Vítězslav Novák

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ON UNIVERSAL QUASI-ORDERED SETS

VÍTĚZSLAV NOVÁK, Brno

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Let $F(\alpha, m)$ denote a type of a set of all sequences of type α formed from elements of a set of cardinality m together with the relation $\leqq$ defined as follows : $\{a_\lambda \mid \lambda < \alpha\} \leqq \{b_\lambda \mid \lambda < \alpha\}$ if and only if the sequence $\{a_\lambda \mid \lambda < \alpha\}$ is a subsequence of the sequence $\{b_\lambda \mid \lambda < \alpha\}$. In this paper the following theorem is proved. For every quasi-ordered set G of cardinality $\leqq \aleph_\alpha$ there exists a subset G' isomorphic with G in a set of type $F(\omega_\alpha . 2, \aleph_\alpha)$. This theorem improves a result of M. NOVOTNÝ in [6].

A *quasi-ordered* set is a non-empty set together with a reflexive and transitive binary relation (see for instance [1]). Two quasi-ordered sets G, G' are called *isomorphic* if a one-one mapping f of the set G onto G' exists such that $x, y \in G, x \leqq y \Leftrightarrow \Leftrightarrow f(x) \leqq f(y)$. A quasi-ordered set G is called an *m-universal* quasi-ordered set (where $m > 0$ is a cardinality) if for every quasi-ordered set H with card $H \leqq m$ there exists a subset $G' \subseteqq G$ isomorphic with H.

Let G be an ordered set (i.e. a non-empty set together with a reflexive, antisymmetric and transitive binary relation). Let B be a chain of type **2** (i.e. a chain containing exactly two points). Let K be a non-empty set. Let $f_\varkappa$ denote a mapping of the set G into B for every $\varkappa \in K$. A system $\{f_\varkappa \mid \varkappa \in K\}$ is called a **2**-*pseudorealizer* of the set G, if $x, y \in G$, $x \leqq y$ is equivalent to $f_\varkappa(x) \leqq f_\varkappa(y)$ for every $\varkappa \in K$. In [5] there is proved that every ordered set G has at least one **2**-pseudorealizer. **2**-*pseudodimension* of the set G is defined as the minimal cardinality of a set K such that $\{f_\varkappa \mid \varkappa \in K\}$ is a **2**-pseudorealizer of G. This cardinal number is denoted **2**-pdim G. In [5] the following theorem is proved: *Let G be an ordered set. Then the following statements are equivalent*:

1) **2**-pdim $G \leqq m$

2) *There exists an antichain* [1]) *M with* card $M = m$ *such that* $G \cong G' \subseteqq B^M$ [2]).

Let $\alpha > 0$ be an ordinal number, let $\{a_\lambda \mid \lambda < \alpha\}$, $\{b_\lambda \mid \lambda < \alpha\}$ be sequences of

[1]) By an antichain we understand a set ordered so that every two its distinct elements are incomparable.

[2]) $G \cong G'$ denotes that the sets G, G' are isomorphic. B^M denotes the Birkhoff's cardinal power (see for instance [1] or [2]).

type α. The sequence $\{a_\lambda \mid \lambda < \alpha\}$ is called a *subsequence* of the sequence $\{b_\lambda \mid \lambda < \alpha\}$ if there exists a strictly increasing sequence $\{\beta_\lambda \mid \lambda < \alpha\}$ of type α formed from ordinal numbers less than α such that $a_\lambda = b_{\beta_\lambda}$ for every $\lambda < \alpha$.

Let M be a non-empty set, let $\alpha > 0$ be an ordinal number. Denote $F(\alpha, M)$ the set of all sequences of type α formed from the elements of the set M together with the relation $\leqq$ defined as follows: $\{a_\lambda \mid \lambda < \alpha\} \leqq \{b_\lambda \mid \lambda < \alpha\}$ if and only if the sequence $\{a_\lambda \mid \lambda < \alpha\}$ is a subsequence of the sequence $\{b_\lambda \mid \lambda < \alpha\}$. It is easy to prove that the relation $\leqq$ is reflexive and transitive so that $F(\alpha, M)$ is a quasi-ordered set. This relation, however, is not antisymmetric in general as it is shown in [6]. $F(\alpha, M)$ is therefore generally not an ordered set. If N is a set with card N = card M then clearly $F(\alpha, N)$ is isomorphic with $F(\alpha, M)$ so that the type of the set $F(\alpha, M)$ depends only on the cardinality m of the set M. We denote this type by $F(\alpha, m)$.

