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REPRESENTATION OF FINITE-LENGTH MODULAR LATTICES

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A canonic representation of modular lattices of finite length is given, in terms of simple nondistributive and finite distributive lattices. A fundamental characteristic, the defect of lattice, is defined and some of its properties exhibited.

There is a fundamental theorem stating that every distributive lattice is the subdirect product of replicas of **2**, the 2-element lattice [1, ch. IX, th. 6]. This completely characterises the structure of distributive lattices, at least if we are only concerned with the algebraic (finitary) properties of the lattice operations. (On the other hand, if infinite meets and joins are studied, there remains an infinite variety of further properties.)

Naturally, the next step would be a representation theorem for modular lattices, and a reasonable first step in this direction is representation of finite-length modular lattices. For complemented finite length modular lattices, we already have an elegant representation: a direct product of a finite Boolean algebra and a finite system of projective geometries [1, ch. VIII, th. 6]. It may be then said that the results of the present paper notice the consequences of omitting the complementation hypothesis; however, interest centers more on unicity of representation (for the direct representation of complemented lattices, this is trivial).

The main results, theorems 4 and 5, may be stated as follows (all for lattices of finite length):

1. Every modular lattice L is the subdirect product of a distributive D and a modular M;

2. M may be taken as the subdirect product of a finite system of simple nondistributive lattices M_j; with this condition, the decomposition of L – and hence also D and M – is determined uniquely by L;

3. the subdirect representation of M by the M_j's may be taken in a certain economic fashion (determined by a relation between certain lattice-theoretic characteristics

of M and M_j); with this condition, the representation of M is determined uniquely by either of M or L.

For assumed results and most of notation and terminology, the reader is referred to [1]. If L is a lattice, then the set of all congruence relations of L (ordered naturally, [1], ch. II, §§ 5, 6) forms a distributive lattice which will be denoted by Θ_L or $\Theta(L)$, and called the *congruence lattice* of L (an unsatisfactory translation of *Kongruenzverband*). If L is a lattice and $\theta \in \Theta_L$, then the lattice of equivalence classes of L modulo θ will be denoted by L/θ. Homomorphism will always mean a lattice-homomorphism. The length of a lattice L is the maximum cardinality of subchains if this is finite, and ∞ if not; it will be denoted by l_L or $l(L)$, the length of Θ_L by λ_L or $\lambda(L)$, etc. For conciseness, *m.l.f.l.* will mean 'modular lattice of finite length'.

Between lattices, $L_1 \approx L_2$ means that L_1 is isomorphic to L_2; and

$$L = \mathbf{P}L_a \quad \text{or} \quad L \leqq \mathbf{P}L_a$$

(and also $L = L_1 \times \ldots \times L_n$ or $L \leqq L_1 \times \ldots \times L_n$) will be used to mean that L is the direct (cardinal in [1]) product, or a subdirect product, respectively, of the L_a's. This notation is rather unfortunate in one respect: a subdirect decomposition is not determined by the system $\{L_a\}$, but by a system of "decomposing" homomorphisms $L \to L_a$ — or equivalently, by a system of "decomposing" congruence relations $\{\theta_a\}$ on L, whereupon $L_a \approx L/\theta_a$, $\bigwedge \theta_a = O$. (An instance of the difficulties this may lead to appears in [3], th. 12.) By a system $\{L_a\}$, say, we mean a mapping of the indices a into the set with elements L_a; in a related sense, e.g. "there are n distributive lattices in the system $\{L_a\}$" will mean that there are n distinct indices a such that L_a is a distributive lattice.

1. FIRST RESULTS

In this section some relations between the lengths l_L and λ_L are considered. As working apparatus, the following two basic theorems will be used, often without further reference.

The congruence relations on a m.l.f.l. L are in 1 – 1 correspondence with the sets of classes of projective prime quotients which they annul. Hence Θ_L is a finite Boolean algebra [1, ch. V, th. 10]. In particular, an atom of Θ_L corresponds to a single class of projective prime quotients in L, and there are precisely $\lambda_L - 1$ of these atoms.

In a modular lattice, any two finite chains between the same end-points possess refinements such that each quotient in either refinement is projective to some quotient in the other (cf. [1], ch. V, corollary to th. 5). In particular, in a m.l.f.l., each prime quotient in a given maximal chain is projective to some prime quotient in any other maximal chain. For completeness we also recall that in modular lattices all maximal chains have the same length [1, ch. V, th. 3].

There follows immediately the

Lemma 1. *Let C be a maximal chain in a m. l. f. l. L. Then there are* $\lambda_L - 1$ *not mutually projective prime quotients in C, and any prime quotient in L is projective to one of these.*

Lemma 2. *For modular lattices L,* $\lambda_L \leqq l_L$.
(Proof. It suffices to consider only the case of l_L finite, whereupon the result follows from lemma 1.)

Lemma 3. *If L is distributive, then* $\lambda_L = l_L$.

Proof. In view of lemma 2 it suffices to show that, if there is a chain $x_1 < x_2 < \ldots < x_m$ in L, then there exist $\theta_1 > \theta_2 > \ldots > \theta_m$ in Θ_L. Now, either $L = \mathbf{1}$ and the assertion is trivially true; or L is subdirectly decomposable into replicas of **2** [1, ch. IX, th. 6], and hence there exist homomorphisms $h_i : L \rightarrow \mathbf{2}$ with

$$O = h_i(x_i) < h_i(x_{i+1}) = I \quad \text{for} \quad 1 \leqq i < m\,,$$

and $h_m(x_m) = O$. With each h_i associate a $\mu_i \in \Theta_L$ in the usual manner, defining $x \equiv y\,(\mu_i)$ iff $h_i(x) = h_i(y)$; and set $\Theta_i = \bigwedge_{j<i} \mu_j$, $\theta_1 = I$. Then these θ_i are as asserted, since evidently $\theta_i \geqq \theta_{i+1}$ and since it is readily verified that $x_j \equiv x_{j+1}(\theta_i)$ iff $i \leqq j$ (for $1 \leqq i \leqq m$, $1 \leqq j < m$).

Lemma 4. *If L is a m. l. f. l. and* $\lambda_L = l_L$, *then L is distributive.*

Proof. Assume L is not distributive; then L must have the 4-element modular nondistributive lattice of fig. 1 as sublattice (cf. [1], ch. IX, th. 2 and ch. V, th. 2). Construct a maximal refinement C of $\{o, a\}$ with the same end-points. Now use [1, ch. V, th. 6] to obtain mappings

$$x \rightarrow x \vee b\,, \quad x \rightarrow x \wedge c\,, \quad x \rightarrow x \vee a$$

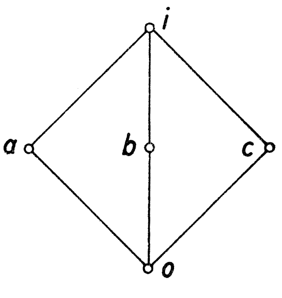

Fig. 1.

