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INDUCTION IN FORMAL LANGUAGES. SOME PROPERTIES OF REDUCING TRANSFORMATIONS AND OF ISOLABLE SETS

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1. INTRODUCTION AND SUMMARY

In this paper we consider languages in the sense of [3] satisfying some additional conditions (see Section 2), particularly the condition of non-cyclicity (see [5]). The class of these languages will be denoted by $\mathscr{C}_0$.

In Section 1 we shall show that in order to prove some assertion about languages from $\mathscr{C}_0$ we can use the induction with respect to maximal length of derivation which is often a very useful means for proving. Moreover, a strengthening of the theorem on structural induction (Theorem 6.7, [3]) is given.

One of the most important concepts in [3] is the concept of a reducing transformation, which is very useful for the study of structural unambiguity. In Section 2 some sufficient conditions for the existence of a reducing transformation are given and some properties of reducing transformations are proved. Especially, the closure of a reducing transformation and the product of two reducing transformations are, under certain assumptions, reducing transformations, too.

In [3] it has been proved that if $\mathscr{A}$ is an isolable set of nonterminal symbols of the language $\mathscr{L}$, then $\mathscr{L}$ is structurally unambiguous (s. u.) if and only if a language $\mathscr{L}_0$ is, which is simpler than $\mathscr{L}$. This method (and hence the concept of an isolable set, too) has been shown to be very useful for the investigation of structural unambiguity of the language (see [7]) which is a slight modification of ALGOL 60. Some sufficient conditions for the existence of isolable sets are given in [3] and [6]. In the last section some necessary and sufficient conditions for the existence of isolable sets and their properties are proved.

The present paper uses notations and definitions of [3]. The reader should be familiar with sections 1 to 9, [3].

In the present paper we shall consider only non-cyclic languages $\mathscr{L}$ (i.e. such languages that there is no text t derivable from the same text t), for which the sets $\mathbf{d}\mathscr{L}$ and $\{\alpha;\ A \in \mathbf{d}\mathscr{L},\ \alpha \in \mathscr{L}A - \sigma_t\mathscr{L}\}$ are finite. (It has been proved [5] that every cyclic language is structurally ambiguous (s. a.).) Denote the class of such languages by $\mathscr{C}_0$.

In this section it is proved that for every grammatical element there is a derivation with maximal length. Hence, in order to prove some assertion about the set $\mathbf{g}\mathscr{L}$ it is possible to use the induction on maximal length of derivation. The results obtained in this section allow to strengthen the theorem on structural induction (Theorem 6.7 [3]).

Definition 2.1. Let $g \in \mathbf{g}\mathscr{L}$. The set of all structures $[\alpha, \tau]$ (such that $\alpha \neq [g1]$ [1]) of g in $\mathscr{L}$ will be denoted by $S_{\mathscr{L}}g$ ($\bar{S}_{\mathscr{L}}g$). (If there is no danger of misunderstanding the symbol specifying the language will be deleted.)

Denote Qg the set

$$\{g_1;\ [\alpha, \tau] \in \bar{S}_{\mathscr{L}}g,\ i \in \mathbf{d}\alpha,\ g_1 = [\alpha i, \tau i] \in \mathbf{g}\mathscr{L}\}$$

Lemma 2.2. *If $g \in \mathbf{g}\mathscr{L}$, then an integer n exists such that $\lambda\sigma < n$ for every $[g1]$-derivation σ of $g2$.*

Proof. Suppose conversely that a grammatical element g exists such that for every n there is a $[g1]$-derivation σ of $g2$ such that $\lambda\sigma \geqq n$. Since $\sigma i_1 \neq \sigma i_2$ for $i_1 \neq i_2$ ($\mathscr{L}$ is non-cyclic), the set $\{u;\ u \to g2,\ u \in \mathbf{t}\mathscr{L}\}$ has at least $n - 1$ elements. Since n is arbitrary, the set $\{u;\ u \to g2,\ u \in \mathbf{t}\mathscr{L}\}$ is infinite which contradicts Lemma 2.10, [5].

Definition 2.3. Let $g \in g\mathscr{L}$. Denote

$$\mu g = \max \{\lambda\sigma;\ \sigma \text{ is a } [g1]\text{-derivation of } g2\}\,.$$

(With respect to Lemma 2.2, μg is a well defined number and the definition is meaningful.)

Remark 2.4. In order to prove some assertion V on the set $\mathbf{g}\mathscr{L}$, it is sufficient, by Lemma 2.2 and Definition 2,3, to prove that Vg holds if Vg_1 holds for all g_1 such that $\mu g_1 < \mu g$.

