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ON A DOMAIN OF THE TYPE $\mathfrak{P}$

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In [1], domains of type $\mathfrak{P}$ were defined. We shall recall this definition:

Let $\Omega \subset E_n$ be a bounded open domain in euclidean n-space E_n; then $\Omega \times (-\infty, 0)$ is the set of all pairs (x, t) with $x \in \Omega$ and $t < 0$. Let $k \geqq 0$, $l \geqq 0$ be integers. Then M denotes the space of all complex-valued infinitely differentiable functions φ with compact support in $\Omega \times (-\infty, \infty)$, such that $\partial^\alpha \varphi / \partial t^\alpha (x, 0) = 0$ for $\alpha = 0, 1, \ldots, l - 1$. Let $\mathscr{M}$ be the closure of M, under the norm

$$\|\varphi\| = \Big(\sum_{|i| \equiv k} |D^i \varphi|^2_{L_2(\Omega x(-\infty,\infty))} + |\varphi|^2 \Big)^{\frac{1}{2}},$$

where

$$D^i = \frac{\partial^{|i|}}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}, \quad |i| = i_1 + \ldots + i_n,$$

$$|\varphi| = \left(\int_\Omega \int_{-\infty}^{\infty} |\eta|^{2l+1} \, |\mathfrak{F}_t \varphi(x, \eta)|^2 \, \mathrm{d}x \, \mathrm{d}\eta \right)^{\frac{1}{2}}$$

and where $\mathfrak{F}_t \varphi(x, \eta) = (1/2\pi) \int_{-\infty}^{\infty} \varphi(x, t) \, e^{i\eta t} \, \mathrm{d}t$ is the Fourier transform of the function φ in the direction t. Let $l' = l + \frac{1}{2}$.

By ${}^0_R W_2^{(k,l')}(\Omega \times (-\infty, 0))$ we denote the space of all functions u for which there exists a function $\tilde{u}$ such that $\tilde{u} \in \mathscr{M}$ and $u(x, t) = \tilde{u}(x, t)$ for $(x, t) \in \Omega \times (-\infty, 0)$. This latter space is a Banach space with respect to the norm

$$\|u\|_R^{(-\infty,0)} = \inf_{\tilde{u} \in \mathscr{M},\ \tilde{u}|_{\Omega \times (-\infty,0)} = u} \|\tilde{u}\| = u,$$

and it is a Hilbert space with respect to a suitable scalar product.

Let us denote by $\mathscr{D}(\Omega \times (a, b))$ the space of all infinitely differentiable complex-valued functions with compact support in $\Omega \times (a, b)$.

Put

$$\langle u, \varphi \rangle_{-\infty}^0 = (-1)^{l+1} \int_\Omega \int_{-\infty}^0 \bar{u} \, \frac{\partial^{2l+1} \varphi}{\partial t^{2l+1}} \, \mathrm{d}\,\Omega \, \mathrm{d}t$$

for $\varphi \in \mathscr{D}(\Omega \times (-\infty, 0))$ and $u \in {}_R^0W_2^{(k,l')}(\Omega \times (-\infty, 0))$, and

$$|u|_P^{(\infty,0)} = \sup_{\|\varphi\|_R^{(-\infty,0)} \leqq 1} |\langle u, \varphi\rangle_{-\infty}^0| .$$

${}_P^0W_2^{(k,l')}(\Omega \times (-\infty, 0))$ will denote the space of all functions $u \in {}_R^0W_2^{(k,l')}(\Omega \times (-\infty, 0))$ for which $|u|_P^{(-\infty,0)} < +\infty$. This space is a Banach space with respect to the norm

$$\|u\|_P^{(-\infty,0)} = ((\|u\|_R^{(-\infty,0)})^2 + (|u|_P^{(-\infty,0)})^2)^{\frac{1}{2}}$$

(see [1]).

It is known [1], that $\mathscr{D}(\Omega \times (-\infty, 0))$ is dense in ${}_R^0W_2^{(k,l')}(\Omega \times (-\infty, 0))$. Hence one may define $\langle u, v\rangle_{-\infty}^0$ for all $u \in {}_P^0W_2^{(k,l')}(\Omega \times (-\infty, 0))$ and $v \in {}_R^0W_2^{(k,l')}(\Omega \times \times (-\infty, 0))$ as $\lim_{\varphi \to v} \langle u, \varphi\rangle_{-\infty}^0$, where $\varphi \in \mathscr{D}(\Omega \times (-\infty, 0))$ and $\varphi \to v$ in the norm of ${}_R^0W_2^{(k,l')}(\Omega \times (-\infty, 0))$.

Now define $\Omega \in \mathfrak{P}^{(k,l')}$ iff $\operatorname{Re} \langle u, u\rangle_{-\infty}^0 \geqq 0$ for all $u \in {}_P^0W_2^{(k,l')}(\Omega \times (-\infty, 0))$, and then set $\mathfrak{P} = \bigcap_{\substack{k \geqq 0 \\ l \geqq 0}} \mathfrak{P}^{(k,l')}$.

$\mathfrak{N}^{(0),1}$ denotes the set of all bounded open domains Ω whose boundary $\Omega^{\cdot}$ can be described locally by functions satisfying a Lipschitz condition (for the precise definition see [2]).

The main aim of this paper is to prove that $\mathfrak{N}^{(0),1} \subset \mathfrak{P}$.

In 14–17 it is shown that $\operatorname{Re} \langle u, u\rangle_{-\infty}^0 \geqq 0$ for $u \in {}_P^0W_2^{(0,l')}(\Omega \times (-\infty, 0))$, employing some properties of solutions of ordinary differential equations. In the proof of Theorem 19 we make use of the fact that $\langle u, u\rangle_{-\infty}^0$ depends continuously on u as u varies continuously in the sense of both the strong topology in ${}_R^0W_2^{(k,l')}(\Omega \times (-\infty, 0))$ and the weak topology in ${}_P^0W_2^{(k,l')}(\Omega \times (-\infty, 0))$. One first approximates u by a function with suitable support, and then apply a regularization technique in the direction of the space variables (cf. 5–12). Thus one obtains a $u \in {}_P^0W_2^{(0,l')}(\Omega \times \times (-\infty, 0))$. Using 14–17 we have $\mathfrak{N}^{(0),1} \subset \mathfrak{P}$.