Let $\{a_\lambda \mid \lambda < \alpha\}$ be a sequence of type α. Let $G = \{x \mid$ there exists an ordinal number $\lambda < \alpha$ such that $a_\lambda = x\}$. Put for every $x \in G$ $m_x(\{a_\lambda \mid \lambda < \alpha\}) = \text{card}\,\{\lambda \mid \lambda \in W(\alpha), a_\lambda = x\}$. [3])

We shall need the following two lemmas.

Lemma 1. *Let G be a non-empty set such that* card $G \leqq \aleph_\alpha$. *Then the elements of the set G can be written in the form of a sequence of type* ω_α, $\{a_\lambda \mid \lambda < \omega_\alpha\}$, *such that* $m_x(\{a_\lambda \mid \lambda < \omega_\alpha\}) = \aleph_\alpha$ *for every* $x \in G$.

Proof. Let H be a set with card $H = \aleph_\alpha$. Put $K = G \times H$. Then card K = card G . card $H = \aleph_\alpha$. Let us write the elements of the set K in the form of a simple[4]) sequence of type ω_α, i.e. $K = \{k_\lambda \mid \lambda < \omega_\alpha\}$. Then $k_\lambda = [x, y]$ where $x \in G$, $y \in H$, for every $\lambda < \omega_\alpha$. Now put for every $\lambda < \omega_\alpha$ $a_\lambda = x$ where $[x, y] = k_\lambda$. Then $\{a_\lambda \mid \lambda < \omega_\alpha\}$ is a sequence of type ω_α formed from the elements of the set G and having the desired property for, if $x_0 \in G$, then card $\{[x_0, y] \mid y \in H\} = \aleph_\alpha$ and therefore card $\{\lambda \mid \lambda \in W(\omega_\alpha), k_\lambda = [x_0, y]\,(y \in H)\} = \aleph_\alpha =$ card $\{\lambda \mid \lambda \in W(\omega_\alpha), a_\lambda = x_0\} = m_{x_0}(\{a_\lambda \mid \lambda < \omega_\alpha\})$.

Lemma 2. *Let G be a set with* card $G = m$ *where* $2 \leqq m \leqq \aleph_\alpha$. *Let* $\mathcal{S}$ *be the set of all sequences of type* ω_α *formed from the elements of the set G and such that* $m_x(\{a_\lambda \mid \lambda < \omega_\alpha\}) = \aleph_\alpha$ *for any sequence* $\{a_\lambda \mid \lambda < \omega_\alpha\} \in \mathcal{S}$ *and any element* $x \in G$. *Then* card $\mathcal{S} = 2^{\aleph_\alpha}$.

Proof. Let $\mathcal{T}$ denote the set of all sequences of type ω_α formed from the elements of the set G. Then card $\mathcal{T} = m^{\aleph_\alpha} = 2^{\aleph_\alpha}$. As $\mathcal{S} \subseteqq \mathcal{T}$, we have card $\mathcal{S} \leqq 2^{\aleph_\alpha}$. Let $\{c_\lambda \mid \lambda < \omega_\alpha\}$ be a given fixed sequence from $\mathcal{S}$, i.e. such a sequence that $m_x(\{c_\lambda \mid \lambda < \omega_\alpha\}) = \aleph_\alpha$ for any $x \in G$. Put for any sequence $\{b_\lambda \mid \lambda < \omega_\alpha\} \in \mathcal{T}$ $\varphi(\{b_\lambda \mid \lambda < \omega_\alpha\}) = \{a_\lambda \mid \lambda < \omega_\alpha\}$ where $\{a_\lambda \mid \lambda < \omega_\alpha\}$ is a sequence of type ω_α defined in the following way:

$$a_\lambda = \begin{cases} b_\nu & \text{for } \lambda = 2\nu \\ c_\nu & \text{for } \lambda = 2\nu + 1 \end{cases}$$

[3]) W(α) denotes the set of all ordinal numbers less than α (see [4]).

