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ON THE BEST STABILIZING CONTROL UNDER A GIVEN CLASS OF PERTURBATIONS

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1.

Consider a linear control system

$$\frac{dx}{dt} = Ax + Bu \tag{1.1}$$

where x is an n-vector, u is an m-vector, A is an $n \times n$- and B an $n \times m$-matrix. Denote $b_1, \ldots, b_m$ the columns of B.

It is known ([7]) that if the system (1.1) is controllable, i.e. if at least n of the vectors $b_1, \ldots, b_m, Ab_1, \ldots, Ab_m, \ldots, A^{n-1}b_1, \ldots, A^{n-1}b_m$ are linearly independent, then there exists a bounded control function $u(x)$ such that the solution of the system

$$\frac{dx}{dt} = Ax + B\,u(x)$$

starting at the origin is trivial and every solution starting in a sufficiently small neighbourhood of the origin reaches the origin in a finite time.

Let us now consider a system

$$\frac{dx}{dt} = Ax + Bu + p \tag{1.2}$$

where x, u, A, B satisfy similar conditions as in (1.1), p are n-dimensional "perturtions" about which we know only that they are measurable and uniformly bounded by a small constant. It is shown in [8] that if $m = n$, B is nonsingular and the perturbations are sufficiently small, we are able to keep the solution of (1.2) at the origin and bring every solution of (1.2) to the origin from a sufficiently small neighbourhood of it in a finite time by an appropriate control function $u(x)$ which is independent of the perturbations.

If $m < n$, such a control need not exist.

In this article there is discussed the problem, whether it is possible in the case $m = 1, n = 2$ to keep the solution of (1.2) in some connected compact region containing the origin by a bounded measurable control $u(x)$ which is independent on the perturbations, if only the perturbations are sufficiently small and whether it is possible to choose a control function such that this region is in a certain sense minimal.

This problem leads to a discontinuous control function and, hence, to a differential system with discontinuous right-hand sides. Therefore, for preserving the real sense of the solution of the problem it is necessary to generalize the notion of the solution of a differential system ([4], [5]).

There are used the notions and methods of the theory of contingens-equations ([1], [3], [5], [6]).

The problems considered in this article may be exactly formulated as follows:

Let us consider a control system

$$(\sigma_\varepsilon) \qquad \begin{aligned} \frac{dx^1}{dt} &= a_1^1 x^1 + a_2^1 x^2 + b^1 u + r^1 , \\ \frac{dx^2}{dt} &= a_1^2 x^2 + a_2^2 x^2 + b^2 u + r^2 , \end{aligned}$$

in vector form

$$\frac{dx}{dt} = Ax + bu + r$$

under the following assumptions:

1. $a_j^i, b^i; i, j = 1, 2$ are constants,

2. $u = u(x)$, the control function, is a measurable function of x^1, x^2, defined almost everywhere in some open domain and satisfying the condition $|u| \leqq 1$,

3. $r = r(t)$ are perturbations, i.e. measurable functions of t, satisfying the condition $r \in \varepsilon R$ [1]), where R is a convex compact set, symmetric with respect to the origin,

4. the system (σ_ε) is controllable, i.e. the vectors b, Ab are linearly independent.

Let $\|x\|$ be the Euclidean norm in the two-dimensional Euclidean space E_2, $V(x, \delta) = \{x' : \|x' - x\| < \delta\}$, conv X the convex closure of X.

The function $x(t)$ is said to be a solution of the system (σ_ε) on the interval $\langle t_1, t_2 \rangle$ if it is absolutely continuous and satisfies the relation

$$\frac{dx}{dt} \in Ax + b \bigcap_{\delta > 0} \bigcap_{\text{mes} N = 0} \text{conv}\, u(V(x(t), \delta) - N) + r(t)$$

for almost every $t \in \langle t_1, t_2 \rangle$, where $r(t)$ is an arbitrary measurable function of t, satisfying the condition $r(t) \in \varepsilon R$ (see § 3).

[1]) $f(X) = \{f(x) : x \in X\}$; $X + Y = \{x + y : x \in X, y \in Y\}$.

If we denote

$$U(x) = \bigcap_{\delta > 0} \bigcap_{\text{mes} N = 0} \text{conv}\, u(V(x, \delta) - N),$$

then the function $x(t)$ is a solution of the system (σ_ε) if, and only if it is a solution of the contingens-equation

$$(\Sigma_\varepsilon) \qquad \frac{dx}{dt} \in Ax + b\, U(x) + \varepsilon R$$

(see § 2).

The control $u(x)$ will be said ε-stabilizing, if there exists a connected compact region G containing the origin such that for every solution $x(t)$ with $x(t_0) \in G$ the relation $x(t) \in G$ for $t \geqq t_0$ is valid. The region G will be said (u, ε)-invariant.

Let $|x|$ be a given norm in E_2 ([11], II, 3.1) and let $|X| = \max_{x \in X} |x|$ for an arbitrary compact set $X \subset E_2$. The control $\bar{u}(x)$ will be called the best ε-stabilizing control in the sense of the given norm, if there exists such a $(\bar{u}, \varepsilon)$-invariant set G_0, that for every (u, ε)-invariant set G

$$|G_0| \leqq |G|$$

is valid.

The control $\tilde{u}(x)$ will be called ε_0-invariant best ε-stabilizing, if it is the best ε-stabilizing control for every $\varepsilon \in (0, \varepsilon_0\rangle$.

The aim of this article is to show that for sufficiently small $\varepsilon > 0$ there exists a best ε-stabilizing control in the sense of a given norm (which may be arbitrary) and that under some further assumptions there exists an ε_0-universal best ε-stabilizing control in the sense of a given norm.

In §§ 2,3 there are given some notions and theorems of the theory of contingens-equations and differential systems with discontinuous right-hand sides. In § 4 a control function for a particular system is constructed, which is proved in § 5 to be for sufficiently small $\varepsilon > 0$ the best ε-stabilizing control in the sense of the Euclidean norm. In § 6 there is shown that under certain assumptions an ε_0-universal best ε-stabilizing control exists. In § 7 the case of a general norm is considered. In § 8 there is shown, that the general problem can be transformed to the special problem investigated in §§ 4–7. In § 9 two examples of special systems are solved. In § 10 our problem is considered from the viewpoint of the theory of games.

2.

Let E_k be the k-dimensional Euclidean space.

Definition 2.1. By a multi-valued function $F(x)$, defined in a domain $D \subset E_m$ with the values in E_n we shall denote a mapping which to every point $x \in D$ assigns a non-empty connected compact set $F(x) \subset E_n$.

Let us denote $V(X, \delta) = \{x \in E_n : \varrho(X, x) < \delta\}$ for a set $X \subset E_n$, where $\varrho(X, x) = \inf_{x' \in X} \|x - x'\|$.

Definition 2.2. The multi-valued function $F(x)$ will be called upper semicontinuous in the sense of inclusion (briefly β-continuous) at $x_0 \in D$, if the following condition is satisfied:

For every $\varepsilon > 0$ there exists such a $\delta > 0$ that for every $x \in D$ such that $\|x - x_0\| < \delta$ the relation

$$F(x) \subset V(F(x_0), \varepsilon)$$

is valid.

Remark 2.1. It is easy to see that the function $F(x)$ is β-continuous at $x_0 \in D$ if and only if from $x_n \to x_0$, $y_n \to y_0$, $y_n \in F(x_n)$ it follows that $y_0 \in F(x_0)$.

Definition 2.3. Let $\varphi(t)$ be a continuous function in the neighbourhood of t_0 with values from E_n. The set

$$\text{cont } \varphi(t) = \left\{ \lim_{k \to \infty} \frac{\varphi(t_0 + h_k) - \varphi(t_0)}{h_k} \right\},$$

where $\{h_n\}$ is an arbitrary sequence tending to zero such that the limit (also $+\infty$, $-\infty$) on the right-hand side exists, will be called the contingens of $\varphi(t)$ at t_0.

Definition 2.4. Let $F(t, x)$ be a multi-valued function defined in some domain $D \subset E_{n+1}$ with values from E_n. The symbol

$$\frac{dx}{dt} \in F(t, x). \tag{2.1}$$

will be called the contingens-equation. The function $x(t)$ will be called a solution of (2.1) on the interval $\langle t_1, t_2 \rangle$ if it is defined and continuous on $\langle t_1, t_2 \rangle$ and satisfies the conditions

$$(t, x(t)) \in D, \quad \text{cont } x(t) \subset F(t, x(t))$$

for each $t \in \langle t_1, t_2 \rangle$.

Theorem 2.1. *Let the multi-valued function $F(t, x)$ be defined and β-continuous in $D \subset E_{n+1}$ and let the set $F(t, x)$ be convex in every point $(t, x) \in D$. Then:*

1. *For every $(t_0, x_0) \in D$ there exists a solution $x(t)$ of (2.1) with $x(t_0) = x_0$.*
2. *$x(t)$ is a solution of (2.1) on $\langle t_1, t_2 \rangle$* if and only if

a) *$x(t)$ is absolutely continuous on $\langle t_1, t_2 \rangle$,*
b) *$(t, x(t)) \in D$ for $t \in \langle t_1, t_2 \rangle$,*
c) *$\dot{x}(t) \in F(t, x(t))$ for almost every $t \in \langle t_1, t_2 \rangle$.*

(dx/dt will be frequently denoted by $\dot{x}$.)

Remark 2.2. If further some statement concerning the derivative of a solution of a contingens-equation for $t \in T$ will be given, it will be understood that it is valid for almost every $t \in T$.