1. Theorem. *Let $\Omega \in \mathfrak{N}^{(0),1}$ and $\varrho(x)$ denote the distance from the point x to the boundary $\Omega^{\cdot}$. Then there is a function $\sigma(x)$ infinitely differentiable in Ω and continuous on $\overline{\Omega}$ such that*

$$\varrho(x) \leqq \sigma(x) \leqq C\,\varrho(x), \quad |D^i \sigma(x)| \leqq \frac{C(i)}{[\sigma(x)]^{|i|-1}}$$

for $x \in \Omega$, $|i| \geqq 1$. The constant $C(i)$ depends only on i and Ω.

For the proof see [2].

2. Theorem. *Let $g(t)$ be a measurable function of a real variable on $(0, \infty)$. Let $1 < p$, $\alpha \neq p - 1$ and*

$$\int_0^\infty |g(t)|^p \, t^\alpha \, \mathrm{d}t < +\infty \, .$$

Then

$$\int_0^\infty \left[\int_0^t |g(\tau)| \, \mathrm{d}\tau\right]^p t^{\alpha - p} \, \mathrm{d}t \leqq \left(\frac{p}{|\alpha - p + 1|}\right)^p \int_0^\infty |g(t)|^p \, t^\alpha \, \mathrm{d}t$$

for $\alpha < p - 1$ and

$$\int_0^\infty \left[\int_t^\infty |g(\tau)| \, \mathrm{d}\tau\right]^p t^{\alpha - p} \, \mathrm{d}t \leqq \left(\frac{p}{|\alpha - p + 1|}\right)^p \int_0^\infty |g(t)|^p \, t^\alpha \, \mathrm{d}t$$

for $\alpha > p - 1$.

For the proof see [3].

3. Lemma. *Let f be real-valued infinitely differentiable function of a real variable such that $f(t) = 0$ for $t \leqq \frac{1}{4}$, $f(t) = 1$ for $t \geqq \frac{3}{4}$. Put $f_h(t) = f(t/h)$ for $h > 0$. Clearly, $f_h^{(\alpha)}(t) = h^{-\alpha} f^{(\alpha)}(t/h)$ and $|f_h^{(\alpha)}| \leqq C(\alpha) \, h^{-\alpha}$. On setting $F_h(x) = f_h(\sigma(x))$ for $x \in \Omega$, it results that*

1) $F_h(x) = 1$ *for* $\varrho(x) \geqq h$,
2) $F_h(x) \in \mathscr{D}(\Omega)$,
3) $D^i F_h(x) = 0$ *for* $|i| > 0$ *and* $\varrho(x) \geqq h$,
4) $|D^i F_h(x)| \leqq C(i) \, h^{-i}$ *for* $|i| \geqq 0$ *and* $h < 1$.

Proof. 1)–3) follow from the definition of F_h. To prove 4) first establish the following equality:

$$D^i F_h(x) = \sum_{\alpha, \beta_j} C^i_{\alpha, \beta_j} f_h^{(\alpha)}(\sigma(x)) \prod_{1 \leqq |j| \leqq |i|} (D^j \sigma(x))^{\beta_j} \, . \tag{1}$$

Here

$$\alpha + \sum_{1 \leqq |j| \leqq |i|} (|j| - 1) \, \beta_j \leqq |i| \tag{2}$$

and C^i_{α, β_j} are constants depending only on the indices i, α, β_j.

Obviously (1) holds for $|i| = 0$. Let (1) be true for $|i| = m$. Differentiating (1) with respect to some variable one obtains a sum of members of the form

$$C f_h^{(\alpha + 1)}(\sigma(x)) \prod_{1 \leqq |j| \leqq |i|} (D^j \sigma(x))^{\beta_j}$$

and of the form

$$C f_h^{(\alpha)}(\sigma(x)) \prod_{1 \leqq |j| \leqq |i|,\, j \neq j_0} (D^j \sigma(x))^{\beta_j} \beta_{j_0}(D^{j_0} \sigma(x))^{\beta_{j_0}-1} D^{j_1} \sigma(x)$$

where $|j_1| = |j_0| + 1$.

The sums (2) corresponding to these members are

$$\alpha + 1 + \sum_{1 \leqq |j| \leqq |i|} (|j| - 1)\, \beta_j \leqq |i| + 1$$

and

$$\alpha + \sum_{1 \leqq |j| \leqq |i|} (|j| - 1)\, \beta_j + (|j_0| - 1)(\beta_{j_0} - 1) + (|j_1| - 1)(\beta_{j_1} + 1) \leqq |i| + 1\,.$$

Hence (1) follows by induction (for all i); by (2) and Theorem 1 we then obtain for $h < 1$ that

$$|D^i F_h(x)| \leqq C h^{-(\alpha + \Sigma(|j| - 1)\beta_j)} \leqq C h^{-|i|}$$

as asserted.

4. Theorem. *Let $g(t)$ be absolutely continuous on the interval $\langle 0, \infty)$ and $g(0) = 0$. Then*

$$\int_0^\infty \left|\frac{g(t)}{t}\right|^2 \mathrm{d}t \leqq C \int_0^\infty |g'(t)|^2 \,\mathrm{d}t\,.$$

Proof. This follows from Theorem 2 where

$$\int_0^t g'(\tau)\, \mathrm{d}\tau = g(t)\,, \quad p = 2\,, \quad \alpha = 0\,.$$

5. Theorem. *Let $u \in {}_R^0W_2^{(k,l')}(\Omega \times (-\infty, 0))$, $\Omega \in \mathfrak{N}^{(0),1}$. Put $u_h(x, t) = F_h(x)\, u(x, t)$. Then $u_h \to u$ in ${}_R^0W_2^{(k,l')}(\Omega \times (-\infty, 0))$.*

Proof. Let $\tilde{u} \in \mathcal{M}$, $\tilde{u}|_{\Omega \times (-\infty, 0)} \equiv u$. Then $F_h\tilde{u} \in \mathcal{M}$ and $F_h\tilde{u}|_{\Omega \times (-\infty,0)} = F_h u$.