Lemma 2.5. *Let $A \in \mathbf{d}\mathscr{L}$, $[A] \to t_1 \to t_2$, τ be a t_1-decomposition of t_2, $i \in \mathbf{d}t_1$, $[t_1 i] \to \tau i$. Then $\mu[t_1 i, \tau i] < \mu[A, t_2]$.*

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[1]) Note that g is a sequence of the length two and therefore if $g = [A, t]$ then $g1 = A$ and $g2 = t$.

Proof. Let σ be a $[t_1 i]$-derivation of τi with maximal length. Let σ_1 be an [A]-derivation of t_1. Put $t_3 = t_1^{(1,i-1)} \times \tau i \times t_1^{(i+1,\lambda t_1)}$. Obviously $[A] \to t_1 \to t_3 \overset{\to}{=} t_2$. Let $\bar{\sigma}$ be a derivation defined as follows: $\mathbf{d}\bar{\sigma} = \mathbf{d}\sigma$, $\bar{\sigma}i = t_1^{(1,i-1)} \times \sigma i \times t_1^{(i+1,\lambda t_1)}$. Obviously $\bar{\sigma}$ is a t_1-derivation of t_3. Let σ_2 be a t_3-derivation of t_2. Then $\sigma_0 = \sigma_1^{(1,\lambda\sigma_1-1)} \times \bar{\sigma} \times \sigma_2^{(2,\lambda\sigma_2)}$ is an [A]-derivation of t_2 and obviously $\mu[t_1 i, \tau i] = \lambda\sigma = \lambda\bar{\sigma} < \lambda\sigma_0 \leqq \mu[A, t_2]$.

Corollary 2.6. *If $g \in \mathbf{g}\mathscr{L}$, $g_1 \in Qg$ then $\mu g_1 < \mu g$.*

Corollary 2.7. *There is no infinite sequence σ such that $\sigma i \in \mathbf{g}\mathscr{L}$ for every $i \in \mathbf{d}\sigma$ and, if $i > 1$, $\sigma(i+1) \in Q\sigma i$.*

Proof. Immediate from Corollary 2.6.

Theorem 2.8. *A $g \in \mathbf{g}\mathscr{L}$ exists such that $\mu g = 2$.*

Proof. Let g_0 be such that $\mu g_0 = \inf\{\mu g', g' \in \mathbf{g}\mathscr{L}\}$. Obviously $\mu g_0 \geqq 2$. Suppose that $\mu g_0 > 2$. Then a structure $[\alpha, \tau]$ and $i \in \mathbf{d}\alpha$ exist such that $[\alpha i, \tau i] \in Qg$ (see Lemma 6.4, [3]). By Corollary 2.6, $\mu[\alpha i, \tau i] < \mu g_0$ which contradicts the choice of g_0. Hence $\mu g_0 = 2$ and the Theorem is proved.

Corollary 2.9. *A structurally unambiguous grammatical element exists in $\mathbf{g}\mathscr{L}$.*

Lemma 2.10. *Let $N \subset \mathbf{g}\mathscr{L}$ and let for every $g \in N$ a $g_1 \in Qg \cap N$ exist. Then $N = \Lambda$.*

Proof. By Corollary 2.7.

Theorem 2.11. (*Structural induction*) *Let* M $\subset \mathbf{g}\mathscr{L}$, *let*

(1) $$g \in M \quad \text{if} \quad Qg \in M\,.$$

Then $M = \mathbf{g}\mathscr{L}$.

Proof. If $\mu g = 2$, then $Qg = \Lambda$ and, by (1), $g \in M$. If $\mu g > 2$ and all $\bar{g}$ with $\mu\bar{g} < \mu g$ are in M, then $Qg \subset M$ and $g \in M$ by (1).

Remark 2.12. Condition (2.11.1) is equivalent to

(1) $g \in M$ if every structure $[\alpha, \tau]$ of g is weakly M-regular,

and is weaker than condition (6.7.2), [3]. Note that (6.7.1), [3] may be omitted in Theorem 2.11 since it follows from (1). On the other hand, it is convenient to verify (1) separately for $\mu g = 2$ where $Qg = \Lambda$, and for $\mu g > 2$ where $Qg \neq \Lambda$.

For assertions concerning simultaneous properties of grammatical elements and their structures the following theorem has been used implicitly in many proofs in [3].