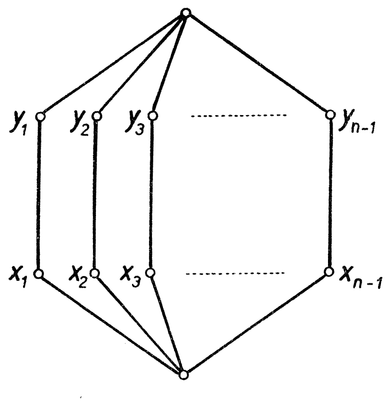

Fig. 2.

which map C into, in turn, $[b, i]$, $[o, c]$, $[a, i]$; the result is a maximal chain C' between end points a, i, and prime quotients of C are mapped into projective prime quotients of C'. Finally take a maximal chain in L prolonging $C \cup C'$. The resulting chain has at least one pair of distinct projective prime quotients. From lemma 1 we then conclude that $\lambda_L - 1 < l_L - 1$, i.e. $\lambda_L < l_L$ as asserted.

Some examples may be illustrative at this stage.

Examples. The modularity assumption is rather essential in lemma 2: if $n > m \geqq 4$ then there is a (nonmodular) lattice L with prescribed $l_L = m$, $\lambda_L = n$. Thus for $m = 4$ such a lattice is easily described by its graph, fig. 2; its congruence lattice is a Boolean algebra whose atoms α_j annul only $[x_j, y_j]$; there are $n - 1$ atoms, and hence $\lambda_L = n$.

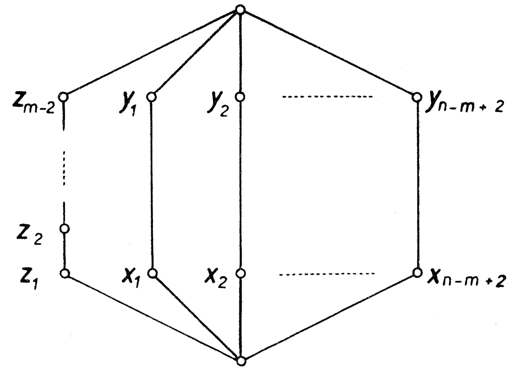

Fig. 3.

For $m \geqq 4$ we may take, analogously, the latitce of fig. 3.(There are obvious modifications for m or n infinite.)

If $2 \leqq n \leqq m$ then there is a modular lattice L with prescribed $l_L = m$, $\lambda_L = n$ (cf. fig. 4).

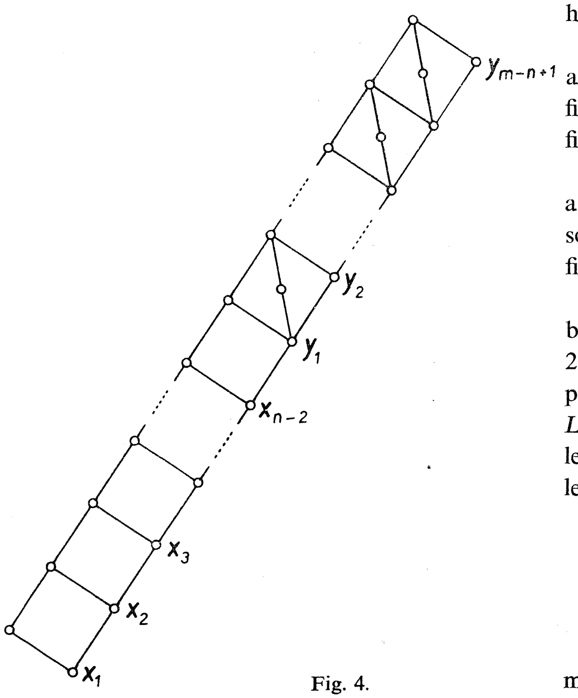

Fig. 4.

From these examples it may be concluded that the estimate $2 \leqq \lambda_L \leqq l_L$ cannot be improved for modular lattices $L \neq \mathbf{1}$, and also that infinite lengths are not allowed in lemma 4.

2. THE DEFECT OF A M. L. F. L.

The results of lemmas 2 to 4 may be conveniently formulated using the

Definition 1. If L is a m. l. f. l., then the defect δ_L (or $\delta(L)$, etc.) of L is defined by

$$\delta_L = l_L - \lambda_L . \tag{2}$$

We then have $0 \leqq \delta_L \leqq l_L - 2$ (exept for the singular case $L = \mathbf{1}$, where $l = \lambda = 1$,

$\delta = 0$); and $\delta_L = 0$ if and only if L is distributive. Thus the defect of a lattice may be thought of as a measure of its departure from distributivity. The "most defective" lattices have $\delta_L = l_L - 2$, i.e. $\lambda_L = 2$; these are precisely the simple m. l. f. l. This section is devoted to the study of elementary properties of δ.

Theorem 1. *Let L_1 be a sublattice of a m. l. f. l. L_2; then $\delta(L_1) \leqq \delta(L_2)$.*

Proof. Denote concepts pertaining to L_1 or L_2 by the corresponding indices. In L_1 take a maximal chain C_1, and in each class of projective prime quotients select one belonging to C_1 (cf. lemma 1); thus there are precisely $\lambda_1 - 1$ distinguished prime quotients in C_1. Now in L_2, prolong C_1 to a maximal chain C_2. In C_2, at least $(l_1 - 1) - (\lambda_1 - 1) = \delta_1$ of its prime quotients are contained within quotients of C_1 distinct from all the distinguished ones; and each of these prime quotients of C_2 is carried by the projectivities obtaining in L_1 into a prime quotient of C_2 contained within some distinguished quotient. Thus there remain, in C_2, $(l_2 - 1) - \delta_1$ prime quotients with the property that any other prime quotient in L_2 is projective to one of these. Hence

$$\lambda_2 - 1 \leqq (l_2 - 1) - \delta_1$$

or $\delta_1 \leqq (l_2 - 1) - (\lambda_2 - 1) \leqq \delta_2$ as it was required to prove.

We have shown that for sublattices L of a fixed m. l. f. l. L_0, the characteristic δ_L is nondecreasing as a function of L. Obviously l_L also has this property; also obviously, λ_L is neither nonincreasing nor nondecreasing. However, for sublattices L with $l_L = l_{L_0}$, λ_L is nonincreasing (this follows from theorem 1 and (2), or from lemma 1 directly); and we have the

Corollary. *If a m. l. f. l. L has a simple sublattice of the same length, then L itself is simple.*

Theorem 1 may be interpreted as treating changes of decrement when the lattice is subjected to a $1-1$ homomorphism (into). Naturally, one then inquires about changes of δ when L is homomorphically mapped onto.