[4]) i.e. $k_{\lambda_1} \neq k_{\lambda_2}$ for $\lambda_1 \neq \lambda_2$.

Then clearly $\varphi(\{b_\lambda \mid \lambda < \omega_\alpha\}) \in \mathscr{S}$ for any sequence $\{b_\lambda \mid \lambda < \omega_\alpha\} \in \mathscr{T}$ and φ is a one-one mapping. This implies card $\mathscr{S} \geqq 2^{\aleph_\alpha}$ and hence card $\mathscr{S} = 2^{\aleph_\alpha}$.

Theorem 1. *Let G be a non-empty ordered set and let* $\mathbf{2}$ – pdim $G \leqq \aleph_\alpha$. *Then the set of type $F(\omega_\alpha, \aleph_\alpha)$ contains a subset isomorphic with G.*

Proof. If the assumptions of Theorem are true then $G \cong G' \subseteq B^M$ where $B = \{0, 1\}$ is a chain of type $\mathbf{2}$ and M is an antichain with card $M = \aleph_\alpha$. The set B^M is isomorphic with the system of all subsets of the set M ordered by a set inclusion[5]).

Let a be any element which does not belong to M. Put for any subset $N \subseteq M$, $N' = N \cup \{a\}$. Then the system $\mathscr{S} = \{N' \mid N \subseteq M\}$ is a system of non-empty sets which is – ordered by a set inclusion – isomorphic with B^M. $\mathscr{S}$ therefore contains a subset $\mathscr{S}'$ isomorphic with G; denote ψ an isomorphism of G onto $\mathscr{S}'$. Now, because card $M' = \text{card}(M \cup \{a\}) = \aleph_\alpha$ it is possible to write the elements of the set M' in the form of a sequence $\{b_\lambda \mid \lambda < \omega_\alpha\}$ of type ω_α such that $m_x(\{b_\lambda \mid \lambda < \omega_\alpha\}) = \aleph_\alpha$ for every $x \in M'$. Let us assign to every set $N' \in \mathscr{S}'$ a sequence $\varphi(N') = \{a_\lambda \mid \lambda < \omega_\alpha\}$ of type ω_α in the following way: $a_0 = b_{\mu_0}$ where μ_0 is the smallest ordinal number such that $b_{\mu_0} \in N'$; suppose that we have defined a_λ for every $\lambda < \lambda_0$ $(\lambda_0 < \omega_\alpha)$. Then we put $a_{\lambda_0} = b_{\mu_{\lambda_0}}$ where μ_{λ_0} is the smallest ordinal number with the following properties: $\mu_{\lambda_0} > \mu_\lambda$ for every $\lambda < \lambda_0$, $\mu_{\lambda_0} < \omega_\alpha$, $b_{\mu_{\lambda_0}} \in N'$. Such an ordinal number always exists for, if μ_{λ_0} were not defined for some $\lambda_0 < \omega_\alpha$, then $m_x(\{b_\lambda \mid \lambda < \omega_\alpha\}) \leqq \leqq \text{card}\{\mu_\lambda \mid \lambda < \lambda_0\} < \aleph_\alpha$ for any element $x \in N'$ which is a contradiction. If $N' \in \mathscr{S}'$ and $\varphi(N') = \{a_\lambda \mid \lambda < \omega_\alpha\}$ then clearly $m_x(\{a_\lambda \mid \lambda < \omega_\alpha\}) = \aleph_\alpha$ for every $x \in N'$. The set $\sum = \{\varphi(N') \mid N' \in \mathscr{S}'\}$ is a subset of a set of type $F(\omega_\alpha, \aleph_\alpha)$ and we shall show that φ is an isomorphism of $\mathscr{S}'$ onto $\sum$. Hence let $N'_1, N'_2 \in \mathscr{S}'$, $N'_1 \subseteq N'_2$ and let $\varphi(N'_1) = \{c_\lambda \mid \lambda < \omega_\alpha\}$, $\varphi(N'_2) = \{d_\lambda \mid \lambda < \omega_\alpha\}$. Let us define the sequence $\{\beta_\lambda \mid \lambda < \omega_\alpha\}$ of type ω_α of ordinal numbers less than ω_α in the following way: $\beta_0 = \mu_0$ where μ_0 is the smallest ordinal number with the property $d_{\mu_0} = c_0$; suppose that we have defined the numbers β_λ for every $\lambda < \lambda_0$ where $\lambda_0 < \omega_\alpha$. Then we define β_{λ_0} as the smallest ordinal number with these properties: $\beta_{\lambda_0} > \beta_\lambda$ for every $\lambda < \lambda_0$, $\beta_{\lambda_0} < \omega_\alpha$, $d_{\beta_{\lambda_0}} = c_{\lambda_0}$. It is easy to see that β_0 is the smallest ordinal number with the property $d_{\beta_0} \in N'_1$ and β_{λ_0} is for every $\lambda_0 < \omega_\alpha$ the smallest ordinal number with these properties: $\beta_{\lambda_0} > \beta_\lambda$ for every $\lambda < \lambda_0$, $\beta_{\lambda_0} < \omega_\alpha$, $d_{\beta_{\lambda_0}} \in N'_1$. From this follows, similarly as in the first part of the proof, that β_λ is defined for every $\lambda < \omega_\alpha$ so that $\{\beta_\lambda \mid \lambda < \omega_\alpha\}$ is a strictly increasing sequence of type ω_α of ordinal numbers less than ω_α and such that $c_\lambda = d_{\beta_\lambda}$ for every $\lambda < \omega_\alpha$. Thus, the sequence $\{c_\lambda \mid \lambda < \omega_\alpha\}$ is a subsequence of the sequence $\{d_\lambda \mid \lambda < \omega_\alpha\}$, i.e. $\{c_\lambda \mid \lambda < \omega_\alpha\} = \varphi(N'_1) \leqq \varphi(N'_2) = \{d_\lambda \mid \lambda < \omega_\alpha\}$. Suppose, on the contrary, that $\varphi(N'_1) = \{c_\lambda \mid \lambda < \omega_\alpha\} \leqq \{d_\lambda \mid \lambda < \omega_\alpha\} = \varphi(N'_2)$ and let $x \in N'_1$. Then there exists an ordinal number $\lambda_0 < \omega_\alpha$ such