Definition 2.5. Let every solution of (2.1) with $x(t_0) = x_0$ be prolongable on the interval $\langle t_0, t_1 \rangle$. By the zone of emission $Z(x_0, t_0, t_1)$ of the equation (2.1) we shall denote the set of points (t, x) belonging to the solutions of (2.1) with $t \in \langle t_0, t_1 \rangle$. By the lateral boundary $H(x_0, t_0, t_1)$ of $Z(x_0, t_0, t_1)$ the closure of the set of boundary points (t, x) of $Z(x_0, t_0, t_1)$ with $t \in (t_0, t_1)$ will be denoted.

Theorem 2.2. *Let the conditions of Theorem* 2.1 *and definition* 2.5 *be satisfied. Then*:

1. $Z(x_0, t_0, t_1)$ *and the intersection of* $Z(x_0, t_0, t_1)$ *with every hyperplane* $t = t'$, $t' \in \langle t_0, t_1 \rangle$ *are connected compact sets.*

2. *For every* $(t', x') \in H(x_0, t_0, t_1)$ *there exists a solution* $x(t)$ *with* $x(t_0) = x_0$, $x(t') = x'$, $(t, x(t)) \in H(x_0, t_0, t_1)$ *for* $t \in \langle t_0, t' \rangle$.

3. *From* $(t', x') \in Z(x_0, t_0, t_1)$, $t_2 \in \langle t'_1, t_1 \rangle$ *it follows that*

$$Z(x', t', t_2) \subset Z(x_0, t_0, t_1) .$$

The proof of this theorem may be found in [6].

Theorem 2.3. *Let the assumptions of Theorem* 2.2 *be satisfied and let the contingens-equation*

$$\frac{dx}{dt} \in G(t, x) \tag{2.2}$$

satisfy analogous conditions as (2.1). *Let for some* $\mu > 0$

$$F(t, x) \subset G(t, x)$$

$$\text{for} \quad (t, x) \in W = [V(Z_G(x_0, t_0, t_1), \mu) - Z_G(x_0, t_0, t_1)] \cap \{(t, x) : t \in (t_0, t_1)\} .$$

(*Denote* Z_F, Z_G *the zones of emission of the equations* (2.1), (2.2), *respectively.*) *Then*

$$Z_F(x_0, t_0, t_1) \subset Z_G(x_0, t_0, t_1) .$$

Proof. Let us suppose that there exists a point $(t', x') \in Z_F(x_0, t_0, t_1) - Z_G(x_0, t_0, t_1)$. Then there exists a solution $x(t)$ of (2.1) such that $x(t_0) = x_0$, $x(t') = x'$, $\text{cont } x(t) \in F(t, x(t))$ for $t \in \langle t_0, t')$ is valid. As $(t', x') \notin Z_G(x_0, t_0, t_1)$ it follows from Theorem 2.2 that there exists a point $(t'', x'') \in Z_G(x_0, t_0, t_1)$ $t_0 \leqq t'' < t'$ and a $\tau > 0$ such that

$$x(t'') = x'' , \quad (t, x(t)) \in W \quad \text{for} \quad t \in (t'', t'' + \tau\rangle, \ \tau > 0 \tag{2.3}$$

For $t \in \langle t'', t'' + \tau \rangle$ we have $\dot{x}(t) \in F(t, x(t)) \subset G(t, x(t))$ and hence, $(\tau, x(\tau)) \in$ $\in Z_G(x'', t'', \tau)$.
Due to Theorem 2.2 we have $(\tau, x(\tau)) \in Z_G(x_0, t_0, t_1)$ which contradicts (2.3). Thus, the theorem is proved.

Now, let $x \in E_1$ and let the assumptions of Theorem 2.3 be satisfied. Denote the lateral boundaries of Z_F, Z_G by H_F, H_G, respectively. It follows from Theorem 2.2 that $H_F(x_0, t_0, t_1)$ consists from the lower boundary $x = h_{1F}(t)$ and upper boundary $x = h_{2F}(t)$, where $h_{1F}(t) = \min_{(t,x) \in Z_F} x$, $h_{2F}(t) = \max_{(t,x) \in Z_F} x$ and both of the functions $h_{1F}(t)$, $h_{2F}(t)$ are solutions of (2.1). Similarly, we denote the lower and upper boundary of $Z_G(x_0, t_0, t_1)$ by $h_{1G}(t)$, $h_{2G}(t)$, respectively.

In a similar way as Theorem 2.3 we may prove

Theorem 2.4. *Let the assumptions of Theorem 2.2 and the similar assumptions on the equation (2.2) be satisfied.*

1. *If $F(t, x) \subset G(t, x)$ for (t, x) such that $t \in (t_0, t_1)$, $h_{1G}(t) - \mu < x < h_{1G}(t)$, $\mu > 0$ then for every $(t, x) \in Z_F(x_0, t_0, t_1)$ it holds $x \geqq h_{1G}(t)$.*

2. *If $F(t, x) \subset G(t, x)$ for (t, x) such that $t \in (t_0, t_1)$, $h_{2G}(t) < x < h_{2G}(t) + \mu$, $\mu > 0$ then for every $(t, x) \in Z_F(x_0, t_0, t_1)$ it holds $x \leqq h_{2G}(t)$.*

Remark 2.3. The analogue of Definition 2.5, Theorems 2.2, 2.3 and 2.4 may easily formulated for $t_1 < t_0$.

Theorem 2.5. *Let $x \in E_1$ and let the assumptions of Theorem 2.2 be satisfied. Let* min $F(t, x)$ *be continuous in some neighbourhood of the lower boundary $h_1(t)$ of the zone of emission $Z(x_0, t_0, t_1)$, $t_1 > t_0$ of the equation (2.1). Then $h_1(t)$ is a solution of the differential equation*

$$(2.4) \qquad \frac{dx}{dt} = \min F(t, x)$$

for $t \in (t_0, t_1)$. Similarly, if max $F(t, x)$ *is continuous in some neighbourhood of $h_2(t)$, then $h_2(t)$ is a solution of the differential equation*

$$(2.5) \qquad \frac{dx}{dt} = \max F(t, x)$$

for $t \in (t_0, t_1)$.

Proof. Suppose that $h_1(t)$ does not satisfy (2.4) for some $t_2 \in (t_0, t_1)$. Let $x_2 =$ $= x(t_2)$. It follows from Theorem 2.2 that $h_1(t)$ is a solution of (2.1) and, therefore, cont $h_1(t_2) \subset F(t_2, x_2)$; as simultaneously $h_1(t)$ does not satisfy at t_2 the equation (2.4), there exists a sequence $\{\tau_n\}$, $\tau_n \to 0$ such that

$$l = \lim_{n \to \infty} \frac{h_1(t_2 + \tau_n) - h_1(t_2)}{\tau_n} > \min F(t_2, h_1(t_2)) .$$

Let an infinite number of τ_n's be positive. From the continuity of the function min F in the neighbourhood of the point (t_2, x_2) we conclude that there exists a solution $\varphi(t)$ of (2.4) with $\varphi(t_2) = x_2$. For sufficiently large n such that $\tau_n > 0$ we have

$$\varphi(t_2 + \tau_n) < h_1(t_2 + \tau_n) . \tag{2.6}$$

As $\varphi(t)$ is a solution of (2.1), we have

$$(t_2 + \tau_n, \varphi(t_2 + \tau_n)) \in Z(x_2, t_2, t_1) \subset Z(x_0, t_0, t_1) ,$$

which contradicts (2.6).

Let now $\tau_n < 0$ for infinitely many n's. As min F is continuous in the neighbourhood of (t_2, x_2), the relations

$$\min F(t, x) < \frac{h_1(t_2 + \tau_N) - h_1(t_2)}{\tau_N}, \quad ((t_2 + \tau_N), h_1(t_2 + \tau_N)) \in V$$

will be satisfied for sufficiently large N and for (t, x) from a sufficiently small neighbourhood V of (t_2, x_2). Denote $\psi(t)$ the solution of (2.4) with $\psi(t_2 + \tau_N) = h_1(t_2 + \tau_N)$. $\psi(t)$ either leaves V below the curve $h_1(t)$ or we have

$$\psi(t_2) = h_1(t_2 + \tau_N) + \int_{t_2+\tau_N}^{t_2} \min F(t, \psi(t))\, \mathrm{d}t < h_1(t_2 + \tau_N) + l\tau_N < h_1(t_2) = x_2 .$$

In both cases we get a contradiction as $\psi(t)$ is a solution of (2.1) and, therefore, must be contained in $Z(x_0, t_0, t_1)$.

Similarly the second part of the theorem may be proved.

Remark 2.4. Theorem 2.5 may be concluded from the maximum principle of L. S. Pontrjagin ([7]).

Theorem 2.6. *Let $\sum$ be a differentiable $n - 1$-dimensional manifold defined by the equation $s(x) = 0$ in an open domain $V \subset E_n$. Denote $S^+(S^-)$ the set of points of V, for which $s(x) > 0$ $(s(x) < 0)$. Let the multi-valued function $F(t, x)$ satisfy the assumptions of Theorem 2.1 for $(t, x) \in \langle t_1, t_2 \rangle \times V$ and let from $z \in F(t, x)$, $t \in \langle t_1, t_2 \rangle$, $x \in S^+$ follow that*

$$(\operatorname{grad} s(x), z) < 0 . \tag{2.7}$$

Then no solution $x(t)$ of (2.1) leaves $S^- \cup \Sigma$ into S^+ in $\langle t_1, t_2 \rangle$ without leaving V.

