Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 16 (1966), No. 1, 130–136

Persistent URL: http://dml.cz/dmlcz/100717

Чехословацкий математический журнал, т. 16 (19) 1966, Прага

KRONECKER INDEX IN ABSTRACT DYNAMICAL SYSTEMS, III

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(Received April 8, 1965)

In [7] it was shown how the Kronecker index can be defined and used for studying a number of properties of a local dynamical system in R^p. (The results of paper [7] will be referred to directly; e.g. lemma 2.8 is lemma 2.8 in [7]). It may be expected that this notion can be used also in the investigation of local dynamical systems defined on locally Euclidean spaces. In this paper there will be given the definition of the Kronecker index of a point with respect to a local dynamical system on a compact p-manifold, and this index will be used to obtain certain characterisation of critical points of a local dynamical system. (Concerning the definition of a local dynamical system on a topological space P see definition 2.7 and also [4].) We use the notation introduced in previous papers [6], [7].

3.1. Let T be a local dynamical system on P and $G \subset \mathsf{P}$ an open set. Define a local dynamical system T' on G as follows: if $x \in G$, $\theta \geqq 0$ (or $\theta \leqq 0$) such that $x\mathsf{T}\theta' \in G$ for each $0 \leqq \theta' \leqq \theta$ ($\theta \leqq \theta' \leqq 0$ respectively), then $x\mathsf{T}'\theta = x\mathsf{T}\theta$; for all other points $(x, \theta) \in G \times \mathsf{R}^1$ let $x\mathsf{T}'\theta$ be undefined. Clearly T' is a local dynamical system on G. Its escape time α'_x (or β'_x) is defined as $\alpha'_x = \sup \{\theta : x\mathsf{T}\theta' \in G \text{ for } 0 \leqq \theta' \leqq \theta\}$ ($\beta'_x = \inf \{\theta : x\mathsf{T}\theta' \in G \text{ for } \theta \leqq \theta' \leqq 0\}$ respectively). The local dynamical system T' is said to be the *relativisation* of the local dynamical system T to the set G.

If U is a p-dimensional element, T a local dynamical system on U, $f : U \approx \mathsf{R}^p$, then T_f denotes a local dynamical system on R^p defined by the relation $f(x\mathsf{T}\theta) = f(x)\mathsf{T}_f\theta$.

3.2. In this section we define the Kronecker index of a point with respect to a local dynamical system T on P. The underlying idea is as follows. To each point $x \in \mathsf{P}$ there is a neighbourhood G homeomorphic to R^p. Let $g : G \approx \mathsf{R}^p$ and let a local dynamical system T' be the relativisation of T to G. Then T'_g is a local dynamical system on R^p. If $x \in G$ is a non-critical or an isolated critical point of T, then $g(x)$ is also a critical or isolated critical point, respectively, of the local dynamical system T'_g on R^p; hence, the index $\text{ind}_{\mathsf{T}_g'}\, g(x)$ is defined (see definition 2.15). Now it is natural to define $\text{ind}_{\mathsf{T}}\, x = \text{ind}_{\mathsf{T}_g'}\, g(x)$. Of course, it will be necessary to show that this integer does not depend on the choice of G and g.

Definition 3.1. Let T be a local dynamical system on a p-manifold P, $x \in \mathsf{P}$, G its neighbourhood homeomorphic to R^p such that $G - \{x\}$ contains no critical points of T. Let the local dynamical system T' be the relativisation of T to G, $g : G \approx \mathsf{R}^p$, $g(x) = 0$. The *Kronecker index* $\operatorname{ind}_{\mathsf{T}} x$ of the point x with respect to T is defined as the number $\operatorname{ind}_{\mathsf{T}_g'} 0$.