[5]) If we assign to every element $f \in B^M$ a subset $\varphi(f) = \{x \mid x \in M, f(x) = 1\} \subseteq \mathbf{M}$, then it is clear that φ is an isomorphism of B^M onto a system of all subsets of the set M ordered by a set inclusion.

that $c_{\lambda_0} = x$ and exists an ordinal number $\beta_{\lambda_0} < \omega_\alpha$ such that $c_{\lambda_0} = d_{\beta_{\lambda_0}}$ so that $d_{\beta_{\lambda_0}} = x$ and hence $x \in N'_2$. This implies $N'_1 \subseteqq N'_2$. Further it is clear that φ is a one-one mapping and therefore φ is an isomorphism of $\mathscr{S}'$ onto $\sum$. From this it follows that a composite mapping $\varphi\psi$ is an isomorphism of G onto $\sum$ and the theorem is proved.

Theorem 2. *A quasi-ordered set of type $F(\omega_\alpha \,.\, 2, \aleph_\alpha)$ is an $\aleph_\alpha$-universal quasi-ordered set.*

Proof. Let G be a quasi-ordered set such that card $G \leqq \aleph_\alpha$. For two elements $x, y \in G$ put $x \equiv y$, if and only if $x \leqq y$, $y \leqq x$. It is known ([1]) that the relation $\equiv$ is an equivalence relation which defines a decomposition $\bar{G}$ of G in such a way that $X \in \bar{G} \Rightarrow x \leqq y$ for all elements $x \in X$, $y \in X$. Further, the set $\bar{G}$ can be ordered in the following way: $X, Y \in \bar{G} \Rightarrow X \leqq Y$ if and only if $x \leqq y$ for any $x \in X$, $y \in Y$. Now card $\bar{G} \leqq \aleph_\alpha$ so that $\mathbf{2}$ – pdim $\bar{G} \leqq \aleph_\alpha$. [6]) According to Theorem 1 $\bar{G}$ is isomorphic with a certain subset $\sum$ of a set $F(\omega_\alpha, M)$ where card $M = \aleph_\alpha$. Let ψ be an isomorphism of $\bar{G}$ onto $\sum$. Let $N = \{a, b\}$ where $a \bar{\in} M$, $b \bar{\in} M$ be any set with card $N = 2$. Now we shall distinguish two cases:

1) $\alpha = 0$. Then let ϑ denote the subset of the set $F(\omega_0, N)$ containing all those sequences $\{b_\lambda \mid \lambda < \omega_0\}$ for which $m_a(\{b_\lambda \mid \lambda < \omega_0\}) = \aleph_0$, $m_b(\{b_\lambda \mid \lambda < \omega_0\}) = \aleph_0$.

2) $\alpha > 0$. In this case let ϑ be the subset of the set $F(\omega_\alpha, N)$ containing all those sequences $\{b_\lambda \mid \lambda < \omega_\alpha\}$ for which $b_\lambda = a(\lambda < \omega_\alpha) \Rightarrow b_{\lambda+1} = b$, $b_\lambda = b \Rightarrow b_{\lambda+1} = a$.

In both cases we have card $\vartheta = 2^{\aleph_\alpha}$. Proof:

1) $\alpha = 0$. Then the statement follows from Lemma 2.

2) $\alpha > 0$. Then it holds: for any limit ordinal number $\lambda_0 < \omega_\alpha$ and any sequence $\{a_\lambda \mid \lambda < \omega_\alpha\} \in \vartheta$ we can have either $a_{\lambda_0} = a$ or $a_{\lambda_0} = b$. As card $\{\lambda \mid \lambda < \omega_\alpha$, λ is a limit ordinal number$\} = \aleph_\alpha$, we have clearly card $\vartheta = 2^{\aleph_\alpha}$.

In both cases there is $\{a_\lambda \mid \lambda < \omega_\alpha\} \in \vartheta$, $\{b_\lambda \mid \lambda < \omega_\alpha\} \in \vartheta \Rightarrow \{a_\lambda \mid \lambda < \omega_\alpha\} \leqq$ $\leqq \{b_\lambda \mid \lambda < \omega_\alpha\}$ in $F(\omega_\alpha, N)$. Proof:

1) $\alpha = 0$. Let $\{a_\lambda \mid \lambda < \omega_0\} \in \vartheta$, $\{b_\lambda \mid \lambda < \omega_0\} \in \vartheta$. Let β_0 be the smallest ordinal number such that $b_{\beta_0} = a_0$. If we have defined β_n for every $n < n_0$ ($n_0 < \omega_0$) then we define β_{n_0} as the smallest ordinal number with these properties: $\beta_{n_0} > \beta_n$ for every $n < n_0$, $\beta_{n_0} < \omega_0$, $b_{\beta_{n_0}} = a_{n_0}$. Such an ordinal number β_{n_0} is always defined since otherwise there would be $m_a(\{b_\lambda \mid \lambda < \omega_0\}) < \aleph_0$ or $m_b(\{b_\lambda \mid \lambda < \omega_0\}) < \aleph_0$. Thus $\{\beta_\lambda \mid \lambda < \omega_0\}$ is a strictly ascending sequence of type ω_0 of ordinal numbers less than ω_0 and such that $a_\lambda = b_{\beta_\lambda}$ for every $\lambda < \omega_0$.

This implies $\{a_\lambda \mid \lambda < \omega_0\} \leqq \{b_\lambda \mid \lambda < \omega_0\}$ in $F(\omega_0, N)$.

2) $\alpha > 0$. If $\{a_\lambda \mid \lambda < \omega_\alpha\} \in \vartheta$, $\{b_\lambda \mid \lambda < \omega_\alpha\} \in \vartheta$, then we put $\beta_\lambda = \lambda$ in case that $a_{\omega.\gamma} = b_{\omega.\gamma}$, and $\beta_\lambda = \lambda + 1$ in case that $a_{\omega.\gamma} \neq b_{\omega.\gamma}$ for λ satisfying $\omega \,.\, \gamma \leqq \lambda <$ $< \omega \,.\, (\gamma + 1)$ for every $\gamma < \omega_\alpha$. Then $\{\beta_\lambda \mid \lambda < \omega_\alpha\}$ is a strictly ascending sequence

[6]) It is not difficult to prove that $\mathbf{2}$— pdim $G \leqq$ card G for any ordered set G.

of type ω_α of ordinal numbers less than ω_α and such that $a_\lambda = b_{\beta_\lambda}$ for every $\lambda < \omega_\alpha$, i.e. $\{a_\lambda \mid \lambda < \omega_\alpha\} \leqq \{b_\lambda \mid \lambda < \omega_\alpha\}$ in $F(\omega_\alpha, N)$.