It is easy to prove that $F_h\tilde{u} \to \tilde{u}$ in $L_2(\Omega \times (-\infty, \infty))$ and $|\tilde{u} - F_h\tilde{u}|^2 \to 0$. Indeed,

$$|\tilde{u} - F_h\tilde{u}|^2 = \int_\Omega \int_{-\infty}^\infty |\eta|^{2l+1} (1 - F_h(x))^2 |\mathfrak{F}_t\, \tilde{u}(x, \eta)|^2 \,\mathrm{d}x \,\mathrm{d}\eta \leqq$$

$$\leqq C \int_{x \in \Omega, \varrho(x) < h} \int_{-\infty}^\infty |\eta|^{2l+1} |\mathfrak{F}_t\, \tilde{u}(x, \eta)|^2 \,\mathrm{d}x \,\mathrm{d}\eta = C(h)\,,$$

where $C(h) \to 0$ as $h \to 0$.

Let $[X_r, x_{rn}]$ be a local coordinate system corresponding to U_r, and a_r functions describing the boundary of Ω. We denote by Δ_r the projection of U_r into the space of the first $(n-1)$ variables. Also omit the index r and write u instead of $u\varphi_r$.

Put $F_h\tilde{u} = \tilde{u}_h$. Clearly

$$\|\tilde{u} - \tilde{u}_h\| \leqq C \sum_r \|(\tilde{u}\varphi_r) - (\tilde{u}\varphi_r)_h\| ,$$

and if $\|\tilde{u}\varphi_r - (\tilde{u}\varphi_r)_h\| \to 0$ as $h \to 0$ for all r then $\|\tilde{u} - \tilde{u}_h\| \to 0$ as $h \to 0$. Thus it is sufficient to prove our theorem for $u\varphi_r$.

Let $|i| \leqq k$. Then

$$D^i \tilde{u}_h(x, t) = F_h(x)\, D^i \tilde{u}(x, t) + \sum_{j<i} C_j D^{i-j} F_h(x)\, D^j \tilde{u}(x, t) ,$$

where $i - j = (i_1 - j_1, \ldots, i_n - j_n)$ and $j < i$ means that $j_\alpha \leqq i_\alpha$ with $j_\alpha < i_\alpha$ for at least one α. It is easily seen that $F_h(x)\, D^i \tilde{u}(x, t) \to D^i \tilde{u}(x, t)$ in $L_2(\Omega \times (-\infty, \infty))$ as $h \to 0$. We shall prove now that

$$\lim_{h\to 0} |D^j \tilde{u}(x, t)\, D^{i-j} F_h(x)|^2_{L_2(\Omega \times (-\infty, \infty))} = 0$$

for $j < i$:

$$I_h = \int_\Omega \int_{-\infty}^{\infty} |D^j \tilde{u}(x, t)\, D^{i-j} F_h(x)|^2 \,\mathrm{d}x \,\mathrm{d}t =$$

$$= \int_{-\infty}^{\infty} \int_{U\cap\Omega} |D^j \tilde{u}(x, t)\, D^{i-j} F_h(x)|^2 \,\mathrm{d}t \,\mathrm{d}x =$$

$$= \int_{-\infty}^{\infty} \int_{\Delta} \int_{-\infty}^{a(X)} |D^j \tilde{u}(X, x_n, t)\, D^{i-j} F_h(X, x_n)|^2 \,\mathrm{d}t \,\mathrm{d}X \,\mathrm{d}x_n .$$

$D^j \tilde{u}(x, t)$ with $j < i$ is absolutely continuous for almost every line $X = \text{const.}$, $t = \text{const.}$, and vanishes for $x_n = a(X)$, if suitably changed on set of measure zero.

On the other hand,

$$C_1\, \varrho(x) \leqq |a(X) - x_n| \leqq C_2\, \varrho(x)$$

for $x \in U$ because a satisfies a Lipschitz condition. Thus for $h < 1$ by Lemma 3

$$I_h \leqq C \int_{-\infty}^{\infty} \int_{x\in\Omega, \varrho(x)<h} |D^j \tilde{u}(X, x_n, t)\, h^{-|i-j|}|^2 \,\mathrm{d}t \,\mathrm{d}X \,\mathrm{d}x_n \leqq$$

$$\leqq C \int_{-\infty}^{\infty} \int_{x\in\Omega, \varrho(x)<h} |D^j \tilde{u}(X, x_n, t)\, (\varrho(x))^{|j|-|i|}|^2 \,\mathrm{d}t \,\mathrm{d}X \,\mathrm{d}x_n .$$

But by Theorem 4

$$\int_{-\infty}^{\infty} \int_{x\in\Omega} \left|D^j \tilde{u}(X, x_n, t)\right|^2 \left|\varrho(x)\right|^{2|j|-2|i|} \mathrm{d}t \, \mathrm{d}X \, \mathrm{d}x_n \leqq$$

$$\leqq C \int_{-\infty}^{\infty} \int_{\Delta} \int_{-\infty}^{a(X)} \left|D^j \tilde{u}(X, x_n, t)\right|^2 \left|a(X) - x_n\right|^{2|j|-2|i|} \mathrm{d}t \, \mathrm{d}X \, \mathrm{d}x_n \leqq$$

$$\leqq C \int_{-\infty}^{\infty} \int_{\Delta} \int_{-\infty}^{a(X)} \left| \frac{\partial^{|i|-|j|}}{\partial X_n^{|i|-|j|}} D^j \tilde{u}(X, x_n, t)\right|^2 \mathrm{d}t \, \mathrm{d}t \, \mathrm{d}X \, \mathrm{d}x_n \leqq$$

$$\leqq C \sum_{|j| \leqq k} \int_{-\infty}^{\infty} \int_{\Omega} \left|D^j \tilde{u}(x, t)\right|^2 \mathrm{d}t \, \mathrm{d}x < +\infty$$

and

$$\int_{-\infty}^{\infty} \int_{x\in\Omega, \varrho(x) < h} \left|D^j \tilde{u}(X, x_n, t)\right|^2 \left|\varrho(x)\right|^{2|j|-2|i|} \mathrm{d}t \, \mathrm{d}x \to 0$$

as $h \to 0$.