Theorem 2.13. Let $M \subset \mathbf{g}\mathscr{L}$, let $N_0 \subset N = \{[g, \alpha, \tau]; g \in \mathbf{g}\mathscr{L}, [\alpha, \tau] \in Sg\}$. *Suppose that*

(1) $g \in M$ *if* $[g, \alpha, \tau] \in N_0$ *for some* α, τ

and

(2) $[g, \alpha, \tau] \in N_0$ *if* $[g, \alpha, \tau] \in N$ *and* $[\alpha i, \tau i] \in M$ *as soon as* $[\alpha i] \to \tau i$.

Then $N_0 = N$.

Proof. If $g \in \mathbf{g}\mathscr{L}$ and $Qg \subset M$, then $g \in M$ by (2) and (1). Hence $M = \mathbf{g}\mathscr{L}$ by Theorem 2.11 and $N_0 = \mathrm{N}$ by (2).

3. REDUCING TRANSFORMATIONS

The concept of a reducing transformation is very important for the study of structural unambiguity (see [3]). It has been proved (Theorem 2.12, [5]) that if only languages from $\mathscr{C}_0$ are considered, then two conditions in the definition of a reducing transformation are always satisfied. This permits us to simplify Theorem 9.6, [3], (see Theorem 3.1).

Generally, the reducing transformation reduces a given grammatical element $g = [A, t]$ in such a way that some parts of the text t are replaced by metasymbols. Often it is easier to verify that a given transformation is reducing if for every gramatical element $g = [A, t]$ at most one part of the text t is replaced by a metasymbol. We shall call such reducing transformation simple. Sufficient conditions for the existence of a simple (5)-reducing transformation are given in Theorem 3.2.

In what follows some properties of reducing transformations are proved. Especially, if we consider only reducing transformations such that $\varrho g \in \mathbf{g}\mathscr{L}$ for all $g \in \mathbf{g}\mathscr{L}$, then the product of two reducing transformations and the closure (see Def. 3.7) of a reducing transformation are reducing transformations, too.

Theorem 3.1. *Let* V, R *be transformations defined on* $\mathbf{g}\mathscr{L}$. *For every* $g = [A, t] \in \mathbf{g}\mathscr{L}$ *let the following two conditions hold*:

(1) *Rg is a decomposition of t, Vg is a sequence,* $\lambda Vg = \lambda Rg$.

(2) *For every structure* $[\alpha, \tau]$ *of g there is an index-decomposition x_0 of Vg such that the decompositions* $\xi = \delta(Vg, x_0)$ *and* $\zeta = \delta(Rg, x_0)$ *satisfy*

(2a) $\tau = \zeta \otimes Rg$

and, for every $i \in \mathbf{d}\alpha$, *at least one of conditions* (2b1), (2b2) *and* (2b3) *holds*:

(2b1) $\lambda \xi i = 1,\ [\alpha i] \rightrightarrows \xi i \to \tau i$,

(2b2) $[g1] \neq \alpha,\ [\alpha i] \neq \tau i,\ \xi i = V[\alpha i, \tau i],\ \zeta i = R[\alpha i, \tau i]$,

(2b3) $\xi i = \tau i,\ \zeta i = \delta_p(\tau i)$.

Then conditions (9.1.1) *and* (9.1.4) *in* [3] *hold for every* $g \in \mathbf{g}\mathscr{L}$ *and for each of its structures* $[\alpha, \tau]$. *Moreover, if conditions* (9.1.3) *and* (9.1.6) *in* [3] *hold with* $\varrho g = [g1, Vg]$, *then* ϱ *is a reducing transformation.*

Proof. The first assertion of the Theorem can be proved similary as in the proof of Theorem 9.6, [3]. For proving (9.1.1) and (9.1.4) in [3] the condition $\delta_0[\alpha i, \xi i] < \delta_0[\alpha i, \tau i]$ in (9.6.2b1), [3] was not used.) The second assertion of Theorem follows from Theorem 2.12, [5].

Theorem 3.2. *Let $M \subset \mathbf{g}\mathscr{L}$ and let f_0, f_1, v be transformations defined on $N \supset M$. Let for every $g \in M$, $f_0 g \in \mathbf{d}g2$, $f_1 g \in \mathbf{d}g2$, $f_0 g \leqq f_1 g$ and let for every $[\alpha, \tau] \in Sg$, $x = \iota\tau$, $xi \leqq f_0 g < x(i + 1)$ either*

(1) $f_0 g = xi,\ f_1 g = x(i + 1) - 1,\ [\alpha i] \rightrightarrows [vg] \to \tau i$

or

(2) $[g1] \neq \alpha,\ [\alpha i, \tau i] \in M,\ vg = v[\alpha i, \tau i],\ f_s g = f_s[\alpha i, \tau i] + xi - 1,\ s = 0, 1.$

Finally let

(3) $vg = g1$ *if* $[g1] \Rightarrow g2,\ g \in M.$

Then a simple (5)-reducing transformation ϱ exists such that $M = \{g; \varrho g \neq g\}$.