Now because card $X \leqq \aleph_\alpha$ for every $X \in \bar{G}$, it is possible to define a one-one mapping χ_X of the set X into ϑ; this mapping is clearly an isomorphism. Finally let us assign to every element $x \in G$ the sequence $\varphi(x) = \{c_\lambda \mid \lambda < \omega_\alpha \,.\, 2\}$ of type $\omega_\alpha \,.\, 2$ in the following way: there exists just one $X \in \bar{G}$ such that $x \in X$. Then $\psi(X) = \{a_\lambda \mid \lambda < \omega_\alpha\} \in \sum$, $\chi_X(x) = \{b_\lambda \mid \lambda < \omega_\alpha\} \in \vartheta$. We define $\varphi(x) = \{c_\lambda \mid \lambda < \omega_\alpha \,.\, 2\}$ so:

$$c_\lambda = \begin{cases} a_\lambda & \text{for } \lambda < \omega_\alpha \\ b_\mu & \text{for } \lambda = \omega_\alpha + \mu,\ \mu < \omega_\alpha \end{cases}$$

The set $\{\varphi(x) \mid x \in G\}$ is a subset of a set of type $F(\omega_\alpha \,.\, 2, \aleph_\alpha)$. We shall show that φ is an isomorphism. Let $x, y \in G$, $x \leqq y$. Then $x \in X$, $y \in Y$ where $X, Y \in \bar{G}$, $X \leqq Y$. Denote $\psi(X) = \{a^x_\lambda \mid \lambda < \omega_\alpha\}$, $\psi(Y) = \{a^y_\lambda \mid \lambda < \omega_\alpha\}$, $\chi_X(x) = \{b^x_\lambda \mid \lambda < \omega_\alpha\}$, $\chi_Y(y) = \{b^y_\lambda \mid \lambda < \omega_\alpha\}$. As ψ is an isomorphism, the sequence $\psi(X) = \{a^x_\lambda \mid \lambda < \omega_\alpha\}$ is a subsequence of the sequence $\psi(Y) = \{a^y_\lambda \mid \lambda < \omega_\alpha\}$, i.e. there exists a strictly ascending sequence $\{\beta_\lambda \mid \lambda < \omega_\alpha\}$ of type ω_α of ordinal numbers less than ω_α such that $a^x_\lambda = a^y_{\beta_\lambda}$ for every $\lambda < \omega_\alpha$. As $\chi_X(x) \in \vartheta$, $\chi_Y(y) \in \vartheta$, the sequence $\chi_X(x) = \{b^x_\lambda \mid \lambda < \omega_\alpha\}$ is a subsequence of the sequence $\chi_Y(y) = \{b^y_\lambda \mid \lambda < \omega_\alpha\}$, i.e. there exists a strictly ascending sequence $\{\gamma_\lambda \mid \lambda < \omega_\alpha\}$ of ordinal numbers less than ω_α such that $b^x_\lambda = b^y_{\gamma_\lambda}$ for every $\lambda < \omega_\alpha$. If we put

$$\delta_\lambda = \begin{cases} \beta_\lambda & \text{for } \lambda < \omega_\alpha \\ \omega_\alpha + \gamma_\mu & \text{for } \lambda = \omega_\alpha + \mu,\ \mu < \omega_\alpha \end{cases}$$

then $\{\delta_\lambda \mid \lambda < \omega_\alpha \,.\, 2\}$ is a strictly ascending sequence of type $\omega_\alpha \,.\, 2$ of ordinal numbers less than $\omega_\alpha \,.\, 2$ such that $c^x_\lambda = c^y_{\delta_\lambda}$ for every $\lambda < \omega_\alpha \,.\, 2$. This implies that $\varphi(x) = \{c^x_\lambda \mid \lambda < \omega_\alpha \,.\, 2\}$ is a subsequence of $\varphi(y) = \{c^y_\lambda \mid \lambda < \omega_\alpha \,.\, 2\}$, i.e. $\varphi(x) \leqq \varphi(y)$.