Theorem 2. *If L is a m. l. f. l. then $\delta(L/\theta)$ is a nonpositive valuation on Θ_L.*

For the proof it will be useful to recall or introduce, more or less provisionally, some further notions. Let $d[.]$ be the dimension function on Θ_L [1, ch. I, § 9].

Given a m. l. f. l. L and a congruence relation θ on L, define $\delta(\theta)$ as follows. Take a maximal chain C in L; then let $\delta(\theta)$ be the number of prime quotients of C which θ annuls. (It may be shown directly that $\delta(\theta)$ is independent of the choice of C; this also follows from (5).)

For a m. l. f. l. L, its congruence lattice Θ_L is a Boolean algebra [1, ch. V, th 10]. By our lemma 2, Θ_L has a finite length λ_L; hence Θ_L itself is finite and has precisely $\lambda_L - 1$ atoms α_j and $\lambda_L - 1$ dual atoms β_j.

Next we shall need two lemmas.

Lemma 5. *Given, a m. l. f. l. L. Then, on* Θ_L,
(i) $\delta(.)$ *is a positive valuation with* $\delta(O) = 0$, $\delta(I) = l_L - 1$;
(ii) $\delta(.) - d[.]$ *is a nonnegative valuation with terminal values* 0 *and* δ_L.

Proof. We know that $d[.]$ is a positive valuation with terminal values 0 and $\lambda_L - 1$. From the definition of $\delta(.)$ it is immediate that it is a valuation, obviously nonnegative with terminal values 0, $l_L - 1$. Hence $\delta(.) - d[.]$ is a valuation with $\delta(O) - d[O] = 0$. For every atom α of Θ_L we have $d[\alpha] = 1 \leqq \delta(\alpha)$, and hence

$$\delta(\theta) - d[\theta] = \textstyle\sum_{\alpha \leqq \theta}(\delta(\alpha) - d[\alpha]) \geqq 0 \,.$$

Since $d[.]$ is positive, so is $\delta(.)$. Finally,

$$\delta(I) - d[I] = (l_L - 1) - (\lambda_L - 1) = \delta_L .$$

This completes the proof.

Lemma 6. *If* θ *is a congruence relation on a m. l. f. l. L, then*

$$\delta(L/\theta) = \delta_L - (\delta(\theta) - d[\theta]) \,. \tag{3}$$

Proof. From lemma 5 it follows that, for any m. l. f. l. L,

$$\delta_L = \delta(I) - d[I] = \textstyle\sum_\alpha(\delta(\alpha) - d(\alpha]) = \sum_\alpha(\delta(\alpha) - 1) \,,$$

summing over all atoms α of Θ_L. We may then apply this to the lattice L/θ,

$$\delta(L/\theta) = \textstyle\sum_\gamma(\delta(\gamma) - 1)$$

summing over all atoms γ in $\Theta(L/\theta)$ ($\delta(\gamma)$ is of course the corresponding function on $\Theta(L/\theta)$, not on Θ_L). Now, these atoms γ correspond, via a $1-1$ homomorphism, to elements $\theta \vee \alpha$ of Θ_L with α an atom, $\theta \wedge \alpha = O$; the next step is to prove that $\delta(\gamma) = \delta(\alpha)$. Recall the definition of $\delta(.)$, take a maximal chain C in L, and the corresponding maximal chain C' in L/θ; consider any prime quotient $[x, y]$ in C. If θ annuls $[x, y]$, then by assumption $\theta \wedge \alpha = O$ it contributes 0 towards $\delta(\alpha)$ and does not occur in C'. If $x \not\equiv y\ (\theta)$, then $[x, y]$ is in C', and is annuled by γ iff it is annuled by α in C; thus its contributions towards $\delta(\gamma)$ and $\delta(\alpha)$ coincide.

Observing there are precisely $\lambda_L - 1 - d[\theta]$ such α's, one may continue,

$$\delta(L/\theta) = \sum_{\alpha \wedge \theta = O} \delta(\alpha) - \sum_{\alpha \wedge \theta = O} 1 = (l_L - 1 - \sum_{\alpha \leqq \theta} \delta(\alpha)) - (\lambda_L - 1 - d[\theta]) \,,$$

having used

$$l_L - 1 = \textstyle\sum \delta(\alpha) = \displaystyle\sum_{\alpha \wedge \theta = O} \delta(\alpha) + \sum_{\alpha \leqq \theta} \delta(\alpha) \,;$$

and thus

$$\delta(L/\theta) = (l_L - 1 - \delta(\theta)) - (\lambda_L - 1 - d[\theta]) = \delta_L - (\delta(\theta) - d[\theta]) \,.$$

This proves (3).

As for the proof of theorem 2, it suffices to apply Lemma 5 to (3).

Corollary 1. *We have the following formulae (all for m. l. f. l.):*

(4) $$\delta(L/\theta) \leqq \delta_L ,$$

(5) $$\lambda(L/\theta) = \lambda_L - d[\theta] , \quad l(L/\theta) = l_L - \delta(\theta) ,$$

(6) $$\delta(L_1 \times \ldots \times L_n) = \textstyle\sum_n^1 \delta(L_j) .$$

Proof. (4) follows directly from theorem 2: $O \leqq \theta$ implies $\delta_L = \delta(L/O) \geqq \delta(L/\theta)$. Since Θ_L is a Boolean algebra, and $\Theta(L/\theta)$ is isomorphic to the dual ideal of all $\eta \geqq \theta$ in Θ_L, we have $\lambda(L/\theta) = \lambda_L - d[\theta]$. Hence and from (2), (3) there follows $l(L/\theta) = l_L - \delta(\theta)$.

For (6), consider $L = L_1 \times L_2$; there exist $\theta_k \in \Theta_L$ such that

$$L_k \approx L/\theta_k , \quad \theta_1 \wedge \theta_2 = O , \quad \theta_1 \vee \theta_2 = I .$$

From theorem 2 we then have that

$$\delta(L/(\theta_1 \vee \theta_2)) + \delta(L/(\theta_1 \wedge \theta_2)) = \delta(L/\theta_1) + \delta(L/\theta_2) ,$$

i.e.

$$0 + \delta_L = \delta(L_1) + \delta(L_2) .$$

Since direct products are associative, we immediately obtain (6).

Corollary 2. *If $L = \mathbf{P} L_a$ is a m. l. f. l., then there are at most δ_L nondistributive factors in the system.*

Proof. Let the nondistributive factors be $L(a_i)$, $1 \leqq i \leqq n$, and set $M = \mathbf{P}_{1 \leqq i \leqq n} L(a_i)$. Then, obviously, M is isomorphic to a sublattice of L, so that according to theorem 1 and (6),

$$\delta_L \geqq \delta_M = \textstyle\sum_{i=1}^n \delta(L(a_i)) .$$

By assumption, each $L(a_i)$ is nondistributive, so that $\delta(L(a_i)) \geqq 1$, and the displayed inequality then yields $\delta_L \geqq n$ as asserted.

