and a_n, c_n given by (33), (34) with the additional restriction $\alpha < 1 - \beta$.

Proof of Remark 4 is formally analogous to that of Remark 3. We note only that n^B is well defined as $e^{B \lg n}$, that β satisfying (31) always exist, since $\|n^B\| \leqq n^{\|B\|}$, and that the expansion

$$(1 + a)^B = \sum_{k=0}^{\infty} \binom{B}{k} a^k , \tag{35}$$

is valid for every $|a| < 1$ and for every matrix B, as follows from [3, Section 5.4, Theorem 1′] and from the Weierstrass Theorem (on uniformly convergent series of analytic functions).

Also Remarks 1 and 2 (of Section 2) can be repeated with obvious changes, the counterexample being the one-dimensional $M_n(x) = -\varepsilon|x|$, etc.

Proof of Theorem 2: Similarly as in the proof of Theorem 1, the scheme (26) can be rewritten in the form

$$z_{n+1} = T_n(z_n) + \varepsilon_n , \tag{36}$$

where

$$T_n(r) = Q_n r - \omega_n + a_n D_{c_n} M_{n+1}(Q_n r - \omega_n + \Theta_{n+1}) , \quad r \in R^p , \tag{37}$$

and $\omega_n = \Theta_{n+1} - Q_n \Theta_n$, $z_n = x_n - \Theta_n$, ε_n being the vector with coordinates $(i = 1, 2, \dots, p)$

$$\varepsilon_{n,i} = \frac{a_n}{c_n} [y^*_{2n,i} - M_{n+1}(x^*_{n} + c_n e_i) - (y^*_{2n-1,i} - M_{n+1}(x^*_n - c_n e_i))] .$$

We shall verify, that the scheme (36) satisfies the conditions of the multidimensional version of Dvoretzky's theorem [4, Theor. 2] for the convergence $z_n \to 0$ with probability one. The fulfilment of

$$\mathsf{E}(\varepsilon_n \mid z_1, \dots, z_n) = 0 , \quad \sum_{n=1}^{\infty} \mathsf{E}(\|\varepsilon_n\|^2) < +\infty \tag{38}$$

is obvious from (27), (28), so that it again suffices to prove

$$\|T_n(r)\| \leqq \max \{\alpha_n, \|r\| - \gamma_n\} , \quad n \in N , \quad r \in R^p \tag{39}$$

for some positive α_n, γ_n such that $\alpha_n \to 0$, $\sum_1^{\infty} \gamma_n = +\infty$.

We shall first estimate $\|D_\varepsilon M_n(x)\|$ with help of (29):

$$\|D_\varepsilon M_n(x)\| \leqq p^{\frac{1}{2}} \operatorname*{Max}_{1 \leqq i \leqq p} \frac{|M_n(x + \varepsilon e_i) - M_n(x - \varepsilon e_i)|}{\varepsilon} \leqq \tag{40}$$

$$\leqq \frac{p^{1/2}}{\varepsilon} (A\|x - \Theta_n\| + A\varepsilon + B) \leqq \frac{A_1\|x - \Theta_n\| + B_1}{\varepsilon}, \quad x \in R^p,$$

$$n \in N, \quad 0 < \varepsilon < \tfrac{1}{2}.$$

Setting this into (37), we get

$$\|T_n(r)\| \leqq \left(1 + A_1 \frac{a_n}{c_n}\right) \|Q_n r - \omega_n\| + B_1 \frac{a_n}{c_n}, \quad r \in R^p, \quad c_n < \tfrac{1}{2}. \tag{41}$$

Further, we shall use the expression for $\|T_n(r)\|^2$, calculated from (37):

$$\|T_n(r)\|^2 = \|Q_n r - \omega_n\|^2 \{1 + \Delta + \Delta'\} \tag{42}$$

where

$$\Delta = 2a_n(D_{c_n} M_{n+1}(Q_n r - \omega_n + \Theta_{n+1}), \quad Q_n r - \omega_n)/\|Q_n r - \omega_n\|^2,$$

$$\Delta' = a_n^2 \|D_{c_n} M_{n+1}(Q_n r - \omega_n + \Theta_{n+1})\|^2/\|Q_n r - \omega_n\|^2.$$

The next proposition follows from (30): For every sequence $\varrho_n > 0$, $\varrho_n \to 0$, bounded by a sufficiently small number, there exists a sequence $\eta_n > 0$, $\eta_n \to 0$, such that

$$\frac{(D_\varepsilon M_m(x), x - \Theta_m)}{\|x - \Theta_m\|^2} < -\varrho_n \tag{43}$$

for all $0 < \varepsilon < \eta_n$, $\|x - \Theta_m\| > \eta_n$, $m \in N$, $n \in N$.