Suppose, on the contrary, that $\varphi(x) \leqq \varphi(y)$, i.e. $\varphi(x)$ is a subsequence of $\varphi(y)$. As $a, b \bar{\in} M$, this implies that $x \in X$, $y \in Y$ and $\psi(X)$ is a subsequence of $\psi(Y)$, i.e. $\psi(X) \leqq \psi(Y)$. As ψ is an isomorphism, this implies $X \leqq Y$ and hence $x \leqq y$.

Finally, it is easy to see that φ is a one-one mapping and therefore φ is an isomorphism and the proof is completed.

If $\aleph_\alpha$ is a regular cardinal number then we are able to prove a stronger result:

Theorem 3. *If $\aleph_\alpha$ is a regular cardinal number then a quasi-ordered set of type $F(\omega_\alpha, \aleph_\alpha)$ is an $\aleph_\alpha$-universal quasi-ordered set.*

Proof. If $\bar{G}$ is an ordered set constructed from G in the same way as in the proof of the Theorem 2 then $\mathbf{2}$ – pdim $\bar{G} \leqq \aleph_\alpha$ so that $\bar{G}$ is isomorphic with a certain system $\mathscr{S}$ of subsets of a set M with card $M = \aleph_\alpha$ ordered by a set inclusion.

Let $a \bar{\in} M$, $b \bar{\in} M$, $a \neq b$ be two elements and put for every set $N \in \mathscr{S}$ $N' = N \cup \{a, b\}$. Then the system $\mathscr{S}' = \{N' \mid N \in \mathscr{S}\}$ is a system of sets such that $2 \leqq$ card $N' \leqq \aleph_\alpha$ for every $N' \in \mathscr{S}'$ which is-ordered by a set inclusion-isomorphic with $\bar{G}$. Denote ψ an isomorphism of $\bar{G}$ onto $\mathscr{S}'$. Let $\sum(N')$ be the set of all sequences