Proof. Let $g = [A, t]$. If $g \in \mathbf{g}\mathscr{L} - M$, then we put $Vg = t$, $Rg = \delta_p t$. If $g \in M$, then we put

(4) $Vg = t^{(1, f_0 g - 1)} \times [vg] \times t^{(f_1 g + 1, \lambda t)}$,

(5) $Rg = \delta_p(t^{(1, f_0 g - 1)}) \times [t^{(f_0 g, f_1 g)}] \times \delta_p(t^{(f_1 g + 1, \lambda t)})$.

We shall show that the conditions of Theorem 3.1 are satisfied. The condition (3.1.1) is obviously satisfied. If $g \notin M$ then (3.1.2a) is satisfied with $x_0 = \iota\tau$ and in this case (3.1.2b3) holds for all $i \in \mathbf{d}\alpha$. Let $g \in M$. If $[\alpha, \tau]$ is a structure of g then obviously Rg is finer than τ. Hence there is an index-decomposition x_0 of Rg (and of Vg, too, because $\lambda Vg = \lambda Rg$) such that the decomposition $\zeta = \delta(Rg, x_0)$ satisfies (3.1.2a). We shall show that ζ and $\xi = \delta(Vg, x_0)$ satisfy, for every $i \in \mathbf{d}\alpha$, at least one of conditions (3.1.2b1) to (3.1.2b3). Put $x = \iota\tau$, and let $xj \leqq f_0 g < x(j + 1)$. If $j \neq i$, then (3.1.2b3) holds. If $j = i$ and (1) holds then (3.1.2b1) holds and if (2) holds then (3.1.2b2) is satisfied. Hence, by Theorem 3.1, we get that conditions (9.1.1) and (9.1.4), [3] are satisfied for every $g \in \mathbf{g}\mathscr{L}$ and each of its structures $[\alpha, \tau]$. The condition (9.1.3) is clearly satisfied if $g \notin M$; if $g \in M$ then $vg = g1$ by (3), $f_0 g = 1$ and $f_1 g = \lambda g2$ (by (1) because for the structure $[[A], [t]] \subset S_{\mathscr{L}} g$ the condition (2) is not satisfied) and hence, by (4), $Vg = [g1]$. Put $\varrho g = [g1, Vg]$ for all $g \in \mathbf{g}\mathscr{L}$. By Theorem 3.1, ϱ is (5)-reducing transformation and, by (4) and (5), ϱ is simple. The equality $M = \{g; \varrho g \neq g\}$ follows, since $\mathscr{L}$ is a non-cyclic language, from the definition of transformations V, R and ϱ. This completes the proof of the Theorem.

Definition 3.3. Let ϱ_1 and ϱ_2 be transformations defined on $\mathbf{g}\mathscr{L}$. Denote $J\varrho_1 = \{g; \varrho_1 g = g\}$. A transformation ϱ_1 is said to be complete if $\varrho_1 \varrho_1 g = \varrho_1 g$ for every $g \in \mathbf{g}\mathscr{L}$. We shall say that ϱ_1 is weakly equivalent with ϱ_2, if $J\varrho_1 = J\varrho_2$.

Theorem 3.4. *Let ϱ_1 and ϱ_2 be reducing transformations. Then a reducing transformation ϱ_3 exists such that $J\varrho_3 = J\varrho_1 \cap J\varrho_2$.*

Proof. Let ϱ_i, $i = 1, 2$ be induced by reducing pairs $\langle V_i, R_i \rangle$. Let V_3, R_3 and ϱ_3 be transformations defined on $\mathbf{g}\mathscr{L}$ in the following way:
If

(1) either $\varrho_2 g \notin \mathbf{g}\mathscr{L}$ or $\varrho_2 g \in \mathbf{g}\mathscr{L}$, $\varrho_2 g \neq g$, $\varrho_1 \varrho_2 g \notin \mathbf{g}\mathscr{L}$

then

(1a) $V_3 g = V_2 g$, $R_3 g = R_2 g$, $\varrho_3 g = [g1, V_3 g] = \varrho_2 g$.

If (1) does not hold, then

(2) $V_3 g = V_1 \varrho_2 g$, $R_3 g = R_1 \varrho_2 g \otimes R_2 g$, $\varrho_3 g = [g1, V_3 g]$.

According to (2) we get that

(3) $\varrho_3 g = \varrho_1 \varrho_2 g$ if (1) does not hold.