Thus $\lim_{h\to 0} \left\|\tilde{u} - \tilde{u}_h\right\| = 0$ and $\lim_{h\to 0} \left\|u - u_h\right\|_R^{(-\infty,0)} = 0$.

6. Corollary. *Under the hypotheses of Theorem 5*

$$\left\|u_h\right\|_R^{(-\infty,0)} \leqq C\left\|u\right\|_R^{(-\infty,0)}$$

where the constant C does not depend on h.

Proof. This follows easily from the proof of Theorem 5 and of Theorem 2.5 in [1].

7. Corollary. *If* $v \in {}_R\overset{0}{W}{}_2^{(k,l')}(\Omega \times (-\infty, 0))$, *then, under the hypotheses of Theorem* 5,

$$\langle u_h, v\rangle_{-\infty}^0 = \langle u, v_h\rangle_{-\infty}^0, \quad \left\|u_h\right\|_P^{(-\infty,0)} \leqq C\left\|u\right\|_P^{(-\infty,0)}$$

and $\lim_{h\to 0} \langle u_h, v\rangle = \langle u, v\rangle$.

Proof. If $\varphi \in \mathscr{D}(\Omega \times (-\infty, 0))$, then

$$\langle u_h, \varphi\rangle_{-\infty}^0 = \langle F_h u, \varphi\rangle_{-\infty}^0 = \langle u, F_h \varphi\rangle_{-\infty}^0 \tag{3}$$

and

$$\left|u_h\right|_P^{(-\infty,0)} = \sup_{\|\varphi\|_R(-\infty,0) \leqq 1} \left|\langle u_h, \varphi\rangle_{-\infty}^0\right| = \sup \left|\langle u, F_h\varphi\rangle_{-\infty}^0\right| \leqq$$

$$\leqq \left|u\right|_P^{(-\infty,0)} \sup_{\|\varphi\|_R(-\infty,0) \leqq 1} \left\|F_h\varphi\right\|_R^{(-\infty,0)} \leqq C\left|u\right|_P^{(-\infty,0)}.$$

This inequality implies the first one in the assertion. Taking $\varphi \to v$ one obtains $\langle u_h, v\rangle^0_{-\infty} = \langle u, v_h\rangle^0_{-\infty}$, and by Theorem 5 $\lim_{h\to 0} \langle u, v_h\rangle^0_{-\infty} = \langle u, v\rangle^0_{-\infty}$.

This completes the proof.

8. Theorem. *Let* $u \in {}^0_R W_2^{(k,l')}(\Omega \times (-\infty, 0))$ *and* $u(x, t) = 0$, *whenever* $\varrho(x) < \varepsilon$ or $x \notin \Omega$. *Put*

$$\mathscr{R}_h u(x, t) = \frac{1}{\varkappa h^n} \int_{|x-y|<h} \exp \frac{|x - y|^2}{|x - y|^2 - h^2}\, u(y, t)\, \mathrm{d}y$$

where

$$\varkappa = \int_{|x|<1} \exp \frac{|x|^2}{|x|^2 - 1}\, \mathrm{d}x$$

and $h < \varepsilon$. *Then* $\mathscr{R}_h u \to u$ *in* ${}^0_R W_2^{(k,l')}(\Omega \times (-\infty, 0))$.

Proof. Let $\tilde{u} \in \mathscr{M}$, $\tilde{u}|_{\Omega\times(-\infty,0)} \equiv u$. It is well known that if $|i| \leqq k$ then $D^i \mathscr{R}_h \tilde{u} \to D^i \tilde{u}$ in $L_2(\Omega \times E_1)$ as $h \to 0$. On the other hand, $\mathfrak{F}_t \mathscr{R}_h \tilde{u}(x, \eta) = \mathscr{R}_h \mathfrak{F}_t \tilde{u}(x, \eta)$ and $|\eta|^{l'} \mathscr{R}_h \mathfrak{F}_t \tilde{u}(x, \eta) = \mathscr{R}_h |\eta|^{l'} \mathfrak{F}_t \tilde{u}(x, \eta)$. Thus

$$|\mathscr{R}_h \tilde{u} - \tilde{u}|^2 = \int_\Omega \int_{-\infty}^{\infty} |\mathscr{R}_h |\eta|^{l'} \mathfrak{F}_t \tilde{u}(x, \eta) - |\eta|^{l'} \mathfrak{F}_t \tilde{u}(x, \eta)|^2\, \mathrm{d}x\, \mathrm{d}\eta\,.$$

The right-hand side of this equality tends to zero as $h \to 0$ because $|\eta|^{l'} \mathfrak{F}_t \tilde{u}(x, \eta) \in L_2(\Omega \times E_1)$.

Thus $\|\mathscr{R}_h \tilde{u} - \tilde{u}\| \to 0$ and $\|\mathscr{R}_h u - u\|_R^{(-\infty,0)} \to 0$ because $\mathscr{R}_h \tilde{u} \in \mathscr{M}$ if $\tilde{u} \in \mathscr{M}$. This completes the proof.