Proof. Suppose the contrary. Then there exists a $t_0 \in \langle t_1, t_2 \rangle$ such that $(t_0, x(t_0)) \in \Sigma$, $(t, x(t)) \in S^+$ for $t \in (t_0, t_0 + \tau\rangle$. From (2.7) it follows that

$$s(x(t_0 + \tau)) = s(x(t_0 + \tau)) - s(x(t_0)) = \int_{t_0}^{t_0+\tau} \left(\operatorname{grad} s(x), \frac{\mathrm{d}x\,(t)}{\mathrm{d}t}\right) \mathrm{d}t < 0$$

which contradicts $x(r_0 + \tau) \in S^+$.

3.

Let us have a system of differential equations

$$\frac{dx}{dt} = f(t, x) \tag{3.1}$$

where $f(t, x)$ is defined in some open domain $D \subset E_{n+1}$ with values from E_n. If $f(t, x)$ is not continuous, a solution through every point in the classical sense need not exist. However, it turns out that in real systems which are described by differential systems in an idealised way there appear solutions having the character of "sliding" along the surfaces of discontinuity of f. They may be described mathematically by solutions in a generalized sense. We shall use the notion of solution due to Filippov ([4]).

Definition 3.1. The function $x(t)$ will be said to be a solution of the system (3.1) on the interval $\langle t_1, t_2 \rangle$ if it is absolutely continuous on $\langle t_1, t_2 \rangle$ and satisfies the relations $(t, x(t)) \in D$ for $t \in \langle t_1, t_2 \rangle$

$$\frac{dx}{dt} = \bigcap_{\delta > 0} \bigcap_{\text{mes} N = 0} \text{conv} f(t, V(x(t), \delta) - N)$$

for almost every $t \in \langle t_1, t_2 \rangle$.

Theorem 3.1. *Let $f(x)$ be a measurable function, bounded almost everywhere in a closed domain $D \subset E_n$. The function $x(t)$ is a solution of the differential system*

$$\frac{dx}{dt} = f(x) \tag{3.2}$$

if, and only if it is a solution of the contingens-equation

$$\frac{dx}{dt} \in F(x), \tag{3.3}$$

where

$$F(x) = \bigcap_{\delta > 0} \bigcap_{\text{mes} N = 0} \text{conv} f(V(x, \delta) - N). \tag{3.4}$$

The function $F(x)$ satisfies the assumptions of Theorem 2.1.

Proof. The equivalence of the system (3.2) and the equation (3.3) follows from Theorem 2.2 if we prove that F satisfies the assumptions of this theorem. This follows from

Lemma 3.1. *Let $f(x)$ be measurable and bounded almost everywhere on a measurable set $A \subset E_n$ with positive measure. Denote A_0 the set of points of density of A. The multi-valued function F given by* (3.4) *is defined and β-continuous on A_0 and the sets $F(x)$ are convex for every $x \in A_0$.*

Proof. Denote A_1 the set of the points of asymptotic continuity of F on A. We have $\operatorname{mes}(A - A_0) = 0$, $\operatorname{mes}(A - A_1) = 0$. Denote $N_0 = A - (A_0 \cup A_1)$. For $x \in A_0$, $F(x)$ is a non empty compact set and,

$$F(x) = \bigcap_{\delta>0} \operatorname{conv} f(V(x, \delta) - N_0) = \operatorname{conv} \bigcap_{\delta>0} \overline{f(V(x, \delta) - N_0)}$$

(see [4], Lemma 1).

As from the β-continuity of a multi-valued function $\Phi(x)$ evidently follows the β-continuity of the function $\operatorname{conv} \Phi(x)$, it suffices to prove the β-continuity of the function

$$F_0(x) = \bigcap_{\delta>0} \overline{f(V(x, \delta) - N_0)}$$

on A_0.

Let $x_0 \in A_0$ and suppose that F_0 is not β-continuous at x_0. Then there exists a $\mu > 0$ such that for every $k > 0$ there exists a $x_k \in A_0$, $x_k \in V(x_0, k^{-1})$ such that $F_0(x_k) \not\subset V(F_0(x_0), \mu)$. For k and x_k there exists a $\eta_k > 0$ such that $V(x_k, \eta_k) \subset V(x_0, k^{-1})$ and an y_k such that $y_k \in f(V(x_k, \eta_k) - N_0)$, $\varrho(F_0(x_0), y_k) > \mu - k^{-1}$. The sequence $\{y_k\}$ is bounded and therefore it has a point of accumulation y for which we have

$$y \in \bigcap_{K=1}^{\infty} \overline{\{y_k\}_{k=K}^{\infty}} \subset \bigcap_{k=1}^{\infty} \overline{f(V(x_0, k^{-1}) - N_0)} = \bigcap_{\delta>0} \overline{f(V(x_0, \delta) - F_0(x_0)} = F_0(x_0)\,.$$

There exists a subsequence $\{y_{k_\nu}\}$ of $\{y_k\}$ such that $y_{k_\nu} \to y$. We have

$$\varrho(F_0(x_0), y_{k_\nu}) \leqq \|y_{k_\nu} - y\| + \varrho(F_0(x_0), y)$$

$$\varrho(F_0(x_0), y) \geqq \varrho(F_0(x_0), y_{k_\nu}) - \|y_{k_\nu} - y\| \geqq \mu - k_\nu^{-1} - \|y_{k_\nu} - y\|\,.$$

From $k_\nu^{-1} \to 0$ and $\|y_{k_\nu} - y\| \to 0$ we get $\varrho(F_0(x_0), y) \geqq \mu$ which contradicts $y \in F_0(x_0)$.

4.

In §§ 4–7 we shall consider the special control system

(s_ε)
$$\dot{y}^1 = y^2 + p^1\,,$$
$$\dot{y}^2 = \alpha y^1 + \beta y^2 + u + p^2\,, \quad (p^1, p^2) \in \varepsilon P$$

in vector form

$$\dot{y} = My + e_2 u + p\,,$$

where

$$M = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad p = \begin{pmatrix} p^1 \\ p^2 \end{pmatrix}.$$

In addition to the assumptions imposed on the system (σ_ε) in § 1 we shall assume that

$$\max_{p \in P} |p^1| = 1 . \tag{4.1}$$

We shall further denote

$$l = \max_{p \in P} |p^2| .$$

The contingens-equation, corresponding to (s_ε) will be

$$\dot{y} \in My + e_2\, U(y) + \varepsilon P . \tag{S_ε}$$

Consider the contingens-equations

$$\dot{y} \in My + e_2 + \varepsilon P , \tag{S_ε^+}$$

$$\dot{y} \in My - e_2 + \varepsilon P . \tag{S_ε^-}$$

Lemma 4.1. *For every $\delta \in (0, 1)$ there exists an $\varepsilon(\delta) > 0$ such that every solution $y(t)$ of (S_ε^+) starting at a point $y_0 = (y_0^1, -\varepsilon)$ such that $-\alpha y_0^1 < 1 - \delta$, intersects the line $y^2 = \varepsilon$ and it holds $\dot{y}^2(t) > 0$ as far as $|y^2| \leqq \varepsilon$.*

Proof. Suppose $\alpha > 0$ (the cases $\alpha = 0$, $\alpha < 0$ may be treated similarly). Let $\varepsilon < \delta K^{-1}$ where $K = \max \{4(l + |\beta|), 4(2\alpha)^{\frac{1}{2}}\}$. Denote C_ε the set $|y^2| \leqq \varepsilon$, $y^1 \geqq -\alpha^{-1} + \delta(2\alpha)^{-1}$.

For $y(t) \in C_\varepsilon$ we have

$$\dot{y}^1 = y^2 + p^1 \geqq -2\varepsilon , \tag{4.2}$$

$$\dot{y}^2 = \alpha y^1 + \beta y^2 + 1 + p^2 \geqq \tfrac{1}{2}\delta - (l + |\beta|)\, \varepsilon \geqq \tfrac{1}{4}\delta > 0 . \tag{4.3}$$

Let $y_0^1 > -\alpha^{-1} + \delta\alpha^{-1}$, $y(0) = y_0$. Suppose that $y(t)$ intersects the segment $y^1 = -\alpha^{-1} + \delta \,.\, 2^{-1}\alpha^{-1}$, $|y^2| \leqq \varepsilon$ which we shall denote by Δ_ε. Let t_1 be the least number such that $y(t_1)$ belongs to Δ_ε. Then from (4.2) it follows that $t_1 \geqq \delta(4\alpha\varepsilon)^{-1}$. From (4.3) it follows that

$$y^2(t_1) \geqq -\varepsilon + \frac{\delta^2}{4}(4\alpha\varepsilon)^{-1} \geqq -\varepsilon + \frac{\delta^2}{16\alpha\varepsilon^2}\, \varepsilon > \varepsilon$$

contrary to $y(t_1) \in \Delta_\varepsilon$. Thus, the lemma is proved.

Lemma 4.2. *Let $y(t)$ be a solution of (S_ε^+), $\varepsilon \leqq \varepsilon(\delta)$, starting at a point $y_0 = (y_0^1, -\varepsilon)$, $-\alpha y_0^1 < 1 - \delta$. The function $y^1 = y^1(t^{-1}(y^2))$ [1] is a solution of the*

[1]) By $t^{-1}(y^2)$ we denote the inverse function to $y^2(t)$.

contingens-equation

$$(R_\varepsilon^+) \qquad \frac{dy^1}{dy^2} \in \left\{ \frac{y^2 + p^1}{\alpha y^1 + \beta y^2 + 1 + p^2} = f^+(y, p) : \quad p \in \varepsilon P \right\}$$

for $y^2 \in \langle -\varepsilon, +\varepsilon \rangle$.