Moreover, from (1a) and (2) we obtain

(4) if either $\varrho_1 g \neq g$ or $\varrho_2 g \neq g$, then $\varrho_3 g \neq g$.

Using these results we get that $J\varrho_3 = J\varrho_1 \cap J\varrho_2$. Now, we are going to show that $\langle V_3, R_3 \rangle$ is a reducing pair. By Theorem 2.12, [5], it is sufficient to show that conditions (9.1.1), (9.1.3), (9.1.4) and (9.1.6), [3] are satisfied for every $g \in \mathbf{g}\mathscr{L}$ and any $[\alpha, \tau] \in Sg$. Since ϱ_2 is a reducing transformation, these conditions are certainly satisfied if condition (1) holds; thus, we may assume that (1) does not hold, i.e.

(5) $\varrho_2 g \in \mathbf{g}\mathscr{L}$ and either $\varrho_2 g = g$ or $\varrho_1 \varrho_2 g \in \mathbf{g}\mathscr{L}$.

Obviously, condition (9.1.3) is true and (9.1.6) follows from the fact that $J\varrho_3 = J\varrho_1 \cap J\varrho_2$. Since $\langle V_i, R_i \rangle$, $i = 1, 2$ are reducing transformations, $R_1 \varrho_2 g$ is a $V_1 \varrho_2 g$-decomposition of $(\varrho_2 g)\, 2$ and $R_2 g$ is a $V_2 g$-decomposition of $g2$. Obviously, $(\varrho_2 g)\, 2 = V_2 g$ and, by Lemma 6.1, [3], $R_1 \varrho_2 g \otimes R_2 g$ is a $V_1 \varrho_2 g$-decomposition of $g2$. Thus, $R_3 g$ is a $V_3 g$-decomposition of $g2$ and (9.1.1) holds. Hence, in order to prove the Theorem, it is sufficient to show that condition (9.1.4) holds, too. Let $g \in \mathbf{g}\mathscr{L}$ and $[\alpha, \tau] \in Sg$.

First suppose that $\varrho_2 g = g$. Then $R_2 g = \delta_p g2$. Since (9.1.4) holds for $\langle V_1, R_1 \rangle$, an α-decomposition ξ of $g2$ exists such that $\tau = \xi \otimes R_1 g$. Then $\tau = \xi \otimes R_1 g = \xi \otimes (R_1 \varrho_2 g \otimes R_2 g) = \xi \otimes R_3 g$ and (9.1.4) holds.

Now suppose that $\varrho_2 g \neq g$ and $V_2 g = \alpha$. In this case condition (9.1.4) is satisfied with $\xi = \delta_p \alpha$, since $R_1 \varrho_2 g = \delta_p \alpha$ (note that $\varrho_1 \varrho_2 g \in \mathbf{g}\mathscr{L}$); hence, $\tau = \xi \otimes (R_1 \varrho_2 g \otimes R_2 g) = \xi \otimes R_3 g$.

Finally suppose that $\varrho_2 g \neq g$ and $V_2 g \neq \alpha$. Since (9.1.4) holds for $\langle V_2, R_2 \rangle$, an α-decomposition ξ_2 of $V_2 g$ exists such that $\tau = \xi_2 \otimes R_2 g$. Since $V_2 g \neq \alpha$, $[\alpha, \xi_2] \in \bar{S}[g1, V_2 g]$. Hence by (9.1.4) an α-decomposition ξ_1 of $V_1 \varrho_2 g$ exists for V_1 and R_1 such that $\xi_2 = \xi_1 \otimes R_1 \varrho_2 g$. Thus, $\tau = (\xi_1 \otimes R_1 \varrho_2 g) \otimes R_2 g = \xi_1 \otimes R_3 g$ and (9.1.4) holds.

Corollary 3.5. *Let ϱ_1, ϱ_2 be reducing transformations such that $\varrho_i g \in \mathbf{g}\mathscr{L}$ for all $g \in \mathbf{g}\mathscr{L}$, $i = 1, 2$. Then $\varrho_1\varrho_2$ is a reducing transformation and $J\varrho_1\varrho_2 = J\varrho_1 \cap J\varrho_2$.*

Proof. The transformation ϱ_3 defined in the proof of the preceding theorem is, by that theorem, a reducing transformation and $J\varrho_3 = J\varrho_1 \cap J\varrho_2$. Since $\varrho_i g \in \mathbf{g}\mathscr{L}$ for all $g \in \mathbf{g}\mathscr{L}$ and $i = 1, 2$, there is no $g \in \mathbf{g}\mathscr{L}$ such that (3.4.1) holds; hence, by (3.4.3), $\varrho_3 = \varrho_1\varrho_2$.