Corollary 3. *If L is a subdirect product of L_j ($1 \leqq j \leqq n$, all m. l. f. l.), then*

(7) $$\max \delta(L_j) \leqq \delta_L \leqq \textstyle\sum_1^n \delta(L_j) .$$

Proof. The first inequality follows from (4). The second from theorem 1, $\delta_L \leqq \delta(L_1 \times \ldots \times L_n)$, and (6).

3. PLAIN M. L. F. L.

For conciseness of formulation in the representation theorems, we will introduce two new notions, that of a *plain lattice* and of an *exact decomposition*. (In a sense, these generalise, respectively, simple lattices and direct decompositions.) The aim of this and the following sections is the study of these notions.

Definition 2. A lattice L is called plain if every homomorphism of L into a distributive lattice is constant.

Equivalent formulations: every homomorphism of L into a given distributive lattice not **1** (e.g., into **2**) is constant; or, I is the only congruence relation θ on L for which L/θ is distributive. Obviously the only plain distributive lattice is **1**. Some elementary properties of plain lattices are formulated in the following lemma; the proofs are trivial.

Lemma 7. *Every simple lattice not* **2** *is plain. The homomorphic image of a plain lattice is plain. If L is a lattice of finite length and has a plain sublattice of the same length , then L itself is plain.*

The last statement is a counterpart to the corollary to theorem 1. Now let us return to our main interest, m. l. f. l.

Lemma 8. *A m. l. f. l. L(not* **1***) is plain if and only if it is the subdirect product of nondistributive simple m. l. f. l. In the positive case the set of factors is determined uniquely.*

Part of the proof may be conveniently separated out:

Lemma 9. *Every m. l. f. l. L is the subdirect product of* $\lambda_L - 1$ *simple lattices; the set of factors is determined uniquely.*

Definition 3. The decomposition of a m. l. f. l. L to be constructed will be termed the canonic decomposition of L.

Proof of lemma 9. Every lattice L is the subdirect product of subdirectly irreducible lattices [1, ch. VI, th. 10]. Interpreting this in the congruence lattice, L/θ is subdirectly irreducible iff

$$(8) \qquad \bigwedge\theta_a = \theta \quad \text{implies some} \quad \theta_a = \theta$$

[1, l. c.]. Now, if L is a m. l. f. l., then Θ_L is a finite Boolean algebra and (8) holds iff θ is a dual atom, i.e. iff L/θ is simple. This proves the first part of the lemma.

Now let a m. l. f. l. L be the subdirect product of simple L_a; let $\{\theta_a\}$ be the congruence relations defining the product, $L_a = L/\theta_a$ and $\bigwedge\theta_a = O$. Since L/θ_a is simple, θ_a is a dual atom of Θ_L; since $\bigwedge\theta_a = O$, the set of all θ_a must consist of all dual atoms of Θ_L. This completes lemma 9.

Proof of lemma 8. First let L be a plain m. l. f. l. not **1**. There is a subdirect decomposition into simple factors (lemma 9); since L is plain, so are the factors (lemma 8). Omit trivial factors **1**, if necessary. Then each factor is nondistributive (lemma 8 again); thus the factors are nondistributive simple m. l. f. l.

Now, conversely, let a m. l. f. l. $L \leqq \mathbf{P}L_a$ with the L_a simple nondistributive. To prove L is plain, consider a homomorphism $k : L \to 2$. Assume k is nonconstant. Then the congruence relation θ_k defined on L by $x \equiv y\,(\theta_k)$ iff $kx = ky$ has $\theta_k < I$, so that there is a dual atom $\beta \geqq \theta_k$. By unicity (lemma 9), $L_a = L/\beta$ for some a. On L_a, define $gt = kx$ if t is the a-th cooordinate of x (i.e. if $x \in t$, t an equivalence class of L modulo β). From $\beta \geqq \theta_k$ it follows that g is defined unequivocally, and obviously it is a nonconstant homomorphism $g : L_a \to \mathbf{2}$. But L_a is simple non-distributive, i.e. plain; this contradiction completes the proof.

It may be said that lemma 8 characterises plain m. l. f. l. constructively; however it also makes possible a simple descriptive characterisation: The plain m. l. f. l. form the least set (of lattices) which contains all nondistributive simple m. l. f. l. and is closed under the operation of taking finite subdirect products (or of taking subdirect products of finite length). This follows from the

Corollary. *For a m. l. f. l. let* $L \leqq \mathbf{P}L_a$ *with the* L_a *plain ; then* L *itself is plain.*

Proof. According to lemma 8, $L_a \leqq \mathbf{P}_b L_{ab}$ with the L_{ab} simple nondistributive; hence $L \leqq \mathbf{P}_{a,b} L_{ab}$; again from lemma 8, L is plain.

Plain m. l. f. l. may also be characterised in terms of their congruence lattices:

Lemma 10. *Let* L *be a m. l. f. l.;* L *is plain if and only if* $\delta(L/\theta)$ *is a negative valuation on* Θ_L.

Proof. In any case, $\delta(L/\theta)$ is a nonpositive valuation (theorem 2). It is a negative valuation iff $\delta(\alpha) - d[\alpha] > 0$ on every atom α of Θ_L (lemma 6), or, symmetrically, if

$$\delta(\beta) - d[\beta] < \delta_L \quad (= \delta(I) - d[I])$$

for every dual atom β.

Now, if L is plain, then L/θ is nondistributive for every $\theta < I$, and thus

$$0 < \delta(L/\theta) = \delta_L - (\delta(\theta) - d[\theta]) , \quad \text{i.e.} \quad \delta(\theta) - d[\theta] < \delta_L$$

for $\theta < I$, and in particular for all dual atoms β.

Conversely if $\delta(L/\theta)$ is a negative valuation, then in a decomposition $L \leqq \mathbf{P}L/\theta_a$ into simple factors every θ_a is a dual atom; hence $\delta(\theta_a) - d[\theta_a] < \delta_L$ by assumption, and therefore $\delta(L/\theta_a) > 0$ (lemma 6), i.e. L/θ_a is nondistributive, L is plain (lemma 8). This proves lemma 10 in entirety.

4. EXACT DECOMPOSITIONS

Definition 4. A subdirect decomposition $L \leqq \mathbf{P}L/\theta_a$ of a m. l. f. l. will be termed exact if

$$\theta'_a = \bigwedge_{b \neq a} \theta_b \quad \text{for all } a .$$

(θ'_a is, of course, the complement of θ_a in Θ_L; also see remarks at the end of the paper.)