Proof. Due to Lemma 4.1, we have $\dot{y}^2(t) > 0$ for $|y^2| \leqq \varepsilon$. Hence, the function $t^{-1}(y^2)$ is defined for $|y^2| \leqq \varepsilon$. Thus, for almost every y^2, $|y^2| \leqq \varepsilon$ we have

$$\frac{dy^1}{dy^2} = \frac{\dot{y}^1}{\dot{y}^2} \in f^+(y, P).$$

Denote

$$m_\varepsilon^+(y) = \min_{p \in \varepsilon P} f^+(y, p) = \min_{p \in \varepsilon P} \frac{y^2 + p^1}{\alpha y^1 + \beta y^2 + 1 + p^2}.$$

Further, denote D_ε the set of points y satisfying the inequality

$$(4.4) \qquad -(\alpha y^1 + \beta y^2) < 1 - l\varepsilon,$$

and $D_{\varepsilon K}$ the set of points satisfying the inequalities

$$(4.5) \qquad -(\alpha y^1 + \beta y^2) \leqq 1 - (l + K^{-1})\,\varepsilon, \ |y^2| \leqq K.$$

Lemma 4.3. 1. *The function $m_\varepsilon^+(y)$ is a continuous function of the variables y^1, y^2, ε in the domain $D = \{(y^1, y^2, \varepsilon) : \varepsilon > 0, y \in D_\varepsilon\}$.*

2. *In the domain $D_{\varepsilon K}$ $m_\varepsilon^+(y)$ satisfies the Lipschitz condition with respect to y^1 with the constant*

$$L = \frac{|\alpha|\,(K + \varepsilon)}{(K^{-1}\varepsilon)^2}.$$

3. *In D_ε a measurable function $p_m(y)$ exists such that*

$$(4.6) \qquad p_m(y) \in \varepsilon P, \quad f^+(y, p_m(y)) = m_\varepsilon^+(y).$$

4. *Let $0 < \varepsilon_1 < \varepsilon_2$. Then $m_{\varepsilon_2}^+(y) < m_{\varepsilon_1}^+(y) < 0$ for $y \in D_\varepsilon$ such that $y^2 < \varepsilon$.*

Proof. 1. Let $(y^1, y^2, \varepsilon) \in D$. Then $m_\varepsilon^+(y)$ is defined. Let $(y_n^1, y_n^2, \varepsilon) \to (y^1, y^2, \varepsilon)$. For sufficiently large n we have evidently $(y_n^1, y_n^2, \varepsilon) \in D$. From $\{m_{\varepsilon_n}^+(y_n)\}$ we may select a subsequence which converges to a (a may be a real number or $\pm\infty$).

Suppose $a > m_\varepsilon^+(y)$. For sufficiently large n we have

$$m_{\varepsilon_n}^+(y_n) > \frac{a + m_\varepsilon^+(y)}{2}.$$

Denote $p_0 \in \varepsilon P$ such, that

$$m_\varepsilon^+(y) = f^+(y, p_0).$$

Evidently

$$f^+\left(y_n, \frac{\varepsilon_n}{\varepsilon} p_0\right) = \frac{y_n^2 + \varepsilon^{-1}\varepsilon_n p_0^1}{\alpha y_n^1 + \beta y_n^2 + 1 + \varepsilon^{-1}\varepsilon_n p_0^2} \to \frac{y^2 + p_0^1}{\alpha y^1 + \beta y^2 + 1 + p_0^2} = m_\varepsilon^+(y)$$

and therefore,

$f^+(y_n, \varepsilon^{-1}\varepsilon_n p_0) < \frac{1}{2}(a + m_\varepsilon^+(y)) < m_{\varepsilon_n}^+(y_n)$ for sufficiently large n. As $\varepsilon_n\varepsilon^{-1}p_0 \in \varepsilon_n P$ we got a contrary to

$$m_{\varepsilon_n}^+(y_n) = \min_{p\in\varepsilon_n P} f^+(y_n, p) .$$

Suppose $a < m_\varepsilon^+(y)$. Denote p_n such a perturbation that

$$f^+(y_n, p_n) = m_{\varepsilon_n}^+(y_n) .$$

From $\{p_n\}$ we select a convergent subsequence $\{p_{n_k}\}$ and denote $\tilde{p} = \lim_{k\to\infty} p_{n_k}$. We have $\tilde{p} \in \varepsilon P$ and

$$f^+(y, \tilde{p}) = \lim_{k\to\infty} f^+(y_{n_k}, p_{n_k}) = a < m_\varepsilon^+(y)$$

contrary to the definition of $m_\varepsilon^+(y)$.

2. Let $(y_1^1, y_1^2), (y_2^1, y_1^2) \in D_{\varepsilon K}$ and let

$$m_\varepsilon^+(y_1^1, y_1^2) \geqq m_\varepsilon^+(y_2^1, y_1^2) .$$

Denote $p_1, p_2 \in \varepsilon P$ such perturbations that

$$f^+(y_1^1, y_1^2, p_1) = m_\varepsilon^+(y_1^1, y_1^2) , \quad f^+(y_2^1, y_1^2, p_1) = m_\varepsilon^+(y_2^1, y_1^2) .$$

Then

$$\left|m_\varepsilon^+(y_1^1, y_1^2) - m_\varepsilon^+(y_2^1, y_1^2)\right| = m_\varepsilon^+(y_1^1, y_1^2) - m_\varepsilon^+(y_2^1, y_1^2) =$$
$$= f_\varepsilon^+(y_1^1, y_1^2, p_1) - f_\varepsilon^+(y_2^1, y_1^2, p_2) \leqq f_\varepsilon^+(y_1^1, y_1^2, p_2) - f_\varepsilon^+(y_2^1, y_1^2, p_2) =$$
$$= \frac{(y_1^2 + p_2^1)\,\alpha(y_2^1 - y_1^1)}{(\alpha y_1^1 + \beta y_1^2 + 1 + p_2^2)(\alpha y_2^1 + \beta y_1^2 + 1 + p_2^2)} \leqq \frac{(K + \varepsilon)\,|\alpha|}{(K^{-1}\varepsilon)^2}\,|y_2^1 - y_1^1| ,$$

which completes the proof.

3. Denote

$$P_m(y) = \{p \in \varepsilon P : f^+(y, p) = m_\varepsilon^+(y)\} .$$

We prove that $P_m(y)$ is a β-continuous multi-valued function in D_ε. The set $P_m(y)$ is evidently compact, convex and non-empty. Hence, due to Remark 2.1 it is sufficient to prove that from $y_n \to y$, $p_n \to \tilde{p}$, $p_n \in P_m(y_n)$ it follows that $\tilde{p} \in P_m(y)$.

Suppose $\tilde{p} \notin P_m(y)$ and let $p_0 \in P_m(y)$. Then $f^+(y, \tilde{p}) > f^+(y, p_0)$. Hence, as $f^+(y, p)$ is evidently continuous, we have for sufficiently large $n, f^+(y_n, p_n) > > f^+(y_n, p_0)$ contrary to the definition of p_n.

Denote

$$p_m(y) = \min \operatorname{lex} P_m(y)$$

i.e.

$$p_m^1(y) = \min_{p \in P_m(y)} p^1\,, \quad p_m^2(y) = \min_{p \in P_m(y),\, p^1 = p_m{}^1(y)} p^2\,.$$

Due to Ważewski's lemma ([2]), $p_m(y)$ is measurable.

4. Due to the assumption (4.1) there exists a number $\bar{p}_2$ such that $(-\varepsilon, \bar{p}^2) \in \varepsilon P$. Denote further $p_1 \in \varepsilon_1 P$ so that $f^+(y, p_1) = m_{\varepsilon_1}^+(y)$. Then for $y^2 < \varepsilon$ we have

$$0 > \frac{y^2 - \varepsilon}{\alpha y^1 + \beta y^2 + 1 + \bar{p}^2} \geqq m_{\varepsilon_1}^+(y) =$$

$$= \frac{y^2 + p_1^1}{\alpha y^1 + \beta y^2 + 1 + p_1^2} > \frac{y^2 + \varepsilon_2 \varepsilon_1^{-1} p_1^1}{\alpha y^1 + \beta y^2 + 1 + \varepsilon_2 \varepsilon_1^{-1} p_1^2} \geqq m_{\varepsilon_2}^+(y)$$

which was to be proved.

Corollary 4.1. Denote $\Gamma_\varepsilon^+(y_0^1)$ the lower boundary of the zone of emission $Z(y_0^1, -\varepsilon, \varepsilon)$ of the equation (R_ε^+). Let $y^1 = \gamma_\varepsilon^+(y^2, y_0^1)$ be its equation. From Theorem 2.6 and Lemma 4.3 it follows that $y^1 = \gamma_\varepsilon^+(y^2, y_0^1)$ is a solution of the differential equation

$$(mR_\varepsilon^+) \qquad \frac{\mathrm{d}y^1}{\mathrm{d}y^2} = m_\varepsilon^+(y)\,.$$

Corollary 4.2. Exactly one solution of the differential equation (mR_ε^+) passes through every point of D_ε.

Corollary 4.3. The solutions of (mR_ε^+) depend continuously on the initial values and on the parameter ε in the domain D. (See [10], chapter I.)

Remark 4.1. The following example shows that the function $m_\varepsilon^+(y)$ need not be differentiable:

Let $\alpha = 0$, $\beta = 0$, $P = \{p : |p^1| \leqq 1, |p^2| \leqq 1\}$. Then

$$m_\varepsilon^+(0, y^2) = \min_{p \in \varepsilon P} \frac{y^2 + p^1}{1 + p^2} = \begin{cases} \dfrac{y^2 - \varepsilon}{1 - \varepsilon}\,, & y^2 < \varepsilon\,, \\[2ex] \dfrac{y^2 - \varepsilon}{1 + \varepsilon}\,, & y^2 > \varepsilon\,, \end{cases}$$

$[\partial m_\varepsilon^+(0, y^2)]/\partial y^2$ evidently does not exist at the point $y^2 = \varepsilon$.