Remark 3.6. It can be the case that $\varrho_1\varrho_2 \neq \varrho_2\varrho_1$.

Example: Let

$$\mathbf{d}\mathscr{L} = \{A, B, C\}, \quad \mathscr{L}A = \{[B, C]\}, \quad \mathscr{L}B = \{[E]\}, \quad \mathscr{L}C = \{[F]\}.$$

Let $\varrho_1 g = g = \varrho_2 \varrho$ if $g \neq [A, [E, F]]$ and let $\varrho_1[A, [E, F]] = [A, [B, F]]$, $\varrho_2[A, [E ,F]] = [A, [E, C]]$. Then

$$\varrho_1\varrho_2[A, [E, F]] = [A, [E, C]] \neq [A, [B, F]] = \varrho_2\varrho_1[A, [E, F]].$$

Definition 3.7. The closure $\nu\varrho$ of a reducing transformation ϱ is the transformation defined on $\mathbf{g}\mathscr{L}$ as follows: $(\nu\varrho)\, g = \varrho^i g$ where i is the smallest integer such that $\varrho^i g \in J\varrho$ or $\varrho^i g \notin \mathbf{g}\mathscr{L}$; such an integer exists by Theorem 2.12, [5].

Theorem 3.8. *Let ϱ be a reducing transformation such that $\varrho g \in \mathbf{g}\mathscr{L}$ for all $g \in \mathbf{g}\mathscr{L}$. Then $\nu\varrho$ is a complete reducing transformation weakly equivalent with ϱ.*

Proof. The relation $J\varrho = J\nu\varrho$ follows from the fact that $\mathscr{L}$ is a non-cyclic language and $(\varrho g)\, 2 \supseteqq g2$ for every $g \in \mathbf{g}\mathscr{L}$. This relation also shows that $\nu\varrho$ satisfies condition (9.1.6), [3]. Indeed, let $g \in \mathbf{g}\mathscr{L}$, $g_1 \in Qg$. If $(\nu\varrho)\, g_1 \neq g_1$, then $g_1 \notin J\nu\varrho$ and hence $g_1 \notin J\varrho$. Since ϱ is a reducing transformation, $\varrho g \neq g$, i.e., $g \notin J\varrho = J\nu\varrho$ and we have $(\nu\varrho)\, g \neq g$.

By Corollary 3.5, ϱ^i is a reducing transformation for every $i = 1, 2, \ldots$ Let ϱ^i be induced by $\langle V_i, R_i \rangle$. Let $i(g)$ be the smallest integer such that $\varrho^{i(g)} g = (\nu\varrho)\, g$. Put $\bar{V}g = V_{i(g)}g$, $\bar{R}g = R_{i(g)}g$. Then $\nu\varrho g = \varrho^{i(g)} g$ for every $g \in \mathbf{g}\mathscr{L}$. and, as we show, $\nu\varrho$ is induced by $\langle \bar{V}, \bar{R} \rangle$. Indeed, conditions (9.1.1), (9.1.3) and (9.1.4), [3] hold for any fixed $g \in \mathbf{g}\mathscr{L}$, for R_i, V_i and any i. In particular, they hold for $i = i(g)$ and consequently, they hold for $\bar{V}$ and $\bar{R}$. By Theorem 2.12, [5], $\nu\varrho$ is a reducing transformation and obviously, $\nu\varrho$ is complete.

Definition 3.9. If ϱ is a reducing transformation, then we shall denote $\chi\varrho$ the transformation defined as follows:

(1) $(\chi\varrho)\, g = \varrho g$ *if* $\mu g > 2$, *and* $(\chi\varrho)\, g = g$ *if* $\mu g = 2$.

Lemma 3.10. *If ϱ is a reducing transformation, then so is $\chi\varrho$ and $(\chi\varrho)\, g \in \mathbf{g}\mathscr{L}$ for all $g \in \mathbf{g}\mathscr{L}$.*