9. Corollary. *Under the hypotheses of Theorem* 8,

$$\|\mathscr{R}_h u\|_R^{(-\infty,0)} \leqq C \|u\|_R^{(-\infty,0)}\,.$$

Proof. This assertion is an immediate consequence of

$$|\mathscr{R}_h f|_{L_2(\Omega\times E_1)} \leqq C |f|_{L_2(\Omega\times E_1)}\,,$$

Theorem 2.5 in [1] and the equality $K\mathscr{R}_h u = \mathscr{R}_h K u$ where K denotes the canonical prolongation.

10. Corollary. *Let* u *satisfy the hypotheses of Theorem* 8, $v \in {}^0_R W_2^{(k,l')}(\Omega \times (-\infty, 0))$ *and* $h < \varepsilon/3$. *Then*

$$\|\mathscr{R}_h u\|_P^{(-\infty,0)} \leqq C \|u\|_P^{(-\infty,0)} \tag{4}$$

$$\langle \mathscr{R}_h u, v\rangle^0_{-\infty} = \langle u, \mathscr{R}_h v\rangle^0_{-\infty} \tag{5}$$

and

$$\lim_{h\to 0} \langle \mathscr{R}_h u, v\rangle_{-\infty}^{0} = \langle u, v\rangle_{-\infty}^{0} \, .$$

Proof. Let $\psi \in \mathscr{D}(\Omega)$, $\psi(x) = 0$ for $\varrho(x) < \varepsilon/3$ and $\psi(x) = 1$ for $\varrho(x) > 2\varepsilon/3$. Then

$$(6) \qquad \langle \mathscr{R}_h u, \varphi\rangle_{-\infty}^{0} = \langle \mathscr{R}_h u, \psi\varphi\rangle_{-\infty}^{0} = \langle u, \mathscr{R}_h \psi\varphi\rangle_{-\infty}^{0} = \langle u, \mathscr{R}_h \varphi\rangle_{-\infty}^{0}$$

and

$$\left|\mathscr{R}_h u\right|_P^{(-\infty,0)} = \sup_{\|\varphi\|_R^{(-\infty,0)} \leqq 1} \left|\langle u, \mathscr{R}_h \varphi\rangle_{-\infty}^{0}\right| \leqq \left|u\right|_P^{(-\infty,0)} \sup_{\|\varphi\|_R^{(-\infty,0)} \leqq 1} \left\|\mathscr{R}_h \varphi\right\|_R^{(-\infty,0)} \leqq$$
$$\leqq C\left|u\right|_P^{(-\infty,0)}$$

by Corollary 9, this implies (4). Taking $\varphi \to v$ in (6) one obtains (5), and by Theorem 8

$$\lim_{h\to 0} \langle \mathscr{R}_h u, v\rangle_{-\infty}^{0} = \lim_{h\to 0} \langle u, \mathscr{R}_h \psi v\rangle_{-\infty}^{0} = \langle u, \psi v\rangle_{-\infty}^{0} = \langle u, v\rangle_{-\infty}^{0} \, .$$

This completes the proof.

11. Theorem. *Let* $u \in {}_P^0W_2^{(k,l')}(\Omega \times (-\infty, 0))$. *Then*

$$\lim_{h\to 0} \langle u_h, u_h\rangle_{-\infty}^{0} = \langle u, u\rangle_{-\infty}^{0} \, .$$

Proof. By Corollaries 6 and 7,

$$\left|\langle u_h, u_h\rangle_{-\infty}^{0} - \langle u, u\rangle_{-\infty}^{0}\right| \leqq$$
$$\leqq \left|\langle u_h, u\rangle_{-\infty}^{0} - \langle u, u\rangle_{-\infty}^{0}\right| + \left|\langle u_h, u_h - u\rangle_{-\infty}^{0}\right| \leqq$$
$$\leqq \left|\langle u, u_h - u\rangle_{-\infty}^{0}\right| + \left|\langle u_h, u_h - u\rangle_{-\infty}^{0}\right| \leqq$$
$$\leqq \left(\left|u\right|_P^{(-\infty,0)} + \left|u_h\right|_P^{(-\infty,0)}\right) \left\|u_h - u\right\|_R^{(-\infty,0)} \leqq C\left\|u\right\|_P^{(-\infty,0)} \left\|u_h - u\right\|_R^{(-\infty,0)} \, ,$$

where $\left\|u_h - u\right\|_R^{(-\infty,0)} \to 0$ as $h \to 0$.

12. Theorem. *Under the hypotheses of Theorem* 8,

$$\lim_{h\to 0} \langle \mathscr{R}_h u, \mathscr{R}_h u\rangle_{-\infty}^{0} = \langle u, u\rangle_{-\infty}^{0} \, .$$

Proof. This case may be treated in a similar manner as in Theorem 11.

13. Theorem. *Under the hypotheses of Theorem* 8, $\mathscr{R}_h u \in {}_P^0W_2^{(0,l')}(\Omega \times (-\infty, 0))$ *for* $h < \varepsilon/3$ *and*

$$(7) \qquad \left|\langle \mathscr{R}_h u, \varphi\rangle_{-\infty}^{0}\right| \leqq C(h) \left|u\right|_P^{(-\infty,0)} \left|\left|\left|\varphi\right|\right|\right|_R^{(-\infty,0)} \, ,$$

where $|||.|||$ *is the norm* $\|.\|$ *in the case* $k = 0$, $\varphi \in \mathscr{D}(\Omega \times (-\infty, 0))$.