Lemma 4.4. *For sufficiently small $\varepsilon > 0$ there exists exactly one $y_\varepsilon^1 > 0$ such that*

$$(4.7) \qquad \gamma_\varepsilon^+(\varepsilon, y_\varepsilon^1) = -y_\varepsilon^1\,,$$

i.e. the points of intersection of $\Gamma_\varepsilon^+(y_\varepsilon^1)$ with the lines $y^2 = \pm\varepsilon$ are symmetric with respect to the origin. y_ε^1 is a continuous and increasing function of ε and it holds $y_\varepsilon^1 \to 0$ for $\varepsilon \to 0$.

Proof. Let $\varepsilon > 0$ be so small that

a) $4\varepsilon|\alpha| (|\beta| + l)^{-1} < 1 - 2(|\beta| + l)\,\varepsilon$;

b) the function $y^1 = \gamma_\varepsilon^+(y^2, y_0^1)$ satisfies the equation (mR_ε^+) for $|y_0^1| \leqq \leqq 4\varepsilon(|\beta| + l)^{-1}$;

c) the function $m_\varepsilon^+(y)$ is a continuous function of y and satisfies the Lipschitz condition with respect to y^1 for $|y^1| \leqq 4\varepsilon(|\beta| + l)^{-1}$, $|y^2| < 2\varepsilon$.

The validity of the conditions a), b), c) for sufficiently small $\varepsilon > 0$ follows from the Lemmas 4.1, 4.3.

From Lemma 4.3 it follows that

$$\gamma_\varepsilon^+(\varepsilon, 0) < 0\,, \tag{4.8}$$

$$\gamma_\varepsilon^+(\varepsilon, 4\varepsilon(|\beta| + l)^{-1}) < 4\varepsilon(|\beta| + l)^{-1}\,. \tag{4.9}$$

Let $|y_0^1| \leqq 4\varepsilon(|\beta| + l)^{-1}$. Due to a) we have

$$\gamma_\varepsilon^+(\varepsilon, y_0^1) = y_0^1 + \int_{-\varepsilon}^{+\varepsilon} m_\varepsilon^+(\gamma_\varepsilon^+(y^2, y_0^1), y^2)\,\mathrm{d}y^2 \geqq y_0^1 + \frac{-2\varepsilon}{(|\beta| + l)\,\varepsilon}\,2\varepsilon = y_0^1 - \frac{4\varepsilon}{|\beta| + l}\,.$$

Hence

$$\gamma_\varepsilon^+(\varepsilon, 4\varepsilon(|\beta| + l)^{-1}) \geqq 0\,. \tag{4.10}$$

From c) it follows that $\gamma_\varepsilon^+(\varepsilon, y_0^1)$ depends continuously on y_0^1 for $|y_0^1| \leqq \leqq 4\varepsilon(|\beta| + l)^{-1}$. From this, (4.8) and (4.10) we conclude that there exists a $y_\varepsilon^1 \in \in (0, 4\varepsilon(|\beta| + l)^{-1})$ which satisfies (4.7).

Let $y_1^1 > y_\varepsilon^1$. Due to Corollary 4.2 we have $\gamma_\varepsilon^+(\varepsilon, y_1^1) > \gamma_\varepsilon^+(\varepsilon, y_\varepsilon^1) = -y_\varepsilon^1 > -y_1^1$. Similarly it may be proved that $\gamma_\varepsilon^+(\varepsilon, y_1^1) < -y_1^1$ for $y_1^1 < y_\varepsilon^1$. From this the uniqueness of y_ε^1 follows.

Now, let $\varepsilon_n \to \varepsilon$. If $y_{\varepsilon_n}^1$ does not converge to y_ε^1, then there exists a subsequence of $\{y_{\varepsilon_n}^1\}$ which converges to a number $y_0^1 \neq y_\varepsilon^1$. We have

$$-y_{\varepsilon_n}^1 = y_{\varepsilon_n}^1 + \int_{-\varepsilon_n}^{+\varepsilon_n} m_{\varepsilon_n}^+(\gamma_{\varepsilon_n}^+(y^2, y_{\varepsilon_n}^1), y^2)\,\mathrm{d}y^2\,.$$

It follows from Lemma 4.3 and Corollary 4.3 that

$$-y_0^1 = y_0^1 + \int_{-\varepsilon}^{+\varepsilon} m_\varepsilon^+(\gamma_\varepsilon^+(y^2, y_0^1), y^2)\,\mathrm{d}y^2\,,$$

i.e.

$$\gamma_\varepsilon^+(\varepsilon, y_0^1) = -y_0^1\,,$$

contrary to the uniqueness of y_ε^1.

Let $\varepsilon' > \varepsilon$ and suppose $y^1_{\varepsilon'} \leqq y^1_\varepsilon$. Denote $y^1 = \eta(y^2)$ the solution of (mR^+_ε) with $\eta(-\varepsilon') = y^1_{\varepsilon'}$. From Lemma 4.3 we conclude $\eta(-\varepsilon) < y^1_\varepsilon$. Hence

$$-y^1_{\varepsilon'} = \gamma^+_{\varepsilon'}(\varepsilon, y^1_{\varepsilon'}) < \eta(\varepsilon) \leqq \gamma^+_\varepsilon(\varepsilon, \eta(-\varepsilon)) < \gamma^+_\varepsilon(\varepsilon, y^1_\varepsilon) = -y^1_\varepsilon \leqq -y^1_{\varepsilon'}$$

which is impossible. Thus, we have $y^1_{\varepsilon'} > y^1_\varepsilon$.

From the continuity of $m^+_\varepsilon(y)$ it follows that for $\varepsilon < \varepsilon_0$ it holds $|m^+_\varepsilon(\gamma^+_\varepsilon(y^2), y^2)| \leqq \leqq M$. Hence

$$|2y^1_\varepsilon| \leqq \int_{-\varepsilon}^{+\varepsilon} |m^+_\varepsilon(\gamma^+_\varepsilon(y^2), y^2)| \, dy^2 \leqq 2M\varepsilon .$$

Thus

$$\lim_{\varepsilon \to 0} y^1_\varepsilon = 0 .$$

Further, we shall denote briefly $\Gamma^+_\varepsilon(y^1_\varepsilon) = \Gamma^+_\varepsilon$ and $\gamma^+_\varepsilon(y^2, y^1_\varepsilon) = \gamma^+_\varepsilon(y^2)$. By the transformation $y = -z$ we get from (S^+_ε) the equation (S^-_ε); similarly we get from the equation (R^+_ε) the equation

$$(R^-_\varepsilon) \qquad \frac{dz^1}{dz^2} \in \left\{ f^-(z, p) = \frac{z^2 + p^1}{\alpha z^1 + \beta z^2 - 1 + p^2} : \; p \in \varepsilon P \right\}$$

and from (mR^+_ε) the equation

$$(mR^-_\varepsilon) \qquad \frac{dz^1}{dz^2} = \min \frac{z^2 + p^1}{\alpha z^1 + \beta z^2 - 1 + p^2} = m^-_\varepsilon(z) .$$

Hence, the solutions of the equations $(S^-_\varepsilon), (R^-_\varepsilon), (mR^-_\varepsilon)$ are symmetric with respect to the origin to the solutions of $(S^+_\varepsilon), (R^+_\varepsilon), (mR^+_\varepsilon)$, respectively and the solution $\gamma^-_\varepsilon(y^2, y^1_0) = -\gamma^+_\varepsilon(-y^2, y^1_0)$ of the equation (mR^-_ε) is the lower boundary of the zone of emission $Z(-y^1_0, \varepsilon, -\varepsilon)$ of the equation (R^-_ε). The graph Γ^-_ε of the function $\gamma^-_\varepsilon(y^2) = \gamma^-_\varepsilon(y^2, -y^1_\varepsilon)$ intersects Γ^+_ε at the points $y_\varepsilon = (-\varepsilon, y^1_\varepsilon)$, $-y_\varepsilon = (\varepsilon, -y^1_\varepsilon)$.

We shall prove that $\Gamma^+_\varepsilon, \Gamma^-_\varepsilon$ have no common points such that $|y^2| < \varepsilon$.

Denote $s_0 = \inf_{y \in B_\varepsilon} m^+_\varepsilon(y)$, where $B_\varepsilon = D_\varepsilon \cap \{y : |y^2| \leqq \varepsilon\}$ and D_ε is given by (4.4).

Lemma 4.5. *Let s be a given number, $s_0 < s < 0$. Then, the set of points $y \in B_\varepsilon$ such that $y \in B_\varepsilon$, $m^+_\varepsilon(y) = s$ is an intersection of a straight line with B_ε.*

Proof. Consider the expression

$$m^+_\varepsilon(y^1, -\varepsilon) = \min_{p \in \varepsilon P} \frac{-\varepsilon + p^1}{\alpha y^1 - \beta\varepsilon + 1 + p^2}$$

for $(y^1, -\varepsilon) \in B_\varepsilon$. Evidently

$$(4.11) \qquad m^+_\varepsilon(y^1, -\varepsilon) \to 0 \quad \text{for} \quad \alpha y^1 - \beta\varepsilon \to \infty .$$

We prove that

$$m_\varepsilon^+(y^1, -\varepsilon) \to s_0 \quad \text{for} \quad \alpha y^1 - \beta\varepsilon \to -1 + l_\varepsilon . \tag{4.12}$$

Let $y_1 \in B_\varepsilon$. For y^1 satisfying the inequality

$$-1 + l_\varepsilon < \alpha y^1 - \beta_\varepsilon \leqq \alpha y_1^1 + \beta y_1^2$$

we have

$$m_\varepsilon^+(y^1, -\varepsilon) = \min_{p\varepsilon P} \frac{-\varepsilon + p^1}{\alpha y^1 - \beta\varepsilon + 1 + p^2} \leqq \frac{-\varepsilon + p_m^1(y_1)}{\alpha y^1 - \beta\varepsilon + 1 + p_m^2(y_1)} \leqq$$
$$\leqq \frac{y_1^2 + p_m^1(y_1)}{\alpha y_1^1 + \beta y_1^2 + 1 + p_m^2(y_1)} = m_\varepsilon^+(y_1) ;$$

herefrom we get (4.12) ($p_m(y)$ is given by (4.6)).