Let H be another p-dimensional element with the properties of G in the definition, $h : H \approx \mathsf{R}^p$. First suppose $G \subset H$ and denote $f : h \mid G$, $f(G) = U \subset \mathsf{R}^p$. Let T' a T'' be the relativisations of T to G and H respectively. Evidently, for every point $(x, \theta) \in$ $\in G \times \mathsf{R}^1$ such that $x\mathsf{T}\theta' \in G \subset H$ whenever $0 \leqq \theta' \leqq \theta$, there holds $x\mathsf{T}\theta = x\mathsf{T}'\theta =$ $= x\mathsf{T}''\theta$; and for every point $(y, \theta) \in U \times \mathsf{R}^1$ such that $y\mathsf{T}_f'\theta' \in U$ whenever $0 \leqq \theta' \leqq \theta$ there holds

$$y\mathsf{T}_f'\theta = y\mathsf{T}_h''\theta\,. \tag{1}$$

Now let Q be a $(p - 1)$-pseudomanifold in U such that $0 \in \operatorname{Int} Q$. Using the compactness of Q, there exists (see lemma 2.8) a mapping $\vartheta : Q \to \mathsf{R}^1$ such that the mappings W_h, W_f defined by $W_h(x) = x\mathsf{T}_h''\vartheta(x) - x$, $W_f(x) = x\mathsf{T}_f'\vartheta(x) - x$ for $x \in Q$ are small vector fields on Q and that $x\mathsf{T}_h''\vartheta(x) \in U$, $x\mathsf{T}_f'\vartheta(x) \in U$. Then from (1) there follows $W_f(x) = W_h(x)$ for $x \in Q$, and from definition 2.9 one obtains $\operatorname{ind}_{\mathsf{T}_h'} Q =$ $= \operatorname{ind}_{\mathsf{T}_h''} Q$. Thus (see definition 2.15)

$$\operatorname{ind}_{\mathsf{T}_f'} 0 = \operatorname{ind}_{\mathsf{T}_h''} 0\,. \tag{2}$$

Using the homeomorphism $fg^{-1} : \mathsf{R}^p \approx U$, $fg^{-1}(0) = 0$, theorem 2.17 (topological invariance of the index) and (2), one obtains

$$\operatorname{ind}_{\mathsf{T}_g'} 0 = \operatorname{ind}_{\mathsf{T}_{fg^{-1}g}'} 0 = \operatorname{ind}_{\mathsf{T}_f'} 0 = \operatorname{ind}_{\mathsf{T}_h''} 0\,;$$

hence there follows the independence of definition 3.1 on the choice of G and R in the special case $G \subset H$. In the general case one can use a third element $K \subset G \cap H$ and apply the proposition just proved to $K \subset G$ and $K \subset H$.

3.3. In the remaining part of this paper P will denote a compact triangulable p-manifold, Π its triangulation, $C_k(\Pi)$ the group of k-chains with integral coefficients. If Π_i, Π_j are two triangulations of P, then a mapping $f : \mathsf{P} \to \mathsf{P}$, simplicial with respect to the triangulations Π_i and Π_j, will be denoted by $f : \mathsf{P}_i \to \mathsf{P}_j$. Next, T will be a local dynamical system on P with a finite set of critical points $x_1, x_2, \ldots, x_n$. We shall prove that the number $\sum_{i=1}^{n} \operatorname{ind}_{\mathsf{T}} x_i$ depends only on the topological structure of the space P; furthermore, we obtain the relation between this number and Euler characteristic χ of P [1, VII, § 11] by proving the following theorem.

Theorem 3.2. *Let* P *be a compact triangulable p-manifold,* T *a local dynamical system on* P *with a finite set of critical points* $x_1, x_2, \ldots, x_n$. *Then there holds*

$$\chi = (-1)^p \sum_{i=1}^{n} \operatorname{ind}_{\mathsf{T}} x_i ,$$

where χ *is the Euler characteristic of* P.

Corollary 3.3. *If* P *is a compact triangulable p-manifold with* $\chi \neq 0$, *then every local dynamical system on* P *has at least one critical point* (see also [5]).

There is an evident close relation of the assertion of theorem 3.2 with the known theorem [1, XIV, § 3, theorem 1]: If K is a p-dimensional polyhedron, $f : K \to K$ with only isolated fixed points, then the sum of indices of the fixed points of f is the Lefschetz number of f, up to a multiple of $(-1)^p$. In the proof of theorem 3.2 we shall use some ideas, notions and assertions employed in the proof of the special case of this theorem in [1]. Several of these notions and assertions will be given in the next two sections. Moreover, our formulations are sometimes reformulations; in each such case, however, the equivalence of the assertions is evident. The proof proper of theorem 3.2 will be given in section 3.6.