Proof.

$$\left|\langle \mathscr{R}_h u, \varphi \rangle_{-\infty}^{0}\right| = \left|\langle u, \mathscr{R}_h \varphi\psi \rangle_{-\infty}^{0}\right| \leqq \left|u\right|_P^{(-\infty,0)} \left\|\mathscr{R}_h \varphi\psi\right\|_R^{(-\infty,0)} .$$

If $K\varphi$ is the canonical prolongation (cf. [1]) of φ, then by Theorem 2.5 in [1],

$$(8) \qquad \left\|\mathscr{R}_h \varphi\psi\right\|_R^{(-\infty,0)} \leqq C\left\|\mathscr{R}_h K\varphi\psi\right\| \leqq$$

$$\leqq C\Big(\sum_{|i| \leqq k} \left|\mathscr{R}_h D^i K\varphi\psi\right|^2_{L_2(\Omega \times (-\infty,0))} + \left|\mathscr{R}_h K\varphi\psi\right|^2\Big)^{\frac{1}{2}} .$$

Now

$$(9) \qquad \left|\mathscr{R}_h K\varphi\psi\right|^2 = \int_\Omega \int_{-\infty}^{\infty} |\eta|^{l' \cdot 2} \left|\mathfrak{F}_t \mathscr{R}_h K\varphi\psi(x, \eta)\right|^2 \, \mathrm{d}x \, \mathrm{d}\eta =$$

$$= \int_\Omega \int_{-\infty}^{\infty} \left|\mathscr{R}_h(\psi(x) \, |\eta|^{l'} \, \mathfrak{F}_t K\varphi(x, \eta))\right|^2 \, \mathrm{d}x \, \mathrm{d}\eta .$$

It is known that

$$(10) \qquad \left|\mathscr{R}_h D^i \varphi\right|_{L_2(\Omega \times (-\infty,0))} \leqq C(h) \left|\varphi\right|_{L_2(\Omega \times (-\infty,0))} .$$

By (10), (9) and (8),

$$\left\|\mathscr{R}_h \varphi\psi\right\|_R^{(-\infty,0)} \leqq C(h) \left(\left|K\varphi\right|^2_{L_2(\Omega \times (-\infty,0))} + \left|K\varphi\right|^2\right)^{\frac{1}{2}} \leqq C(h) \left|\left|\left|\varphi\right|\right|\right|^{(-\infty,0)} .$$

This completes the proof of (7). Hence $\mathscr{R}_h u \in {}_P^0W_2^{(0,h')}(\Omega \times (-\infty, 0))$.

14. Theorem. *Let* $v \in {}_P^0W_2^{(0,l')}(\Omega \times (-\infty, 0))$ *and suppose that*

$$(11) \qquad (-1)^l \frac{\partial^{2l+1} v}{\partial t^{2l+1}} + v = 0 \quad on \quad \Omega \times (-\infty, 0)$$

in the sense of distributions. Then $v = 0$.

Proof. Put

$$\mathscr{S}_h \, v(x, t) = \frac{1}{\varkappa_1 h} \int_{|x-y|^2 + |t-s|^2 < h^2} \exp \frac{|x - y|^2 + |t - s|^2}{|x - y|^2 + |t - s|^2 - h^2} \, v(y, s) \, \mathrm{d}y \, \mathrm{d}s$$

where

$$\varkappa_1 = \int_{|x|^2 + |t|^2 < 1} \exp \frac{|x|^2 + |t|^2}{|x|^2 + |t|^2 - 1} \, \mathrm{d}x \, \mathrm{d}t \, , \quad h > 0 \, .$$

Let $\overline{\Omega}^* \subset \Omega$, $\varepsilon > 0$. Then there is a $h_0 = h_0(\Omega^*, \varepsilon)$ such that $\mathscr{S}_h \, v(x, t)$ satisfy (11) on $\Omega^* \times (-\infty, -\varepsilon)$ for all $h < h_0(\Omega^*, \varepsilon)$.

Denote by λ_α the roots of $\lambda^{2l+1} + (-1)^l = 0$ so arranged that $\operatorname{Re} \lambda_\alpha \leqq \operatorname{Re} \lambda_{\alpha+1}$. Then $\mathscr{S}_h \, v(x, t) = \sum_{\alpha=1}^{2l+1} C_\alpha^h(x) \, e^{\lambda_\alpha t}$ on $\Omega^* \times (-\infty, -\varepsilon)$. On the other hand, $\mathscr{S}_h v \to v$

as $h \to 0$ in $L_2(\Omega^* \times (-\infty, -\varepsilon))$. Thus $v(x, t) = \sum_{\alpha=1} C_\alpha(x)\, e^{\lambda_\alpha t}$ a.e. on $\Omega \times (-\infty, 0)$.

For almost all fixed $x \in \Omega$ there is $v(x, t) \in L_2(0, \infty)$. Hence $C_1(x) = \ldots = C_{l+1}(x) = 0$ a.e. in Ω because $\operatorname{Re} \lambda_\alpha < 0$ for $\alpha < l + 1$.

Now $v(x, 0) \in L_2(\Omega), \ldots, (\partial^{l-1} v/\partial t^{l-1})(x, 0) \in L_2(\Omega)$ (see [4]), and $v(x, 0) = \ldots = (\partial^{l-1} v/\partial t^{l-1})(x, 0) = 0$ for a.e. $x \in \Omega$ because $v \in {}_R\overset{0}{W}{}_2^{(0,l')}(\Omega \times (-\infty, 0))$. Thus $\sum_{\alpha=1}^{2l+1} C_\alpha(x)\, \lambda_\alpha^\beta = 0$ for $\beta = 0, \ldots, l - 1$ and, consequently, $C_{l+2}(x) = \ldots = C_{2l+1}(x) = 0$ a.e. in Ω. This completes the proof.

15. Definition. Let $\mathscr{E}(\Omega \times (a, b))$ be the space of all infinitely differentiable function on $\Omega \times (a, b)$ which have continuous partial derivatives of all orders in $\overline{\Omega} \times \langle a, b\rangle$.