Due to the continuity of $m_\varepsilon^+(y^1, -\varepsilon)$ it follows from (4.11), (4.12) that there exists a point $y_s = (y_s^1, -\varepsilon)$ such that $m_\varepsilon^+(y_\varepsilon^1, -\varepsilon) = s$.

Let

$$y^2(1 - s\beta) < s\alpha(y^1 - y_s^1) + \varepsilon(s\beta - 1) . \tag{4.13}$$

Then

$$\varepsilon + y^2 < s[\alpha(y^1 - y_s^1) + \beta(y^2 + \varepsilon)] . \tag{4.14}$$

Simultaneously,

$$-\varepsilon + p_m^1(y_s) = s(\alpha y_s^1 - \beta\varepsilon + 1 + p_m^2(y_s)) . \tag{4.15}$$

Adding (4.15) to (4.14) we get

$$y^2 + p_m^1(y_s) < s[\alpha y^1 + \beta y^2 + 1 + p_m^2(y_s)] ,$$

$$f^+(y, p_m^1(y_s)) = \frac{y^2 + p_m^1(y_s)}{\alpha(y^1 - y_s^1) + \beta y^2 + \alpha y_s^1 + 1 + p_m^2(y_s)} < s .$$

Hence,

$$m_\varepsilon^+(y) = \min_{p\varepsilon P} f^+(y, p) < s .$$

Now, let

$$y^2(1 - s\beta) > s\alpha(y^1 - y_s^1) + \varepsilon(s\beta - 1) , \tag{4.16}$$

i.e.

$$y^2 > s[\alpha(y^1 - y_s^1) + \beta(y^2 + \varepsilon)] - \varepsilon . \tag{4.17}$$

Suppose

$$m_\varepsilon^+(y) \leqq s . \tag{4.18}$$

i.e. $$\frac{y^2 + p_m^1(y)}{\alpha y^1 + \beta y^2 + 1 + p_m^2(y)} \leqq s .$$

Due to (4.17) we have

$$s\alpha y^1 - s\alpha y_s^1 + s\beta y^2 + s\beta\varepsilon - \varepsilon + p_m^1(y) < s(\alpha y^1 + \beta y^2 + 1 + p_m^2(y)) ,$$
$$-\varepsilon + p_m^1(y) < s(\alpha y_s^1 - \beta\varepsilon + 1 + p_m^2(y)) ,$$
$$\frac{p_m^1(y) - \varepsilon}{\alpha y_s^1 - \beta\varepsilon + 1 + p_m^2(y)} = f^+(y_s, p_m(y)) < s$$

contrary to the definition of y_s. Hence, from (4.16) it follows $m_\varepsilon^+(y) > s$, and from (4.13), $m_\varepsilon^+(y) < s$, for $y \in B_\varepsilon$. As $m_\varepsilon^+(y)$ is continuous, $m_\varepsilon^+(y) = s$ for $y \in B_\varepsilon$ if and only if

(4.19) $$y^2(1 - s\beta) = s\alpha(y^1 - y_s^1) + \varepsilon(s\beta - 1)$$

which is an equation of a straight line.

Lemma 4.6

$$\gamma_\varepsilon^+(y^2) < -y_\varepsilon^1\varepsilon^{-1}y^2 \quad for \quad |y^2| < \varepsilon .$$

Proof. Suppose the contrary. Denote T the open segment $y^1 = -y_\varepsilon^1\varepsilon^{-1}y^2$, $|y^2| < \varepsilon$. We have

(4.20) $$\left.\frac{d\gamma_\varepsilon^+(y^2)}{dy^2}\right|_{y^2=\varepsilon} = m_\varepsilon^+(-y_\varepsilon^1) = \min_{p\in P} \frac{\varepsilon + p^1}{-\alpha y_\varepsilon^1 + \beta\varepsilon + 1 + p^2} = 0 ,$$

and therefore, $\gamma_\varepsilon^+(y^2) < -\varepsilon y_\varepsilon^1 y^2$ for y^2 sufficiently near to ε. Hence, there exists a common point y_1 of Γ_ε^+ and T with the largest second coordinate. Evidently,

(4.21) $$m_\varepsilon^+(y^1) = \frac{d\gamma_\varepsilon^+(y^2)}{dy^2} \leqq -\varepsilon^{-1}y_\varepsilon^1 .$$

As Γ_ε^+ has with T also the point $+y_\varepsilon$ in common, there exists a point $y_2 \in T$, $y_2^2 \leqq y_1^2$ such that

(4.22) $$m_\varepsilon^+(y_2) \geqq -\varepsilon^{-1}y_\varepsilon^1 .$$

From (4.20), (4.21), (4.22) and the continuity of $m_\varepsilon^+(y)$ the existence of at least two points of the segment T follows such that

(4.23) $$m_\varepsilon^+(y) = -\varepsilon^{-1}y_\varepsilon^1 .$$

From this due to Lemma 4.5 it follows that (4.23) holds for each $y \in T$. This contradicts (4.20). This completes the proof.

As

$$\gamma_\varepsilon^-(y^2) = -\gamma_\varepsilon^+(-y^2)\,,$$

we have

$$\gamma_\varepsilon^-(y^2) > -\varepsilon^{-1}y_\varepsilon^1 y^2\,.$$

Hence, $\Gamma_\varepsilon^+ \cup \Gamma_\varepsilon^-$ is the boundary of a compact connected set G_ε containing the origin in its interior and such that

$$\gamma_\varepsilon^+(y^2) \leqq y^1 \leqq \gamma_\varepsilon^-(y^2)$$

for every point $(y^1, y^2) \in G_\varepsilon$.

Denote Ψ_ε^- the graph of the function $y^1 = \psi_\varepsilon^-(y^2)$ defined and continuous for $y^2 \in \langle \varepsilon, \varepsilon + \eta \rangle$, $\eta > 0$ and satisfying the conditions $\psi_\varepsilon^-(\varepsilon) = -y_\varepsilon^1$, $\psi_\varepsilon^-(y^2) = -y_\varepsilon^1$ for $y^2 > \varepsilon$. Further denote Ψ_ε^+ the curve symmetric to Ψ_ε^- with respect to the origin.

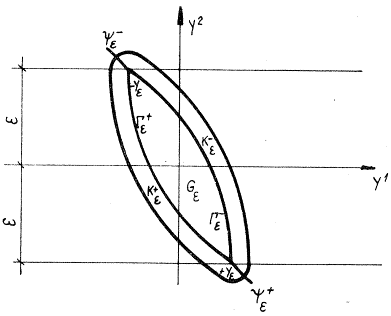

Fig. 1.

Denote

$$K_\varepsilon = V(G_\varepsilon, \eta) - (G_\varepsilon \cup \Psi_\varepsilon^+ \cup \Psi_\varepsilon^-)\,.$$

The curves Ψ_ε^+, Ψ_ε^- divide K_ε into two parts. Denote K_ε^+ the part of K_ε to the boundary of which Γ_ε^+ belongs and K_ε^- the other part. (Fig. 1.)

We define the control function $u_\varepsilon(y)$ as follows

$$u_\varepsilon(y) = \begin{cases} +1 & \text{for} \quad y \in K_\varepsilon^+\,, \\ -1 & \text{for} \quad y \in K_\varepsilon^-\,. \end{cases}$$

Outside K_ε the function u_ε may be defined arbitrarily.

In the following paragraph we shall prove that u_ε is the best ε-stabilizing control for the given class of perturbations in the sense of the Euclidean norm with the minimal (u, ε)-invariant region G_ε.

5.

Theorem 5.1. *u_ε is an ε-stabilizing control and G_ε is an $(u_\varepsilon, \varepsilon)$-invariant region for sufficiently small $\varepsilon > 0$.*

Proof. It follows from Theorem 2.4 that a solution of (S_ε) starting in G_ε cannot leave G_ε in a point distinct from the points $\pm y_\varepsilon$.

Suppose that there exists a solution $\tilde{y}(t)$ which leaves G_ε through the point $-y_\varepsilon$ Let $\tilde{y}(t_0) = -y_\varepsilon$, $y(t) \notin G_\varepsilon$ for $t \in (t_0, t_1\rangle$, $t_1 > t_0$. There are three possibilities:

a) $\tilde{y}(t) \in K_\varepsilon^+$ for $t \in (t_0, t_0 + s)$, $s > 0$.
b) $\tilde{y}(t) \in K_\varepsilon^-$ for $t \in (t_0, t_0 + s)$, $s > 0$.
c) There exists a decreasing sequence $\{t_n\}$, $t_n \to t_0$, $\tilde{y}(t_n) \in \Psi^-$.