Lemma 11. *Each of the following subdirect decompositions of a m. l. f. l.* L *is exact: any direct decomposition, the canonic decomposition into simple factors, the subdirect decomposition corresponding to a neutral element.*

Proof. Let $L = \mathbf{P}L/\theta_a$ be a direct decomposition; then also

$$L \approx L/\theta_a \times \mathbf{P}_{b \neq a} L/\theta_b = L/\theta_a \times L/(\bigwedge_{b \neq a} \theta_b) = L_1 \times L_2$$

is a direct decomposition. Set $\theta = \theta_a$, $\tau = \bigwedge_{b \neq a} \theta_b$; we are to prove that $\theta \vee \tau = I$. Take any $x = [x_1, x_2]$, $y = [y_1, y_2] \in L$. Since the decomposition into $L_1 \times L_2$ is direct, the element $z = [x_1, y_2] \in L$, and thus

$$x \equiv z(\theta), \quad z \equiv y(\tau), \quad \text{i.e.} \quad x \equiv y(\theta\tau).$$

This proves $\theta\tau = I$ in Θ_L; thus

$$\theta \vee \tau = \theta \cup \tau \cup \theta\tau \cup \tau\theta \cup \theta\tau\theta \cup \ldots \geqq \theta\tau = I$$

and hence $\theta \vee \tau = I$, as was to be proved.

As for the canonic decomposition into simple factors, the decomposing congruence relations are all the dual atoms of Θ_L; the exactness condition is then immediate, since Θ_L is a finite Boolean algebra.

(The final part of lemma 11 is only included for completeness and is not needed in the sequel.) Consider the subdirect decomposition $L \leqq L_1 \times L_2$ corresponding to a neutral element $c \in L$ ([3, § 2], [1, ch. II, § 10]; the decomposing congruence relations θ_1, θ_2 are defined by $x \equiv y(\theta_1)$ iff $x \wedge e = y \wedge e$, $x \equiv y(\theta_2)$ iff $x \vee e = y \vee e$). Then $e = [I, O]$. We shall show that, more generally, if $[I, O]$ or $[O, I] \in L \leqq L_1 \times L_2$, then $\theta_1\theta_2\theta_1 = I = \theta_2\theta_1\theta_2$ and thus $\theta_1 \vee \theta_2 = I$, the decomposition is exact. Assume $e = [I, O] \in L$.

Take any $x = [x_1, x_2]$, $y = [y_1, y_2] \in L$. The both $[x_1, O] = x \wedge e$, $[y_1, O] = y \wedge e$ are in L and hence

$$x \equiv x \wedge e(\theta_1), \quad x \wedge e \equiv y \wedge e(\theta_2), \quad y \wedge e \equiv y(\theta_1)$$

so that $x \equiv y(\theta_1\theta_2\theta_1)$. Thus we have shown that $\theta_1\theta_2\theta_1 = I$. (Similarly $\theta_2\theta_1\theta_2 = I$; for the proof use $x \vee e$ and $y \vee e$.) The final step is as in the first part of the proof:

$$\theta_1 \vee \theta_2 = \theta_1 \cup \theta_2 \cup \theta_1\theta_2 \cup \theta_2\theta_1 \cup \theta_1\theta_2\theta_1 \cup \ldots \geqq \theta_1\theta_2\theta_1 = I.$$

This completes the proof of Lemma 11.

Since Θ_L is finite Boolean we have immediately the

Lemma 12. *Exact decompositions are associative in the following sense: if $L = \mathbf{P}_A L/\theta_a$ is exact, L m. l. f. l., and if $A = \cup_b A(b)$ is a decomposition into disjoint summands, then $L \leqq \mathbf{P}_b L/(\bigwedge_{a \in A(b)} \theta_a)$ is exact.*

Lemma 13. *If $L \leqq \mathbf{P}L_a$ is an exact decomposition of a m. l. f. l. L, then the number of nontrivial factors in the system is finite,*

$$\delta_L = \sum \delta(L_a), \tag{10}$$

$$\Theta_L \approx \mathbf{P}\,\Theta(L_a), \tag{11}$$

$$l_L = 1 + \sum(l(L_a) - 1), \quad \lambda_L = 1 + \sum(\lambda(L_a) - 1). \tag{12}$$

Proof. Let $L_a = L/\theta_a$. The congruence lattice Θ_L is finite, so that the number of distinct θ_a is also finite. Now assume some θ_a is repeated, i.e. that $\theta_a = \theta_b$ for some $a \neq b$. Then $\theta'_a = \bigwedge_{c \neq a} \theta_c \leqq \theta_b = \theta_a$, $\theta'_a \leqq \theta_a$, $\theta_a = I$. Thus the only repeated θ_a is I and $L/\theta_i = \mathbf{1}$ is trivial.

As for (10) we may then repeat the proof (but not the statement!) of (6), corollary 1 to theorem 2, for a finite system of factors, e.g. for all non-trivial factors, and the trivial factors contribute $\delta(L_a) = \delta(\mathbf{1}) = 0$.

Formula (11) is simple for two factors: we have $L \leqq L_1 \times L_2$ with $L_1 = L/\theta$, $L_2 = L/\theta'$ (complement in Θ_L). Every element of a Boolean algebra is central; hence $\Theta_L \approx \bar{\theta} \times \bar{\theta}'$ and dually $\Theta_L = \underline{\theta} \times \underline{\theta}' = \Theta(L_1) \times \Theta(L_2)$ (cf. [1, ch. II, § 8], [3, th. 3d]; $\bar{\theta}$ denotes the set of all $\alpha \leqq \theta$, and dually $\underline{\theta}$ the set of all $\alpha \geqq \theta$). The extension to several factors is trivial.

From (11) we conclude immediately the second formula in (12); the first follows hence and from (10). This completes the proof of lemma 13.

Theorem 3. *There is a unique exact decomposition of a m. l. f. l. into simple factors, namely the canonic decomposition.*

Proof. Let θ_a be the congruence relations associated with an exact decomposition $L = \mathbf{P}L_a$ into simple factors. Then each θ_a is a dual atom in Θ_L, and since $\bigwedge \theta_a = O$, the system $\{\theta_a\}$ contains all the dual atoms; easily from exactness, each dual atom occurs precisely once in $\{\theta_a\}$. Thus $L \leqq \mathbf{P}L/\theta_a$ is precisely the canonic decomposition, and $L \leqq \mathbf{P}L_a$ coincides with it up to factor isomorphism $L_a \approx L/\theta_a$.

Corollary 1. *Each exact decomposition of a complemented m. l. f. l. is direct.*

Proof. Let $L \leqq \mathbf{P}L_a$ be exact, L complemented m. l. f. l. Let $L_a \leqq \mathbf{P}_b L_{ab}$ be the canonic decomposition of L_a into simple factors L_{ab}, an exact decomposition according to lemma 11. It follows easily that

$$L \leqq \mathbf{P}_{a,b} L_{ab} \tag{13}$$

is also exact. Now, there also exists a direct decomposition $L$of into simple factors (Dilworth's theorem [1, ch. II, th. 8]). This is exact according to lemma 11, and coincides with (13) according to theorem 3. Hence (13) is direct, and thus so are all the decompositions from which (13) is composed. Thus, indeed, $L \leqq \mathbf{P}L_a$ is direct.