3.4. First recall the notion of a fixed simplex [1, XIV, § 1,1], which generalises the notion of a fixed point of a simplicial mapping.

Let Π_0 be a triangulation of P, Π_1 a subdivision of Π_0, $g : \mathsf{P}_1 \to \mathsf{P}_0$. A simplex $\tau \in \Pi_1$ is said to be a *fixed simplex* of the mapping g if $\tau \subset g(\tau)$. In particular, if $\Pi_1 = \Pi_0$ then τ is a fixed simplex of the mapping g iff $\tau = g(\tau)$.

Now let us introduce several objects which will be needed later.

Let T be a local dynamical system as assumed in theorem 3.2 with isolated critical points $x_1, x_2, \ldots, x_n$. For $i = 1, 2, \ldots, n$ let $U_i \subset \mathsf{P}$ be a p-dimensional element containing x_i and let $U_i \cap U_j = \emptyset$ for $i \neq j$. The triangulation Π_0 of P can be choosed so that all critical points x_i are vertices in Π_0, and that for each i there holds $\overline{st(x_i)} \subset \subset U_i$, where $st(x_i)$ is the open star of the simplex x_i in Π_0 [3, II, definition 3.6]. Set $L = P - \bigcup_{i=1}^{n} st(x_i)$. Since L is compact, there exists a number $A' > 0$ (see the proof of lemma 2.8) such that for every mapping $\vartheta_1 : \mathsf{P} \to (0, A')$ there holds $x\mathsf{T}\vartheta_1(x) \neq x$ for all $x \in L$. Let $A'' > 0$ be such that for every mapping $\vartheta_2 : \mathsf{P} \to (0, A'')$ there is $x\mathsf{T}\vartheta_2(x) \in U$ for all $x \in \overline{st(x_i)}$, $i = 1, 2, \ldots, n$. Take an arbitrary mapping $\vartheta : \mathsf{P} \to \to (0, \min(A', A''))$ and define a mapping $F : \mathsf{P} \to \mathsf{P}$ by the relation $F(x) = x\mathsf{T}\vartheta(x)$. Evidently F has no fixed point on L and $F(\overline{st(x_i)}) \subset U_i$ for all i. In [1] chap. XIV, § 3.5 there is proved the following lemma.

Lemma 3.4. *There exists a triangulation* Π_0 *of* P, *a subdivision* Π_p *of* Π_0 *and a simplicial approximation* $f : \mathsf{P}_p \to \mathsf{P}_0$ *of the mapping* F *such that* f *has no fixed k-simplex for* $0 \leq k < p$.

From the proof of this lemma in [1] there follows (using our assumptions on Π_0, F and the theorem on existence of barycentric subdivisions with an arbitrarily fine mesh [1, III, § 2, theorem IIa]) the following lemma.

Lemma 3.5. *The mappings f, F from lemma* 3.4 *have these additional properties*:

(i) *all fixed p-simplices of f are contained in* $\bigcup_{i=1}^{n} st(x_i)$;

(ii) $f(\overline{st(x_i)}) \subset U_i$ *for* $i = 1, 2, \ldots, n$;

(iii) *there exists a homotopy* $h_\lambda : f|L \simeq F|L$ *such that* $h_\lambda(x) \neq x$ *for* $(\lambda, x) \in I \times L$.

The following lemma is proved in [1, XIV, § 3,2].

Lemma 3.6. *In every fixed p-simplex of the mapping from lemma* 3.4 *there exists precisely one fixed point of f.*

3.5. In this section there will be given a relation between the Euler characteristic of P and the fixed points of f from lemma 3.6. The notions and notation from 2.1 will be used throughout.