$\{a_\lambda \mid \lambda < \omega_\alpha\}$ of type ω_α formed from the elements of the set N' and such that $m_x(\{a_\lambda \mid \lambda < \omega_\alpha\}) = \aleph_\alpha$ for every $x \in N'$. According to Lemma 2 we have card $\sum(N') = 2^{\aleph_\alpha}$ for every $N' \in \mathcal{S}'$. As card $X \leqq \aleph_\alpha$ for every $X \in \bar{G}$ it is possible to define a one-one mapping φ_X of the set X into $\sum[\psi(X)]$. Finally put $\varphi(x) = \varphi_X(x)$ where $x \in X \in \bar{G}$. φ is a one-one mapping of G into $\{\sum(N') \mid N' \in \mathcal{S}'\}$; the latter set is a subset of a set of type $F(\omega_\alpha, \aleph_\alpha)$. We shall show that φ is an isomorphism. Let $x, y \in G$, $x \leqq y$. Then $x \in X$, $y \in Y$ and $X \leqq Y$ in $\bar{G}$. Thus $\varphi(x) = \varphi_X(x) = \{a_\lambda \mid \lambda < \omega_\alpha\}$ where $m_u(\{a_\lambda \mid \lambda < \omega_\alpha\}) = \aleph_\alpha$ for every $u \in \psi(X)$ and $\varphi(y) = \varphi_Y(y) = \{b_\lambda \mid \lambda < \omega_\alpha\}$ where $m_v(\{b_\lambda \mid \lambda < \omega_\alpha\}) = \aleph_\alpha$ for every $v \in \psi(Y)$; at the same time $\psi(X) \subseteqq \psi(Y)$. We define the sequence $\{\beta_\lambda \mid \lambda < \omega_\alpha\}$ of ordinal numbers less than ω_α in the following way: β_0 is the smallest ordinal number such that $a_0 = b_{\beta_0}$; suppose, we have defined β_λ for every $\lambda < \lambda_0$ $(\lambda_0 < \omega_\alpha)$. Then we define β_{λ_0} as the smallest ordinal number with these properties: $\beta_{\lambda_0} > \beta_\lambda$ for every $\lambda < \lambda_0$, $\beta_{\lambda_0} < \omega_\alpha$, $b_{\beta_{\lambda_0}} = a_{\lambda_0}$. β_{λ_0} is defined for every $\lambda_0 < \omega_\alpha$ because $\{\beta_\lambda \mid \lambda < \lambda_0 (\lambda_0 < \omega_\alpha)\}$ is a sequence of type λ_0 $(< \omega_\alpha)$ of ordinal numbers less than ω_α and hence it is not confinal with ω_α (ω_α is a regular ordinal number). Thus, there exists an ordinal number $\gamma < \omega_\alpha$ which is greater than β_λ for every $\lambda < \lambda_0$. From this it follows that there exist ordinal numbers $\beta < \omega_\alpha$ greater than β_λ for every $\lambda < \lambda_0$ and such that $b_\beta = a_{\lambda_0}$ for, otherwise $m_{a_{\lambda_0}}(\{b_\lambda \mid \lambda < \omega_\alpha)\} \leqq$ card $\gamma < \aleph_\alpha$ which is a contradiction. This implies that $\{\beta_\lambda \mid \lambda < \omega_\alpha\}$ is a strictly ascending sequence of type ω_α of ordinal numbers less than ω_α such that $a_\lambda = b_{\beta_\lambda}$ for every $\lambda < \omega_\alpha$, i.e. $\{a_\lambda \mid \lambda < \omega_\alpha\} = \varphi(x) \leqq \varphi(y) = \{b_\lambda \mid \lambda < \omega_\alpha\}$. Suppose, on the contrary, that $\varphi(x) = \{a_\lambda \mid \lambda < \omega_\alpha\} \leqq \{b_\lambda \mid \lambda < \omega_\alpha\} = \varphi(y)$. Let $x \in X$ $(\in \bar{G})$, $y \in Y (\in \bar{G})$ and let $u \in \psi(X)$. Then there exists an ordinal number $\lambda_0 < \omega_\alpha$ such that $u = a_{\lambda_0}$ and an ordinal number $\beta_{\lambda_0} < \omega_\alpha$ such that $b_{\beta_{\lambda_0}} = a_{\lambda_0} = u$. This implies $u \in \psi(Y)$ and hence $\psi(X) \subseteqq \psi(Y)$. As ψ is an isomorphism we have $X \leqq Y$ in $\bar{G}$ and therefore $x \leqq y$ in G. Thus φ is an isomorphism and the theorem is proved.

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Резюме

ОБ УНИВЕРСАЛЬНЫХ КВАЗИУПОРЯДОЧЕННЫХ МНОЖЕСТВАХ

ВИТЕЗСЛАВ НОВАК (Vítězslav Novák), Брно

Пусть $F(\alpha, m)$ (где α — ординальное и m — кардинальное число) — тип множества всех последовательностей типа α, образованных из элементов множества мощности m, квазиупорядоченного отношением $\leqq$, определенным следующим образом: $\{a_\lambda \mid \lambda < \alpha\} \leqq \{b_\lambda \mid \lambda < \alpha\}$ тогда и только тогда, когда последовательность $\{a_\lambda \mid \lambda < \alpha\}$ является подпоследовательностью последовательности $\{b_\lambda \mid \lambda < \alpha\}$. В статье доказывается: Для всякого квазиупорядоченного множества мощности $\leqq\aleph_\alpha$ имеется в множестве типа $F(\omega_\alpha . 2, \aleph_\alpha)$ изоморфное подмножество (Теорема 2.). Если мощность $\aleph_\alpha$ регулярна, то имеет место более сильная теорема: Для всякого квазиупорядоченного множества мощности $\leqq\aleph_\alpha$, где $\aleph_\alpha$ — регулярное кардинальное число, имеется в множестве типа $F(\omega_\alpha, \aleph_\alpha)$ изоморфное подмножество (Теорема 3.).