Corollary 2. *Let $L \leqq \mathbf{P}L_j$ be the canonic decomposition of a m. l. f. l. L into nontrivial simple factors. Then*

$$\sum l(L_j) = 2l_L - \delta_L - 2\,. \tag{14}$$

Proof. The canonic decomposition is exact, hence $\delta_L = \sum \delta(L_j)$; the factors are simple, so that $\delta(L_j) = l(L_j) - 2$, and there are $\lambda_L - 1$ terms; $\lambda_L = l_L - \delta_L$. All this implies our formula.

It is almost obvious that (11) in lemma 13 is a necessary and sufficient condition for exactness. Now notice condition (10); it is unaffected by zero terms, i.e. by the presence of distributive subdirect factors. On the other hand, for plain, m. l. f. l. the condition is quite stringent:

Lemma 14. *Let M be a plain m. l. f. l. Then a subdirect decomposition $M \leqq \mathbf{P}M_a$ is exact if and only if $\delta_M = \sum \delta(M_a)$.*

Proof. Necessity was proved in lemma 13. Let M be a plain m. l. f. l., $M \leqq \mathbf{P}M_a$. We may omit trivial factors $\mathbf{1}$; denote the resulting set of indices by A, its cardinality by n. Let θ_a be the congruence relations defining the decomposition, $M_a = M/\theta_a$, $\bigwedge \theta_a = O$. The proof seems to need a rather elaborate formal apparatus. Let α_j, $1 \leqq j \leqq \lambda_M - 1$, be all the atoms of Θ_M; thus

$$\text{either } \alpha_j \leqq \theta_a \text{ or } \alpha_j \wedge \theta_a = O\,.$$

Define (analogues of Kronecker symbols)

$$\delta_j^a = I \text{ and } d_j^a = 1 \text{ if } \alpha_j \leqq \theta_a\,,$$

$$\delta_j^a = O \text{ and } d_j^a = 0 \text{ if } \alpha_j \wedge \theta_a = O\,.$$

Thus we have $\theta_a = \bigvee_j (\alpha_j \wedge \delta_j^a)$. Also, to every j there is an a with $\delta_j^a = O$, since otherwise $O < \alpha_j \leqq \bigwedge \theta_a = O$. Hence we conclude

$$\sum_a d_j^a \leqq n - 1 \text{ for every } j\,. \tag{15}$$

Next, consider $\delta(M_a) = \delta(M/\theta_a)$ (cf. lemmas 5 and 6):

$$\delta(M_a) = \delta_M - (\delta(\theta_a) - d[\theta_a]) = \delta_M - \sum_j (\delta(\alpha_j \wedge \delta_j^a) - d[\alpha_j \wedge \delta_j^a])$$
$$= \delta_M - \sum_j (\delta(\alpha_j) - d[\alpha_j])\, d_j^a\,.$$

The assumption $\delta_M = \sum \delta(M_a)$ then implies

$$\delta_M = n\delta_M - \sum_{a,j} (\delta(\alpha_j) - d[\alpha_j])\, d_j^a = n\delta_M - \sum_j (\delta(\alpha_j) - d[\alpha_j])\, (\sum_a d_j^a)\,,$$

and from (15) and lemma 5 we conclude

$$(n-1)\,\delta_M = \sum_j (\delta(\alpha_j) - d[\alpha_j])\,(\sum_a d_j^a) \leqq (n-1) \sum_j (\delta(\alpha_j) - d[\alpha_j]) =$$
$$= (n-1)\,(\delta(I) - d[I]) = (n-1)\,\delta_M\,.$$

Since (from lemmas 6 and 10) all the coefficients $\delta(\alpha_j) - d[\alpha_j]$ are positive, it follows hence that all $\sum_a d_j^a = n - 1$. Thus to every j there is precisely one $a \in A$ with $\alpha_j \wedge \theta_a = O$; in other words, the relation $\alpha_j \wedge \theta_a = O$ defines a mapping $j \to a(j)$ of the j's into A.

Next we shall show that this is a mapping *onto* A. We have $\alpha'_j \geqq \theta_{a(j)}$, and thus $\bigwedge_j \theta_{a(j)} = O$. Now, suppress repetitions in $\{\theta_{a(j)}\}_j$, obtaining say $\{\theta_b\}_{b \in B}$. Obviously

$\bigwedge_b \theta_b = O$, the θ_b define a subdirect decomposition, and thus (corollary 3 to theorem 2)

$$\delta_M \leqq \sum_b \delta(M_b) \leqq \sum_a \delta(M_a) = \delta_M .$$

Since all $\delta(M_a) > 0$, we conclude $B = A$, and the mapping $j \to a(j)$ is indeed onto A.

Now return to the statement of our lemma. Since the decomposition is subdirect, we have $\theta_a \wedge \bigwedge_{b \neq a} \theta_b = O$, *i.e.* $\theta'_a \geqq \bigwedge_{b \neq a} \theta_b$. Now assume the decomposition is not exact; then in the last relation we must have $>$ for some a. Thus there is an atom α_j with

$$\theta'_a \geqq \alpha_j , \quad \alpha_j \wedge \bigwedge_{b \neq a} \theta_b = O .$$

But then $\theta_a \wedge \alpha_j = O$, i.e. $a = a(j)$; and therefore, for all $b \neq a$, not $\theta_b \wedge \alpha_j = O$, i.e. $\alpha_j \leqq \theta_b$. We conclude

$$O < \alpha_j \leqq \bigwedge_{b \neq a} \theta_b ,$$

a contradiction. This proves lemma 14.

5. REPRESENTATION THEOREMS

Theorem 4. *To a m. l. f. l. L there is a unique subdirect decomposition*

$$L \leqq D \times M$$

into a distributive D and a plain M. Furthermore, D is finite, the decomposition is exact and thus

$$(16) \qquad \delta_L = \delta_M , \quad l_L = l_D + l_M - 1 , \quad \lambda_L = \lambda_D + \lambda_M - 1 .$$

Proof. Let L be a m. l. f. l. For every homomorphism $h : L \to \mathbf{2}$ define a relation θ_h on L by

$$(17) \qquad x \equiv y\,(\theta_h) \quad \text{iff} \quad hx = hy .$$

Obviously θ_h is a congruence relation. Set $\Delta = \bigwedge_h \theta_h$; thus Δ is the least congruence relation θ on L for which L/θ is distributive.

Define $D = L/\Delta$, $M = L/\Delta'$ (Δ' is the complement of Δ in Θ_L). Obviously $L \leqq D \times M$, D is distributive; since D has finite length $l_D \leqq l_L$, it is a sublattice of the Boolean algebra $\mathbf{2}^{l_D - 1}$, and hence D is finite. From the estimates (7) (corollary 3 to theorem 2) we conclude $\delta_M \leqq \delta_L$. Exactness is immediate; (16) then follows from lemma 13.