In the case a) $\tilde{y}(t)$ satisfies for $t \in (t_0, t_0 + s)$ the equation (S_ε^+) and, therefore, $y^1(t^{-1}(y^2))$ is a solution of (R_ε^-). But for $y^2 \geqq \varepsilon$, $p \in \varepsilon P$ we have

$$f^+(y, p) \geqq \frac{y^2 - \varepsilon}{\alpha y^1 + \beta y^2 + 1 + l\varepsilon} \geqq 0 .$$

From this due to the properties of Ψ_ε^- we get that the zone of emission $Z(-y_\varepsilon^1, \varepsilon, \varepsilon + \eta)$, $\eta > 0$ of (R_ε^+) is contained in K_ε^-. Hence, a) is impossible.

In the case b) $y(t)$ is for $t \in (t_0, t_0 + s)$ a solution of (S_ε^-) and, hence, $\tilde{y}^2(t)$ is decreasing. From this it follows that for t sufficiently near to t_0, $\tilde{y}(t)$ is contained in the zone of emission $Z(-y_\varepsilon^1, \varepsilon, \varepsilon - \eta)$, $\eta > 0$ of (R_ε^-). But from Theorem 2.4 it follows that $Z(-y_\varepsilon^1, \varepsilon, \varepsilon - \eta)$ cannot contain points from K_ε^-. Hence, b) is also impossible.

In the case c) there exists a $s > 0$ such that $\tilde{y}(t) \in V(-y_\varepsilon, \eta)$ for $t \in \langle t_0, t_0 + s)$ and such $t_N, t_{N+1} \in \langle t_0, t_0 + s)$ that

$$\tilde{y}^1(t_{N+1}) > \tilde{y}^1(t_N) . \tag{5.1}$$

If such t_N, t_{N+1} would not exist, it would be $\tilde{y}^1(t_{n+1}) \leqq \tilde{y}^1(t_n)$ for $n \geqq n_0$ sufficiently large. From this it would follow $\tilde{y}^1(t_n) \leqq \tilde{y}^1(t_{n_0})$ for $n \geqq n_0$. But $\tilde{y}(t_{n_0})$ belongs to Ψ_ε^- and is distinct from $-y_\varepsilon$; thus it must be $\tilde{y}^1(t_{n_0}) < -y_\varepsilon^1$. From this it follows that $\tilde{y}(t)$ could not be continuous at t_0.

For $t \in \langle t_{N+1}, t_N \rangle$ we have $\tilde{y}(t) \in H = \{y : y^1 \leqq -y_\varepsilon^1,\ y^2 \geqq \varepsilon\}$. To prove this, observe that as $\tilde{y}(t) \in V(-y_\varepsilon, \eta)$, $y(t) \notin G_\varepsilon$ for $t \in \langle t_{N+1}, t_N \rangle$, $\tilde{y}(t)$ can leave H only by intersecting one of the open segments $T_1 = \{y : y^2 = \varepsilon,\ -y_\varepsilon^1 - \eta < y < -y_\varepsilon^1\}$, $T_2 = \{y : y^1 = -y_\varepsilon^1,\ \varepsilon < y^2 < \varepsilon + \eta\}$. But from Theorem 2.6 it follows that $\tilde{y}(t)$ cannot leave H through T_1. If $\tilde{y}(t)$ would leave H through T_2, it had to return into H through T_2, which leads to a contradiction with Theorem 2.6 again.

As $\tilde{y}(t) \in H$ for $t \in \langle t_{N+1}, t_N \rangle$, we have $\dot{\tilde{y}}^1(t) = \tilde{y}^2 + p^1 \geqq \tilde{y}^2 - \varepsilon \geqq 0$ for $t \in \langle t_{N+1}, t_N \rangle$. Thus, (5.1) is impossible. Hence, c) is also impossible, which completes the proof, as at y_ε the situation is symmetric.

Theorem 5.2. *u_ε is the best ε-stabilizing control and G_ε is the minimal (u, ε)-invariant region in the sense of the Euclidean norm, for sufficiently small $\varepsilon > 0$.*

For the proof we shall need two lemmas.

Lemma 5.1. *Let $G \subset D_\varepsilon$ be an (u, ε)-invariant region. Denote $G' = G \cap \{y : |y^2| \leqq \varepsilon\}$. Then from $y_1 \in G'$, $y_1^1 = \min_{y \in G'} y^1$, $y^2 \in G'$, $y_2^1 = \max_{y \in G'} y^1$ it follows $y_1^2 = \varepsilon$, $y_2^2 = -\varepsilon$.*

Proof. Suppose $y_1^2 < \varepsilon$. Let $y(t_0) = y_1$. There exists such a t_1 that for every solution $y(t)$ starting at y_1 for $t = t_0$ we have $y^2(t) < \varepsilon$ for $t \in \langle t_0, t_1 \rangle$. With a perturbation $p(t)$ such that $p^1(t) = -\varepsilon$ for $t \in \langle t_0, t_1 \rangle$ it will be

$$y^1(t_1) = y^1(t_0) + \int_{t_0}^{t_1} (y^2 - \varepsilon)\,\mathrm{d}t < y^1(t_0)$$

contrary to the assumption. Similarly the second part of the lemma may be proved.

Lemma 5.2. *Let $G \subset D_\varepsilon$ be an (u, ε)-invariant region. Then G contains either a point y_1 with $y_1^2 = \varepsilon$, $y_1^1 \leqq -y_\varepsilon^1$ or a point y_2 with $y_2^2 = -\varepsilon$, $y_2^1 \geqq y_\varepsilon^1$.*

Proof. Due to Lemma 5.1 there exists a point $y_2 \in G$ with $y_2^2 = -\varepsilon$. Suppose $y_2^1 < y_\varepsilon^1$. Let the function $p_m(y)$ be defined by (4.3) Consider the contingens-equations

$$\dot{y} \in My + e_2 J + p_m(y) = F_1(y)\,, \tag{5.2}$$

$$\dot{y} \in My + e_2\, U(y) + p_m(y) = F_2(y)\,, \tag{5.3}$$

where $J = \langle -1, +1 \rangle$.

The right-hand sides of the equations (5.2), (5.3) are evidently β-continuous in y, and every solution of (5.3) is a solution of (S_ε). As $F_2(y) \subset F_1(y)$ due to Theorem 2.3, every solution of (5.3) is a solution of (5.2). From (4.1) the existence of a number p_ε^2 follows such that $(-\varepsilon, p_\varepsilon^2) \in \varepsilon P$.

We have evidently

$$f_\varepsilon^+(y, p_m(y)) \leqq f_\varepsilon^+(y, (-\varepsilon, p_\varepsilon^2)) = \frac{y^2 - \varepsilon}{\alpha y^1 + \beta y^2 + 1 + p_\varepsilon^2}$$

for $y \in D_\varepsilon$. From this it follows $\dot{y}^1(t) = y^2 + p_m^1(y) < 0$ for $y \in D$, $y^2 < \varepsilon$.

From this we conclude similarly as in Lemma 4.2 that if $y(t)$ is a solution of (5.2), the function $y^2(t^{-1}(y^1))$ satisfies the contingens-equation

$$\frac{\mathrm{d}y^2}{\mathrm{d}y^1} \in \frac{\alpha y^1 + \beta y^2 + J + p_m^2(y)}{y^2 + p_m^1(y)}\,. \tag{5.4}$$

Due to the Remark 2.4, the upper boundary of the zone of emission $Z(-\varepsilon, y_2^1, y^1)$, $y^1 < y_2^1$ of the equation (5.4) satisfies the differential equation

$$\frac{\mathrm{d}y^2}{\mathrm{d}y^1} = \min_{v \in \langle -1, +1 \rangle} \frac{\alpha y^1 + \beta y^2 + v + p_m^2(y)}{y^2 + p_m^1(y)} = \frac{\alpha y^1 + \beta y^2 + 1 + p_m^2(y)}{y^2 + p_m^1(y)} \tag{5.5}$$

as far as $y \in D_\varepsilon$, $y^2 < \varepsilon$. Now, it is easy to see that $y^1(t^{-1}(y^2))$ is also a solution of the equation (mR_ε^+). Thus, due to Corollary 4.2, its graph is identical with $\Gamma_\varepsilon^+(y_2^1)$. From Corollary 4.2 it follows further that $\Gamma_\varepsilon^+(y_2^1)$ and Γ_ε^+ cannot have points in common. Hence, for the point of intersection y_3 of $\Gamma_\varepsilon^+(y_2^1)$ with the line $y^2 = \varepsilon$ it holds $y_3^1 < -y_\varepsilon^1$.

Due to Theorem 2.1 there exists a solution $y(t)$ of (5.3) through y_2. As for $y \in D_\varepsilon$ we have $\dot{y}^1(t) = y^2(t) + p_m^1(y(t))$, there are only the following possibilities:

1. $y(t)$ leaves D_ε,
2. $y^1(t) \to -\infty$ for $t \to \infty$,
3. $y^2(t) \to \varepsilon$ for $t \to \infty$ or $y^2(T) = \varepsilon$ for some T.

In the first case G does not satisfy the condition $G \subset D_\varepsilon$ and in the second one G is not an (u, ε)-invariant region. In the third case G contains a point y_1 on the line $y^2 = \varepsilon$ which is contained in the zone of emission $Z(-\varepsilon, y_2^1, y^1)$, $y^1 < y_1^1$ of (5.3) and therefore also in that of (5.2). Hence, for y_1 it holds $y_1^1 < -y_\varepsilon^1$. This completes the proof.