Proof. Let ϱ be induced by $\langle V, R\rangle$. If $\mu g > 2$, then there is an $[\alpha, \tau] \in \bar{S}g$ and, by (9.1.1), $[g1] \Rightarrow \alpha \rightrightarrows Vg$. Thus, $(\chi\varrho)\, g = \varrho g = [g1, Vg] \in \mathbf{g}\mathscr{L}$ and the last assertion of the Lemma holds. Now, let us define transformations V_1, R_1 on $\mathbf{g}\mathscr{L}$ as follows: $V_1 g = Vg$, $R_1 g = Rg$ if $\mu g > 2$; $V_1 g = g2$, $R_1 g = \delta_p g2$ if $\mu g = 2$. It is evident that conditions (9.1.1), (9.1.3), (9.1.4), [3] are satisfied for V_1, R_1 if $\mu g = 2$; if $\mu g > 2$, then this follows from the fact that these conditions hold for V and R. Now let $g \in g\mathscr{L}$, $g_1 \in Qg$ and $(\chi\varrho)\, g_1 \neq g_1$. Then $\mu g_1 > 2$ and $\mu g > 2$, too. Thus, $(\chi\varrho)\, g_1 = \varrho g_1 \neq g_1$. Since ϱ is a reducing transformation, $\varrho g_1 \neq g_1$ implies $\varrho g \neq g$. Since $\mu g > 2$, $\varrho g \neq g$ implies $(\chi\varrho)\, g \neq g$ and (9.1.6) holds, too. Thus, $\chi\varrho$ is a reducing transformation.

Definition 3.11. Two reducing transformation ϱ_1, ϱ_2 are said to be equivalent if $\nu\chi\varrho_1 = \nu\chi\varrho_2$.

Remark 3.12. It can be the case that ϱ is a reducing transformation, $\varrho g \in \mathbf{g}\mathscr{L}$ for all $g \in \mathbf{g}\mathscr{L}$ and there is no simple reducing transformation ϱ_1 equivalent with ϱ.

Example: $\mathbf{d}\mathscr{L} = \{A, B, C, D, E\}$, $\mathscr{L}A = \{[B, C]\}$, $\mathscr{L}B = \{[D]\}$, $\mathscr{L}C = \{[E]\}$, $\mathscr{L}D = \{[F]\}$, $\mathscr{L}E = \{[G]\}$. Let ϱ be defined as follows: $\varrho g = g$ if $g \neq [A, [F, G]] = g_0$ and $\varrho g_0 = [A, [D, E]]$.

Suppose conversely that there is a simple reducing transformation ϱ_1 equivalent with ϱ. Then $\varrho_1\varrho_1 g_0 = [A, [D, E]]$. Hence, either $\varrho_1 g_0 = [A, [D, G]]$ and $\varrho_1[A, [D, G]] = [A, [D, E]]$, or $\varrho_1 g_0 = [A, [F, E]]$ and $\varrho_1[A, [F, E]] = [A, [D, E]]$ In both cases there is a $g_1 \neq g_0$ such that $g_1 \neq \varrho_1 g_1 \in \mathbf{g}\mathscr{L}$ and consequently, $(\nu\chi\varrho_1)\, g_1 \neq g_1 = (\nu\chi\varrho)\, g_1$. Thus, ϱ_1 is not equivalent with ϱ.

4. ISOLABLE SETS

In this section isolable sets and their relationship to structural unambiguity is investigated. Some necessary and sufficient conditions for a set $\mathscr{A}$ to be isolable are given.

Theorem 4.1. *A non-empty subset $\mathscr{A} \subset \mathbf{d}\mathscr{L}$ is isolable if and only if there is a reducing transformation ϱ such that $g_1 1 \in \mathscr{A}$ for no $g \in J\varrho_1$, $g_1 \in Qg$. In this case we shall say that $\mathscr{A}$ is ϱ-isolable.*

Proof. According to Definition 9.7, [3].

Corollary 4.2. *A non-empty subset of an isolable set is isolable.*

Theorem 4.3. *If $\mathscr{A}_1$, $\mathscr{A}_2$ are isolable sets then so is $\mathscr{A}_1 \cup \mathscr{A}_2$.*

Proof. Let $\mathscr{A}_i$ be ϱ_i-isolable for $i = 1, 2$. Let ϱ_3 be defined as in the proof of Lemma 3.4. Then ϱ_3 is a reducing transformation. If $\varrho_3 g = g$ and $g_1 \in Qg$, then, by (3.4.4), $\varrho_1 g = g$ and $\varrho_2 g = g$. Since $\mathscr{A}_i$ is ϱ_i-isolable, an application of Theorem 4.1 gives $g_1 1 \notin \mathscr{A}_1 \cup \mathscr{A}_2$. Thus, again by Theorem 4.1, $\mathscr{A}_1 \cup \mathscr{A}_2$ is ϱ_3-isolable.