16. Theorem. *Let* $v \in {}_R\overset{0}{W}{}_2^{(0,l')}(\Omega \times (-\infty, 0))$ *satisfy*

$$D_0(u, v) = \int_\Omega \int_{-\infty}^0 \frac{\partial^{l+1} \bar{u}}{\partial t^{l+1}} \frac{\partial^l v}{\partial t^l} \,\mathrm{d}x\, \mathrm{d}t + \int_\Omega \int_{-\infty}^0 \bar{u} v \,\mathrm{d}x\, \mathrm{d}t = 0 \tag{12}$$

for all $u \in {}_P\overset{0}{W}{}_2^{(0,l')}(\Omega \times (-\infty, 0)) \cap \mathscr{E}(\Omega \times (-\infty, 0))$. *Then* $v = 0$.

Proof. Put $u \in \mathscr{D}(\Omega \times (-\infty, 0))$ in (12) and obtain that v satisfies

$$(-1)^{l+1} \frac{\partial^{2l+1} v}{\partial t^{2l+1}} + v = 0 \tag{13}$$

on $\Omega \times (-\infty, 0)$ in the sense of distributions. As in the proof of Theorem 14, $v(x, t) = \sum_{\alpha=l+1}^{2l+1} C_\alpha(x)\, e^{\lambda_\alpha t}$ a.e., where λ_α $(\alpha = l + 1, \ldots, 2l + 1)$ are the roots of $\lambda^{2l+1} + (-1)^{l+1} = 0$ with positive real parts. On differentiating (13) one obtains that $(\partial^\beta v)/(\partial t^\beta) \in L_2(\Omega \times (-\infty, 0))$ for all integers β and, consequently,

$$\sum_{\alpha=l+1}^{2l+1} C_\alpha(x)\, \lambda_\alpha^\beta e^{\lambda_\alpha t} = \frac{\partial^\beta v}{\partial t^\beta}(x, t) \in L_2(\Omega \times (-\infty, 0)) .$$

Thus $C_\alpha(x)\, e^{\lambda_\alpha t} \in L_2(\Omega \times (-\infty, 0))$ and consequently $C_\alpha(x) \in L_2(\Omega)$.

Let now $u \in \mathscr{E}(\Omega \times (-\infty, 0)) \cap {}_P\overset{0}{W}{}_2^{(0,l')}(\Omega \times (-\infty, 0))$. Integrating by parts one obtains

$$0 = \int_\Omega \int_{-\infty}^0 \frac{\partial^{l+1} \bar{u}}{\partial t^{l+1}} \frac{\partial^l v}{\partial t^l} \,\mathrm{d}x\, \mathrm{d}t + \int_\Omega \int_{-\infty}^0 \bar{u} v \,\mathrm{d}x\, \mathrm{d}t =$$

$$= \int_\Omega \frac{\partial^l u(x, 0)}{\partial t^l} \sum_{\alpha=l+1}^{2\lambda+1} C_\alpha(x)\, \lambda_\alpha^l \,\mathrm{d}x\, \mathrm{d}t = 0$$

whence

$$(14) \qquad \sum_{\alpha=l+1}^{2l+1} C_\alpha(x)\, \lambda_\alpha^l = 0$$

a.e. in Ω.

Now $v \in {}^0_R W_2^{(0,l')}(\Omega \times (-\infty, 0))$ and thus

$$(15) \qquad \sum_{\alpha=l+1}^{2l+1} C_\alpha(x)\, \lambda_\alpha^\beta = 0$$

a.e. in Ω for $\beta = 0, 1, \ldots, l-1$.

From (14) and (15) we obtain $C_\alpha(x) = 0$ a.e. in Ω and $v = 0$ a.e. on $\Omega \times (-\infty, 0)$.

17. Theorem. *$\mathscr{E}(\Omega \times (-\infty, 0)) \cap {}^0_P W_2^{(0,l')}(\Omega \times (-\infty, 0))$ is dense in*

$${}^0_P W_2^{(0,l')}(\Omega \times (-\infty, 0)).$$

Proof. Let H be the closure of the set

$$\mathscr{H} = \mathscr{E}(\Omega \times (-\infty, 0)) \cap {}^0_P W_2^{(0,l')}(\Omega \times (-\infty, 0)).$$

If $u \in \mathscr{H}$, $\varphi \in \mathscr{D}(\Omega \times (-\infty, 0))$ then

$$(16) \qquad \langle u, \varphi \rangle_{-\infty}^0 = \int_\Omega \int_{-\infty}^0 \frac{\partial^{l+1} \bar{u}}{\partial t^{l+1}} \frac{\partial^l \varphi}{\partial t^l}\, \mathrm{d}x\, \mathrm{d}t\,.$$

If $\varphi \to u$ in ${}^0_R W_2^{(0,l')}(\Omega \times (-\infty, 0))$, then $\partial^l \varphi / \partial t^l \to \partial^l u / \partial t^l$ in $L_2(\Omega \times (-\infty, 0))$. Taking $\varphi \to u$ in (16) one obtains that

$$\langle u, u \rangle_{-\infty}^0 = \int_{-\infty}^0 \int_\Omega \frac{\partial^{l+1} \bar{u}}{\partial t^{l+1}} \frac{\partial^l u}{\partial t^l}\, \mathrm{d}x\, \mathrm{d}t =$$

$$= -\int_{-\infty}^0 \int_\Omega \frac{\partial^l \bar{u}}{\partial t^l} \frac{\partial^{l+1} u}{\partial t^{l+1}}\, \mathrm{d}x\, \mathrm{d}t + \int_\Omega \frac{d^l \bar{u}}{\partial t^l} \frac{\partial^l u}{\partial t^l}\, \mathrm{d}x = -\overline{\langle u, u \rangle_{-\infty}^0} + \int_\Omega \left| \frac{\partial^l u}{\partial t^l} \right|^2 \mathrm{d}x\,.$$

Thus

$$(17) \qquad 2 \operatorname{Re} \langle u, u \rangle_{-\infty}^0 = \int_\Omega \left| \frac{\partial^l u}{\partial t^l} \right|^2 \mathrm{d}x \geqq 0\,.$$