Let $\tau_1, \tau_2, \ldots, \tau_n$ be all the p-simplices of the complex Π_p; hence they form a base in $C_p(\Pi_p)$. Denote by $f_{\#}$ the homomorphism $C_p(\Pi_p) \to C_p(\Pi_0)$ induced by f, and h the homomorphism $C_p(\Pi_0) \to C_p(\Pi_p)$ induced by the subdivision of Π_0; then $f_\circ = hf_{\#}$ is an endomorphism of the group $C_p(\Pi_p)$. Thus for every p-simplex $\tau_k \in \Pi_p$ there holds

$$f_\circ(\tau_k) = \sum_{l=1}^{r} a_{kl}\tau_l\,, \tag{3}$$

with a_{kl} integers, and for the Euler characteristic χ of P there holds (see [1], chap. XIV, § 3 (4) and § 1 (2), (7))

$$\chi = (-1)^p \sum_{k=1}^{r} a_{kk}\,. \tag{4}$$

In the preceding section there is assigned to each critical point x_i ($i = 1, 2, \ldots, n$) of T a neighbourhood heomeomorphic to R^p. Let $\varphi_i : U_i \approx \mathsf{R}^p$. For any fixed k, $1 \leqq k \leqq n$ denote $x = x_k$, $U = U_k$, $\varphi = \varphi_k$. Let $\sigma_1, \sigma_2, \ldots, \sigma_{\alpha_k}$ be all the fixed p-simplices of f contained in $\overline{st(x)}$, $y_1, y_2, \ldots, y_{\alpha_k}$ fixed points of f contained in the fixed p-simplices $\sigma_1, \sigma_2, \ldots, \sigma_{\alpha_k}$. For $j = 1, 2, \ldots, \alpha_k$ let s_j be an arbitrary $(p-1)$-sphere with center in y_j and contained in the interior of σ_j, E_j the open solid p-sphere with boundary s_j, and S the boundary of $st(x)$. The sets $S, s_1, s_2, \ldots, s_{\alpha_k}$ are $(p-1)$-pseudomanifolds [2, I, § 4,3], and obviously

$$s_j \cap S = \emptyset = s_j \cap s_i \quad \text{for} \quad j \neq i\,, \quad j, i = 1, 2, \ldots, \alpha_k\,.$$

Denote $Q = \overline{st(x)} - \bigcup_{j=1}^{\alpha_k} E_j$, $i_0 : S \subset Q$, $i_j : S_j \subset Q$ and define mappings $F_1, f_1 : Q \to$ $\to U$ thus: $F_1(x) = F(x)$, $f_1(x) = f(x)$ for $x \in Q$.

In [1] chap. XIV, § 2,5 there is proved the following lemma.

Lemma 3.7. *Let $\tau_1, \tau_2, \ldots, \tau_r$ be all the p-simplices of Π_p; choose j, $1 \leqq j \leqq r$. If $\tau_j \in \sigma_l$ (i.e. if τ_j is a fixed simplex of f contained in $st(x)$), then for the coefficients a_{jj} in (3) there holds*

$$a_{jj} = \varepsilon_{\varphi(s_l)} \cdot \omega(\varphi(y_l), \varphi f_1 i_l) .$$

If τ_j is not a fixed simplex of f, then $a_{jj} = 0$.

For $l = 1, 2, \ldots, \alpha_k$, $k = 1, 2, \ldots, n$ let σ_l^k be the fixed p-simplices in $\overline{st(x_k)}$, and similarly for y_l^k, s_l^k, i_l^k. From lemma 3.7 and (4) one obtains the following assertion.

Lemma 3.8.

$$\chi = (-1)^p \sum_{k=1}^{n} \sum_{l=1}^{\alpha_k} \varepsilon_{\varphi_k(s_l{}^k)} \cdot \omega(\varphi_k(y_l^k), \varphi_k f_1 i_l^k) .$$

The assertion of this lemma is the required relation between the characteristic χ and the indices of fixed points of the mapping f. Now let us try to find a similar relation between the index of a critical point x of T and the fixed points of f contained in $st(x)$. We use the notions and notation introduced at the beginning of this section.

Define mappings $W: Q \to \mathrm{R}^p$, $V: S \to \mathrm{R}^p$ by the relations

$$W(z) = \varphi f_1(z) - \varphi(z) , \quad V(z) = \varphi F_1 i_0(z) - \varphi i_0(z) .$$

Clearly $W(z) \neq 0$ for $z \in Q$. Let us prove the following lemma.

Lemma 3.9.

$$\mathrm{ind}_{\mathrm{T}}\, x = \sum_{j=1}^{\alpha_k} \varepsilon_{\varphi(s_j)} \cdot \omega(\varphi(y_j), \varphi f_1 i_j) .$$

Before presenting the proof of this lemma, we shall prove two propositions.