Now consider any homomorphism $h : M \to \mathbf{2}$; let $p : L \to M = L/\Delta'$ be the natural homomorphism. Define a congruence relation θ on L by $x \equiv y\,(\theta)$ iff $hpx = hpy$. Obviously $\theta \geqq \Delta'$; and L/θ is either $\mathbf{1}$ or $\mathbf{2}$, i.e. distributive, and therefore $\theta \geqq \Delta$. Thus $\theta \geqq \Delta \wedge \Delta' = I$; hence hp is constant on L, hence h constant on M. In conclusion, any homomorphism $M \to \mathbf{2}$ is constant, i.e. M is plain.

It remains to prove unicity. Let $L \leqq D_1 \times M_1$ with D_1 distributive and M_1 plain; let $D_1 = L/\Lambda$, $M_1 = L/\Lambda^0$, $\Lambda \wedge \Lambda^0 = O$. Since M_1 is plain, no dual atom $\beta \geqq \Lambda^0$

is of the form θ_h (as in (17)); thus $\Lambda^0 \geqq \Delta'$. Since $D_1 = L/\Lambda$ is distributive, $\Lambda \geqq \Delta$. Then from $\Lambda \wedge \Lambda^0 = O$ there follows $\Lambda = \Delta$, $\Lambda^0 = \Delta'$; this completes the proof.

Corollary. *Let L, L_a be m. l. f. l., let $L \leqq D \times M$ and $L_a \leqq D_a \times M_a$ be decompositions as in theorem 4. If L is a subdirect (or exact, direct) product of the L_a, then both*

$$D \leqq \mathbf{P}D_a\,, \quad M \leqq \mathbf{P}M_a$$

are subdirect, exact, direct, respectively.

One aspect of these results may be summarized as follows. Given a m. l. f. l. L, several integer-valued characteristics of L have been introduced:

$$l_L, \lambda_L, \delta_L\,; \quad l_D, \lambda_D, \delta_D\,; \quad l_M, \lambda_M, \delta_M\,.$$

These are not independent ($\delta = l - \lambda$ in each group, (16), distributivity of D implying $\delta_D = 0$, inequalities such as $0 \leqq \delta \leqq l - 2$, (21), etc.); and we may distinguish subsets of "independent" characteristics, e.g. l_L, δ_L, l_M. Then for m. l. f. l. L, $\delta_L = 0$ characterises distributive lattices, $\delta_L = l_L - 2$ characterises simple lattices, $l_L = l_M$ characterises plain lattices (this last follows from theorem 4).

Theorem 5. *To a m. l. f. l. L there is a unique subdirect decomposition*

$$L \leqq D \times M_1 \times \ldots \times M_n$$

with D distributive, M_j simple modular nondistributive, $\delta_L = \sum \delta(M_j)$. Furthermore, D is finite, the decomposition is exact, and

$$\sum l(M_j) = 2(l_L - l_D) - \delta_L\,. \tag{18}$$

Proof. Existence follows from theorem 4, the canonic exact decomposition of M (lemmas 9, 11) and (16), (10).

Next, consider unicity. Let a subdirect decomposition of L be given as described; let $D = L/\Delta$, $M_j = L/\theta_j$. Set $M = L/(\bigwedge \theta_j)$. Then $M \leqq \mathbf{P}M_j$, so that M is plain (lemma 8); hence M is determined uniquely (theorem 4), $\delta_M = \delta_L = \sum \delta(M_j)$. But then $M \leqq \mathbf{P}M_j$ is exact (lemma 14), so that this decomposition is unique (theorem 3).

It remains to prove (18). From (14) (corollary 2 to theorem 3), we have $\sum l(M_j) = 2l_M - \delta_M - 2$. Now $\delta_M = \delta_L$, and from (16) $l_M - 1 = (l_L - 1) - (l_D - 1)$; hence $\sum l(M_j) = 2(l_M - 1) - \delta_M = 2(l_L - l_D) - \delta_L$, i.e. (18). This completes the proof.

Corollary 1. *With the notation of theorems 4 and 5,*

$$\tfrac{1}{2}(l_L - l_D) \leqq \delta_L \leqq l_L - l_D\,, \tag{19}$$

$$\delta_L \leqq \sum l(M_j) \leqq 3\delta_L\,; \tag{20}$$

$$l_M \leqq 1 + 2\delta_L\,. \tag{21}$$

Proof. Consider (18); all M_j are modular nondistributive, hence $l(M_j) \geqq 3$, and there are precisely $\lambda_M - 1$ of these. Hence

$$2(l_L - l_D) - \delta_L \geqq 3(\lambda_M - 1).$$

Now $\lambda_M - 1 = (\lambda_L - 1) - (\lambda_D - 1)$ since we have (12) and the decomposition $L \leqq D \times M$ is exact; also $\lambda_D = l_D$, $\lambda_L = \lambda_L - \delta_L$. Thus $\lambda_M - 1 = \lambda_L - \lambda_D = l_L - l_D - \delta_L$ and therefore $2(l_L - l_D) - \delta_L \geqq 3(l_L - l_D) - 3\delta_L$; this is the first inequality of (19).

Since $D = L/\Delta$, we have $\lambda_D \leqq \lambda_L$, i.e. $l_D \leqq \lambda_L$; and then $\delta_L = l_L - \lambda_L \leqq l_L - l_D$, the second inequality of (19). (It may be noticed that (19) generalizes the statement that $\delta_L = 0$ iff the m. l. f. l. L is distributive.)

If we use (19) to estimate $l_L - l_D$ in (18), (20) results.

From theorem 5 we have $\sum \delta(M_j) = \delta_L$ with precisely $\lambda_M - 1$ summands all $\geqq 1$. Thus $\lambda_M - 1 \leqq \delta_L$; with $\lambda_M = l_M - \delta_M = l_M - \delta_L$, this implies (21).

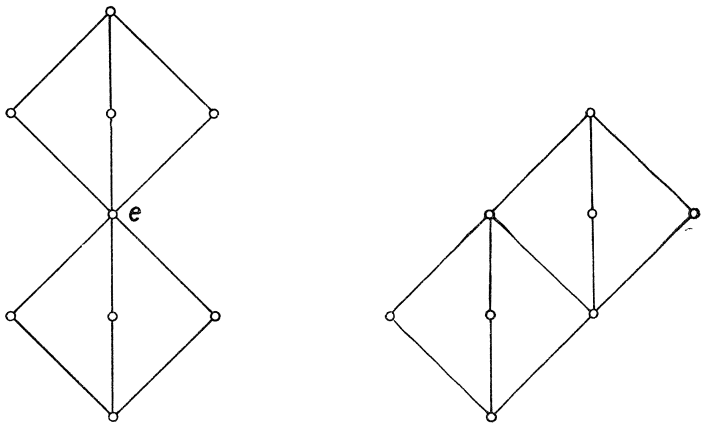

Fig. 5. Fig. 6.