Let us choose ϱ_n such that it holds

$$\sum_1^\infty a_n \varrho_n^2 = +\infty, \quad \|\omega_n\| = o(a_n \varrho_n^2), \quad (\|Q_n\| - 1)^+ = o(a_n \varrho_n), \tag{44}$$

$$(a_n/c_n^2) = o(\varrho_n);$$

let the corresponding sequence η_n be chosen in such a way that it satisfies the conditions

$$\frac{a_n}{c_n} = o(\eta_n), \quad c_n = o(\eta_n), \quad \frac{a_n}{c_n^2 \varrho_n} = o(\eta_n^2). \tag{45}$$

Let $\|Q_n r - \omega_n\| \leqq \eta_n$, let n be sufficiently large; then $\|T_n(r)\| < 2\eta_n$, according to (41) and (45).

Finally, let $\|Q_n r - \omega_n\| > \eta_n$, n be large. Setting $\varepsilon = c_n$, $m = n + 1$, $x = Q_n r - \omega_n + \Theta_{n+1}$ in (43) we get $\Delta < -2a_n\varrho_n$; using (40) and (45) we get further

$$\Delta' \leqq 2\frac{a_n^2}{c_n^2}\left(A_1^2 + \frac{B_1^2}{\|Q_n r - \omega_n\|^2}\right) < \frac{Ca_n^2}{c_n^2\eta_n^2} < a_n\varrho_n .$$

Setting these estimates into (42), we get

$$\|T_n(r)\|^2 \leqq \|Q_n r - \omega_n\|^2 (1 - a_n\varrho_n) ,$$

i.e.

$$\|T_n(r)\| \leqq \|Q_n r - \omega_n\| (1 - \tfrac{1}{2}a_n\varrho_n) .$$

The rest of the proof coincides with the proof of Theorem 1 (cf. (21) and below).

Remark 5. In the one-dimensional Kiefer-Wolfowitz case, the condition $a_n/c_n^2 \to 0$ can be omitted, and the condition (29) can be weakened to

$$|M_n(x + 1) - M_n(x)| < A|x - \Theta_n| + B .$$

Furthermore, $\mathsf{E}(x_1^2) < +\infty$ implies $\mathsf{E}((x_n - \Theta_n)^2) \to 0$. We omit the proof.

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Author's address: Praha 8 - Karlín, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta University Karlovy).

Резюме

СТОХАСТИЧЕСКИЕ АППРОКСИМАЦИИ ПРИ НАЛИЧИИ ТРЕНДА

ВАЦЛАВ ДУПАЧ (Václav Dupač), Прага

Стохастический аппроксимационный метод для нахождения корня уравнения или для отыскания точки максимума функции приспособляется для случая, когда корень или точка максимума изменяются в течение аппроксимационного процесса. Именно, доказывается следующий результат:

Пусть $M_n(x)$, $n = 1, 2, \ldots$ — бэровские функции, пусть Θ_n — корень уравнения $M_n(x) = 0$. Пусть M_n и Θ_n неизвестны, а известны некоторые постоянные $a_n > 0$ и q_n так, что $\sum a_n = +\infty$, $\sum a_n^2 < +\infty$; $\Theta_{n+1} = q_n\Theta_n + o(a_n)$; $(|q_n| - 1)^+ = o(a_n)$. Пусть x_1 — произвольная случайная величина; для $n \geqq 1$ положим $x_{n+1} = x_n^* - a_n y_n^*$, где $x_n^* = q_n x_n$ и y_n^* — случайная величина такая, что $\mathsf{E}(y_n^* \mid x_1, \ldots, x_n) = M_{n+1}(x_n^*)$, $\mathsf{Var}(y_n^* \mid x_1, \ldots, x_n) \leqq \text{const}$. Пусть $M_n(x)$ удовлетворяют условиям: $|M_n(x)| \leqq A|x - \Theta_n| + B$ для всякого $-\infty < x < +\infty$ и $n = 1, 2, \ldots$; $\inf_{n=1,2,\ldots} \inf_{|x-\Theta_n|>\delta} M_n(x)/(x - \Theta_n) > 0$ для всякого $\delta > 0$. Тогда $x_n - \Theta_n \to 0$ п.н. Если, в частности, $\Theta_n = gn^\beta + h$, где β известно, а g и h неизвестны, то можно выбрать $a_n = an^{-\alpha}$, $q_n = 1 + \binom{\beta}{1} n^{-1} + \ldots + \binom{\beta}{r} n^{-r}$, где $\frac{1}{2} < \alpha < 1$, $r = [\alpha + \beta]$.

Аналогичные результаты доказаны для отыскания точки максимума функции нескольких переменных. Статья является продолжением работы автора [1].