Proposition 1.

$$\mathrm{ind}_{\mathrm{T}}\, x = \varepsilon_{\varphi(s)} \cdot \omega(0, W i_0) .$$

Proof of proposition 1: Using lemma 3.5 (iii) it is easily shown that there exists a homotopy $g_\lambda : W i_0 \simeq V$ in $\mathrm{R}^p - \{0\}$. Let a local dynamical system T′ be the relativisation of T to U, T'_φ the local dynamical system on R^p induced by the system T′ and the homeomorphism φ. Then

$$\varepsilon_{\varphi(s)} \cdot \omega(0, W i_0) = \varepsilon_{\varphi(s)} \cdot \omega(0, V) = \mathrm{ind}_{\mathrm{T}_\varphi'}\, \varphi(s) = \mathrm{ind}_{\mathrm{T}_\varphi}\, 0 = \mathrm{ind}_{\mathrm{T}}\, x ,$$

proving proposition 1.

Proposition 2. For $j = 1, 2, \ldots, \alpha_k$ there hold

$$\omega(0, W_j) = \omega(\varphi(y_j), \varphi f_1 i_j).$$

The proof is similar to that of (9) in [7].

Proof of lemma 3.9. Let the orienting generators of the groups $H_{p-1}(S)$, $H_{p-1}(s_1)$, $H_{p-1}(s_2), \ldots, H_{p-1}(s_{\alpha_k})$ be the positive orienting generators $Z_0, Z_1, Z_2, \ldots, Z_{\alpha_k}$ of these groups (see section 2.3); thus $\varepsilon_{\varphi(s)} = \varepsilon_{\varphi(s_1)} = \ldots = \varepsilon_{\varphi(s_{\alpha_k})}$. Using propositions 1 and 2 it suffices to prove the relation $\omega(0, Wi_0) = \sum_{j=1}^{\alpha_k} \omega(0, Wi_j)$.

However, this relation has been proved, under the same assumptions, in the proof of theorem 2.16. This completes the proof of lemma 3.9.

It is now obvious that the proof of theorem 3.2 is contained in the preceding lemmas; and it suffices to summarise the results just obtained.

3.6. For $k = 1, 2, \ldots, n$ denote by $\sigma_1, \sigma_2, \ldots, \sigma_{\alpha_k}$ the fixed p-simplices of f contained in the neighbourhood U_k of the critical point x_k of the local dynamical system T. Then from lemmas 3.8 and 3.9 there follows

$$\sum_{k=1}^{n} \operatorname{ind}_{\mathsf{T}} x_k = \sum_{k=1}^{n} \sum_{l=1}^{\alpha_k} \varepsilon_{\varphi_k(s_l{}^k)} \cdot \omega(\varphi_k(y_l^k), \varphi_k f_1 i_l^k) = (-1)^p \chi,$$

i.e.

$$\chi = (-1)^p \sum_{k=1}^{n} \operatorname{ind}_{\mathsf{T}} x_k.$$

This concludes the proof of theorem 3.2.

Note. Theorem 3.2 and corollary 3.3 also hold for T a local semi-dynamical system (see section 2.7).

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Резюме

ИНДЕКС КРОНЕККЕРА В АБСТРАКТНЫХ ДИНАМИЧЕСКИХ СИСТЕМАХ, III

ЙОСЕФ НАДЬ, (Jozef Nagy), Прага

В работе определяется *индекс Кронеккера* $\text{ind}_{\mathsf{T}}\, x$ точки p-мерного компактного многообразия относительно локальной динамической системы T, определенной на этом мнообразии. Потом доказывается следующая теорема.

Теорема 3.2. *Пусть* P — *компактное триангулируемое p-мерное многообразие,* T — *локальная динамическая система на* P, *имеющая только конечное число критических точек* $x_1, x_2, \ldots, x_n$. *Тогда имеет место соотношение*

$$\chi = (-1)^p \sum_{i=1}^{n} \text{ind}_{\mathsf{T}}\, x_i \,,$$

где χ — *эйлерова характеристика мнообразия* P.