Theorem 4.4. *A non-empty subset $\mathscr{A} \subset \mathbf{d}\mathscr{L}$ is isolable if and only if there is a reducing transformation ϱ such that $J\varrho \subset \mathbf{g}_a\mathscr{L} = \{g;\ Qg \subset g\mathscr{L}_{\mathbf{d}\mathscr{L}-\mathscr{A}}\}$.*

Proof. First suppose that $\mathscr{A}$ is ϱ-isolable. If $J\varrho \not\subset \mathbf{g}_a\mathscr{L}$, then a $g \in J\varrho - \mathbf{g}_a\mathscr{L}$ exists such that $\mu g = \min\{\mu g_0,\ g_0 \in J\varrho - \mathbf{g}_a\mathscr{L}\}$. If $g_1 \in Qg$ then $\varrho g_1 = g_1$ (since $g \in J\varrho$) and, by Corollary 2.6, $\mu g_1 < \mu g$. Thus, $g_1 \in \mathbf{g}_a\mathscr{L}$. Next, since $\mathscr{A}$ is ϱ-isolable, and $\varrho g = g$, an application of Theorem 4.1 gives $g_1 1 \notin \mathscr{A}$. Thus $g_1 \in \mathbf{g}\mathscr{L}_{\mathbf{d}\mathscr{L}-\mathscr{A}}$. Since g_1 is an arbitrary element from Qg, $g \in \mathbf{g}_a\mathscr{L}$ whih contradicts our assumption $g \in J\varrho - \mathbf{g}_a\mathscr{L}$. Thus, $J\varrho \subset \mathbf{g}_a\mathscr{L}$.

Secondly, let ϱ be a reducing transformation and $J\varrho \subset \mathbf{g}_a\mathscr{L}$. If $\varrho g = g$, then $g \in J\varrho \subset \mathbf{g}_a\mathscr{L}$. Thus $g_1 1 \in \mathscr{A}$ for no $g_1 \in Qg$ and an application of Theorem 4.1 shows that $\mathscr{A}$ is ϱ-isolable.

Theorem 4.5. *If $\mathscr{L}$ is a s. u. language, then every non-empty subset of $\mathbf{d}\mathscr{L}$ is isolable.*

Proof. By Theorem 9.5, [3], a reducing transformation ϱ exists such that $J\varrho = \Lambda$ and hence, by Theorem 4.4, every non-empty subset of $\mathbf{d}\mathscr{L}$ is isolable.

Theorem 4.6. *Let $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n$ be subsets of $\mathbf{d}\mathscr{L}$ such that $\bigcup_{i=1}^{n} \mathscr{A}_i = \mathbf{d}\mathscr{L}$. For a language $\mathscr{L}$ to be s. u. it is necessary and sufficient that every $\mathscr{A}_i$ is isolable.*

Proof. The necessity follows from Theorem 4.5. The sufficiency follows from Theorem 4.3 and 9.13, [3].

Theorem 4.7. *Denote $\mathscr{C}_1$ the class of languages $\mathscr{L}$ such that $\mathbf{d}\mathscr{L}$ and $\{\alpha;\ A \in \mathbf{d}\mathscr{L},\ \alpha \in \mathscr{L}A\}$ are finite sets. There is no algorithm to decide, for any given language $\mathscr{L} \in \mathscr{C}_1$, whether or not $\mathbf{d}\mathscr{L}$ is isolable.*

Proof. It has been proved by many authors, [1, 2, 4], that there is no algorithm to decide, for any given language $\mathscr{L} \in \mathscr{C}_1$, whether or not $\mathscr{L}$ is s. u. Now, the assertion of the theorem follows from Theorem 4.5 and Theorem 4.6.

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Резюме

ИНДУКЦИЯ В ФОРМАЛЬНЫХ ЯЗЫКАХ. НЕКОТОРЫЕ СВОЙСТВА РЕДУЦИРУЮЩИХ ПРЕОБРАЗОВАНИЙ И ИЗОЛИРУЕМЫХ МНОЖЕСТВ

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В работе изучаются формальные языка, введенные В. Фабианом в [3]. В первой части доказано, что в нециклических языках для каждого грамматического элемента существует вывод максимальной длины. Это свойство иногда полезно при доказательствах.

В следующих частях работы изучаются редуцирующие преобразования и изолируемые множества (смотри [3]). Рассматривается случай, когда к редуцирующим преобразованиям ϱ_1 и ϱ_2 существует такое редуцирующее преобразование ϱ, которое редуцирует такие и только такие грамматические элементы, которые редуцирует хоть одно из преобразований ϱ_1, ϱ_2. Кроме того, доказывается что язык $\mathscr{L}$ структурно однозначен тогда и только тогда, когда существует разбиение множества всех метасимволов языка $\mathscr{L}$ на изолируемые множества.