On the other hand,

$$(18) \qquad \left| \langle u, u \rangle_{-\infty}^0 - \langle v, v \rangle_{-\infty}^0 \right| \leqq \left| \langle u, u - v \rangle_{-\infty}^0 \right| +$$
$$+ \left| \langle v, u - v \rangle_{-\infty}^0 \right| \leqq \left(|||u|||_P^{(-\infty,0)} + |||v|||_P^{(-\infty,0)} \right) |||u - v|||_R^{(-\infty,0)}\,.$$

In view of (18), the inequality (17) extends to an arbitrary $u \in H$. Put $D_0 u = (-1)^l (\partial^{2l+1} u / \partial t^{2l+1}) + u$. Then, by Theorem 4.2 in [1],

$$|||u|||_P^{(-\infty,0)} \leqq C |D_0 u|_{{}_R W_2^{(0,-l')}}$$

and consequently $D_0 H$ is a complete subspace of the space adjoint to ${}^0_R W_2^{(0,l')}(\Omega \times (-\infty, 0))$. This latter space is reflexive. Let $f(v) = 0$ for all $f \in D_0(H)$, i.e. $D_0(u, v) = 0$ for all $u \in H$. By Theorem 16, $v = 0$. Thus $D_0(H)$ coincides with the space adjoint to ${}^0_R W_2^{(0,l')}(\Omega \times (-\infty, 0))$.

Let now $u \in {}^0_P W_2^{(0,l')}(\Omega \times (-\infty, 0))$. There exists a $u_0 \in H$ such that $D_0 u_0 = D_0 u$, i.e. $D_0(u - u_0) = 0$. Then $u - u_0 \in {}^0_P W_2^{(0,l')}(\Omega \times (-\infty, 0))$ and, by Theorem 14, $u - u_0 = 0$, i.e. $u \in H$. This completes the proof.

18. Corollary. $\operatorname{Re} \langle u, u \rangle_{-\infty}^0 \geqq 0$ for every $u \in {}^0_P W_2^{(0,l')}(\Omega \times (-\infty, 0))$.

19. Theorem. $\mathfrak{N}^{(0),1} \subset \mathfrak{P}$.

Proof. Let $u \in {}^0_P W_2^{(0,l')}(\Omega \times (-\infty, 0)) \cap {}^0_P W_2^{(k,l')}(\Omega \times (-\infty, 0))$. Then, by Corollary 18,

$$\lim_{\varphi \to u} \operatorname{Re} \langle u, \varphi \rangle_{-\infty}^0 \geqq 0\,,$$

where $\varphi \to u$ in ${}^0_R W_2^{(0,l')}(\Omega \times (-\infty, 0))$. If now $\varphi \to u$ in ${}^0_R W_2^{(k,l')}(\Omega \times (-\infty, 0))$, then $\varphi \to u$ in ${}^0_R W_2^{(0,l')}(\Omega \times (-\infty, 0))$ and therefore

$$\operatorname{Re} \langle u, u \rangle_{-\infty}^0 \geqq 0\,. \tag{19}$$

By Theorems 11, 12 and 13, the inequality (19) extends to all functions $u \in {}^0_P W_2^{(k,l')}(\Omega \times (-\infty, 0))$.

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Резюме

ОБ ОБЛАСТЯХ ТИПА $\mathfrak{P}$

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В работе определяется пространство ${}^{0}_{R}W_2^{(k,l+\frac{1}{2})}(\Omega \times (-\infty, 0))$ функций, интегрируемых с квадратом вместе со всеми производными порядка k по пространственным переменным и производной порядка $l + \frac{1}{2}$ по времени в цилиндре $\Omega \times (-\infty, 0)$, таких, что

$$u = \frac{\partial u}{\partial \nu} = \ldots = \frac{\partial^{k-1} u}{\partial \nu^{k-1}} = 0 \quad \text{на} \quad \Omega^{\cdot} \times (-\infty, 0) \quad \text{и} \quad u = \frac{\partial u}{\partial t} = \ldots = \frac{\partial^{l-1}}{\partial t^{l-1}} = 0$$

на $\Omega \times \{0\}$. Далее, определяется пространство ${}^{0}_{P}W_2^{(k,l+\frac{1}{2})}(\Omega \times (-\infty, 0))$ всех функций из пространства ${}^{0}_{R}W_2^{(k,l+\frac{1}{2})}(\Omega \times (-\infty, 0))$, для которых

$$\sup_{\varphi \in B} \left| \int_\Omega \int_{-\infty}^0 \bar{u} \, \frac{\partial^{2l+1} \varphi}{\partial t^{2l+1}} \, \mathrm{d}\Omega \, \mathrm{d}t \right| < +\infty$$

где B — множество всех $\varphi \in \mathscr{D}(\Omega \times (-\infty, 0))$, для которых $\|\varphi\|_{{}^0_R W_2^{(k,l+\frac{1}{2})}(\Omega \times (-\infty,0))} \leqq$ $\leqq 1$. Теперь скажем, что $\Omega \in \mathfrak{P}$, если для всех $u \in {}^{0}_{R}W_2^{(k,l+\frac{1}{2})}(\Omega \times (-\infty, 0))$, и для всех $\varphi_n \in \mathscr{D}(\Omega \times (-\infty, 0))$ таких, что $\varphi_n \to u$ в пространстве ${}^{0}_{R}W_2^{(k,l+\frac{1}{2})}$ $(\Omega \times (-\infty, 0))$ имеет место,

$$(-1)^{l+1} \lim_{n \to \infty} \operatorname{Re} \int_\Omega \int_{-\infty}^0 \bar{u} \, \frac{\partial^{2l+1} \varphi_n}{\partial t^{2l+1}} \, \mathrm{d}\Omega \, \mathrm{d}t \geqq 0 \, .$$

В работе доказано, что область с границей Липшица $(\Omega \in \mathfrak{N}^{(0),1})$ обязательно находится в классе $\mathfrak{P}$.