Example. From (21) we have that for plain m. l. f. l. L, $l_L \leqq 1 + 2\delta_L$. However, this condition does not characterise plain lattices among m. l. f. l. Indeed, there is no relation at all between l_L and δ_L (or equivalently, between l_L and λ_L) which characterises plain lattices among m. l. f. l.

Let S_3 be the lattice of fig. 1, S_4 that of fig. 6. Obviously both are simple m. l. f. l.,

$$l(S_3) = 3, \quad l(S_4) = 4, \quad \text{hence} \quad \delta(S_3) = 1, \quad \delta(S_4) = 2.$$

Set $L_1 = S_3 \times S_3$, $L_2 = S_4 \times \mathbf{2}$. Since direct decompositions are exact,

$$l(L_1) = 5 = l(L_2), \quad \delta(L_1) = 2 = \delta(L_2)$$

(lemma 12, (10) and (12)). By lemma 8, L_1 is plain and L_2 is not.

Corollary 2. *If L is complemented, then the decompositions of theorems 4 and 5 are direct.*

This follows immediately from corollary 1 to theorem 3.

We may now attempt classification of m. l. f. l. according to their defects, and obtain some information about their structure. As an illustration, we will state the two following lemmas.

Lemma 15. *A m. l. f. l. L with defect* 1 *and length l is the subdirect product of a simple modular nondistributive lattice of length* 3 *and a distributive lattice of length* $l - 2$.

Proof. The number n of simple nondistributive factors in the canonic decomposition has $n \geqq 1$, since L is nondistributive ($\delta_L = 1$), and also $n \leqq \delta_M = \delta_L = 1$. Thus there is a unique simple nondistributive factor M; it has $\delta_M = 1$ and hence length 3. (Alternately, (21) may be used.) The length l_D of the distributive factor satisfies (16), i.e. $l = l_D + 3 - 1$.

Lemma 16. *A m. l. f. l. L with defect* 2 *and length l is the subdirect product of either*

(i) *two simple modular nondistributive lattices of length* 3 *and a distributive lattice of length* $l - 4$, *or*

(ii) *a simple modular nondistributive lattice of length* 4 *and a distributive lattice of length* $l - 3$.

The proof is similar to the preceding. If $\lambda_M - 1 = \delta_L$ we have case (i), if $\lambda_M - 1 < < \delta_L$ we have case (ii).

We then have the following:

Lemma 17. *A modular lattice of length* 3 *is either*

(i) *a simple modular (nondistributive) lattice of length* 3, *or*

(ii) **3**, *or*

(iii) **2** × **2**.

Lemma 18. *A modular lattice of length* 4 *is either*

(i) *a simple lattice of length* 4 (*iff* $\delta = 2$), *or*

(ii) *a subdirect product of a simple lattice of length* 3 *and* **2** (*iff* $\delta = 1$), *or*

(iii) *a distributive lattice of length* 4 (*iff* $\delta = 0$).

Lemma 19. *A modular lattice of length* 5 *is either*

(i) *a simple lattice of length* 5 ($\delta = 3$), *or*

(ii) *a subdirect product of two simple lattices of length* 3 ($\delta = 2$), *or*

(iii) *dtto, of a simple lattice of length* 4 *and* **2** ($\delta = 2$ *again*), *or*

(iv) *dtto, of a simple lattice of length* 4 *and* **3** ($\delta = 1$), *or*

(v) *dtto, of a simple lattice of length* 3 *and* $\mathbf{2} \times \mathbf{2}$ ($\delta = 1$), *or*

(vi) *a distributive lattice of length* 5 (*iff* $\delta = 0$).

Several interesting questions, raised by the preceding results, remain open.

1. Given a finite distributive lattice D and a plain m. l. f. l. M, determine all lattices which are subdirects products of D, M.

2. Given a finite system of simple nondistributive m. l. f. l. M_j, determine all (plain) lattices with canonic decomposition $M_1 \times \ldots \times M_n$.

3. Determine all simple modular lattices of given length.

4. How much of the preceding results remains true if the finiteness of length assumption is dropped?

5. Does the defect of a direct product of projective geometries have simple geometrical significance? (The defect of a projective geometry is length $- 2 =$ dimension -1).

Remarks. The referee, M. KOLIBIAR, has been good enough to bring to the author's attention the following references.

1. In [5, p. 100], Maeda defined *canonic* factorisations of general lattices L as those subdirect decompositions $L \leqq \mathbf{P}L/\theta_a$ which have $\theta_a^* = \bigwedge_{b \neq a} \theta_b$ for all a ($\theta_a \in \Theta_L$; θ_a^* denotes the pseudocomplement of θ_a). Since for m. l. f. l., Θ_L is a finite Boolean algebra, one has $\theta_a^* = \theta_a'$; thus the exact decompositions of definition 4 are merely a special case of canonic factorisations.

2. The existence part of theorem 3 (i.e. lemma 9) may also be obtained from Dilworth's theorem [4, th. 3.3]; of course, the proof of this latter also proceeds via [1, ch. VI, th. 10].

3. The second assertion in lemma 11 follows from [6], lemma 4.

4. It may also be noted that, for canonic factorisations, Maeda obtained a theorem [5, th. 2.4] analogous to the Birkhoff factor theorem [1, ch. II, th. 7]; it seems probable that, using this, one might obtain a unicity theorem extending the present theorem 3 to general lattices.

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Резюме

ПРЕДСТАВЛЕНИЕ ДЕДЕКИНДОВЫХ СТРУКТУР КОНЕЧНОЙ ДЛИНЫ

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В работе ищется каноническое представление дедекиндовых структур конечной длины (д. с. к. д.). Назовём структуру L *полупростой* (plain) если не существует структурных гомоморфизмов, изобращающих L на двух-элементную структуру.

Теорема 4. *Существует единственное полупрямое разложение д. с. к. д. L в дистрибуривную структуру D и полупростую структуру M.*

Определим *дефект* д. с. к. д. L как $\delta_L = l_L - \lambda_L$, где l_L — длина L и λ_L — длина структуры отношений конгруэнтности на L. Полупрямое произведение L в систему структур $L_1, \ldots, L_n$ называется *точным*, если $\delta_L = \sum \delta_{Lj}$ („точное произведение" — exact decomposition — используется в работе в другом смысле).

Теорема 5. *Существует единственное точное произведение д. с. к. д. L в дистрибутивную структуру D и конечное число простых недистрибутивных д. с. к. д. M_j.*

В работе имеется также более детальное исследование понятия дефекта (п. 1, 2) полупростой структуры и точного представления (п. 3, 4); приведены также некоторые соотношения между длинами L, M, D, M_j (п. 4).