The proof of Theorem 5.2 is now simple enough. It follows from Lemma 5.1 that G_ε is contained in the rectangle formed by the lines $y^1 = \pm y_\varepsilon^1$, $y^2 = \pm\varepsilon$. From this it follows

$$\|G_\varepsilon\| = \|y_\varepsilon\| . \tag{5.6}$$

Due to Lemma 4.4 we have $y_\varepsilon \to 0$ for $\varepsilon \to 0$. As $D_{\varepsilon_1} \subset D_{\varepsilon_2}$ for $\varepsilon_1 > \varepsilon_2 > 0$, for sufficiently small $\varepsilon > 0$ it holds

$$\|y_\varepsilon\| < \min_{y \notin D_\varepsilon} \|y\| .$$

From Lemma 5.2 it follows that every (u, ε)-invariant set contains a point y_0 with either $y_0 \notin D_\varepsilon$ or $|y_0^2| = \varepsilon$, $|y_0^1| \geqq y_\varepsilon^1$. In both cases we have $\|y_0\| > \|y_\varepsilon\|$ and due to (5.6) also

$$\|G\| \geqq \|y_0\| \geqq \|y_\varepsilon\| = \|G_\varepsilon\| ,$$

which completes the proof.

Theorem 5.3. *For $\varepsilon > 0$ sufficiently small G_ε is the unique minimal (u, ε)-invariant region in the sense of the Euclidean norm.*

Proof. It suffices to prove that if $G \neq G_\varepsilon$ is an (u, ε)-invariant region contained in $D_\varepsilon \cap \{y : |y^2| \leqq \varepsilon\}$, then G contains either a point y_1 with $y_1^1 < -y_\varepsilon^1$, $y_1^1 = \varepsilon$ or a point y_2 with $y_2^1 > y_\varepsilon^1$, $y_2^2 = -\varepsilon$.

As $G \neq G_\varepsilon$, it contains either a point with $y^1 < \gamma_\varepsilon^+(y^2)$ or a point with $y^1 > \gamma_\varepsilon^-(y^2)$. Similarly as in Lemma 5.2 it may be proved that in the first case G contains a point y_1 with $y_1^1 < -y_\varepsilon^1$, $y_1^2 = \varepsilon$ and in the second one a point y_2 with $y_2^1 > +y_\varepsilon^1$, $y_2^2 = -\varepsilon$.

6.

Theorem 6.1. *Let u_ε be for $\varepsilon \leqq \varepsilon_0$ the best ε-stabilizing control with the minimal (u, ε)-invariant region G_ε in the sense of the Euclidean norm. Let for each $\varepsilon_1 \leqq \varepsilon_2 \leqq \varepsilon_0$ hold $y_{\varepsilon_1} \in G_{\varepsilon_2}$ and let $G_{\varepsilon_0} \subset D_{\varepsilon_0}$. Denote*

$$\varphi(\varepsilon) = -y_\varepsilon^1,\ \varepsilon > 0\,,\quad \varphi(0) = 0\,,\quad \varphi(-\varepsilon) = -\varphi(\varepsilon)\,, \tag{6.1}$$

(6.2) $$\tilde{u}(y) = \begin{cases} +1 & \text{for} \quad y \in G_{\varepsilon 0},\ y^1 < \varphi(y^2)\,, \\ -1 & \text{for} \quad y \in G_{\varepsilon 0},\ y^1 > \varphi(y^2)\,. \end{cases}$$

Then $\tilde{u}(y)$ is a ε_0-universal best ε-stabilizing control in the sense of the Euclidean norm. Moreover, for every solution $y(t)$ of (S_ε) starting in $G_{\varepsilon 0}$ we have

(6.3) $$\lim_{t \to \infty} \varrho(G_\varepsilon, y(t)) = 0\,.$$

Remark 6.1. The assumption $y_{\varepsilon_1} \in G_{\varepsilon_2}$ for $\varepsilon_1 \leqq \varepsilon_2 \leqq \varepsilon_0$ is not always satisfied even for ε_0 small as it will be seen from Example 9.1.

Proof. Denote Φ the graph of the function φ, further Φ^+ its part for $y^2 \leqq 0$ and Φ^- its part for $y^2 \geqq 0$. From Lemma 4.4 it follows that $\varphi(y^2)$ is continuous and decreasing for $|y^2| \leqq \varepsilon_0$.

The ε_0-universality of $\tilde{u}$ follows from the fact that in the neighbourhood of every point $\pm y_\varepsilon \Phi$ satisfies the conditions imposed on Ψ_ε.

For the proof of the second part of the theorem denote

$$L_\varepsilon^+ = \{y : y \in G_{\varepsilon_0} - G_\varepsilon,\ y^1 < \varphi(y^2)\}\,,$$
$$L_\varepsilon^- = \{y : y \in G_{\varepsilon_0} - G_\varepsilon,\ y^1 > \varphi(y^2)\}\,.$$

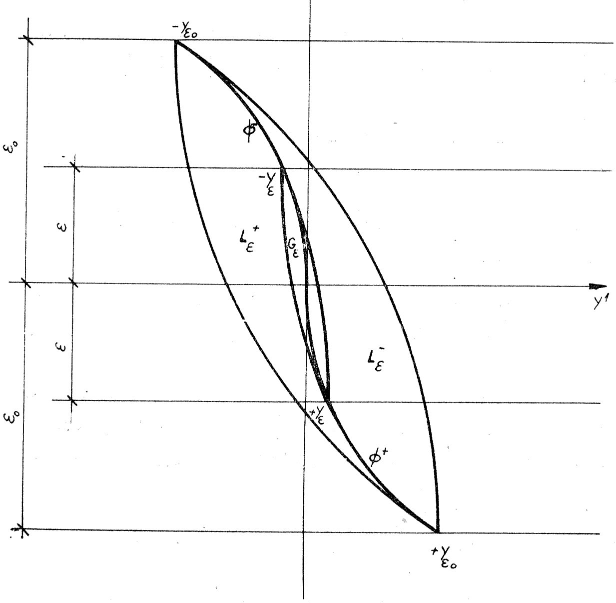

Fig. 2.

Consider a solution $y(t)$ of (S_ε) such that $y(0) \in L_\varepsilon^+$ and suppose that $y(t)$ does not enter into G_ε in a finite time.

$y(t)$ cannot enter into L_ε^- by intersecting Φ^+. Suppose the contrary. Then there exists a t_0 such that $y(t_0) \in \Phi^+$ and $y(t) \in L_\varepsilon^-$ for $t \in (t_0, t_0 + \tau\rangle$, $\tau > 0$. If $\tau > 0$ is sufficiently small, we have

$$\dot{y}^2(t) = \alpha y^1 + \beta y^2 - 1 + p^2 < 0$$

for $t \in \langle t_0, t_0 + \tau \rangle$.

Thus, $y^2(t) < \varepsilon$ for $t \in \langle t_0, t_0 + \tau \rangle$. From this it follows that

$$\dot{y}^1(t) = y^2(t) + p^1 \leqq y^2 + \varepsilon < 0 \quad \text{for} \quad t \in \langle t_0, t_0 + \tau \rangle\,.$$

This is impossible, as $y(t_0 + \tau) \in L_\varepsilon^-$ and φ is decreasing.

If $y \in L_\varepsilon^+$ and $y^1 > y_\varepsilon^1$, then $y^2 < -\varepsilon$ and, therefore,

$$\dot{y}^1(t) \leqq y^2(t) + \varepsilon < 0 . \tag{6.4}$$

Hence, either $y(t) \to y_\varepsilon$ or $y(t)$ leaves Φ^+. In the second case we have $\dot{y}^2 = \alpha y^1 + \beta y^2 + 1 + p^2 > c > 0$ as far as $y \in L_\varepsilon^+$. $y(t)$ is evidently also a solution of $(S_{\varepsilon 0})$ and, therefore, it cannot leave $G_{\varepsilon 0}$. Due to the assumption it cannot enter into G_ε, hence, it intersects Φ^-.

By repeating this consideration and a similar consideration for L_ε^- we conclude that either $y(t)$ converges to one of the points y_ε, $-y_\varepsilon$ or it continues in passing from L_ε^+ to L_ε^- and conversely till infinity and "spirals" around G_ε. As in the first case (6.3) is evidently satisfied, it suffices to consider the second case.

Let $T^+ = \{t : y(t) \in \Phi^+\}$, $T^- = \{t: y(t) \in \Phi^-\}$. The sets T^+, T^- are closed and divide each other into a sequence of compact sets T_n^+, T_n^- $n = 1, 2, 3, \ldots$ such that between any two points of T_n there does not lie any point from T^- and conversely, between any two points of T_n^- there does not lie any point from T^+. Denote $t_{2n-1} = \min T_n^-$, $\tau_{2n-1} = \max T_n^-$, $t_{2n} = \min T_n^+$, $\tau_{2n} = \max T_n^+$, $n = 1, 2, 3, \ldots$ The sequences $\{t_n\}$ and $\{\tau_n\}$ are evidently increasing. From (6.4) it follows that $y^1(\tau_{2n}) \leqq y^1(t_{2n})$ and therefore, as φ is decreasing,

$$y^2(\tau_{2n}) \geqq y^2(t_{2n}), \quad n = 1, 2, 3, \ldots \tag{6.5}$$

Similarly we have

$$y^2(\tau_{2n-1}) \leqq y^2(t_{2n-1}), \quad n = 1, 2, 3, \ldots \tag{6.6}$$

Consider the solution $y^1 = \eta(y^2, y_0)$ of (mR_ε^+) through a point $y_0 \in \Phi$, $y_0^2 < -\varepsilon$. Denoting $\varepsilon' = y_0^2$ we have $y_0 = y_{\varepsilon'}$. The solution of (mR_ε^+) through $y_0 = y_{\varepsilon'}$ is the function $\gamma_{\varepsilon'}^+(y^2)$ which intersects Φ^- at $-y_{\varepsilon'}$.

From Lemma 4.3 it follows that

$$\eta(y^2, y_0) > \gamma_{\varepsilon'}^+(y^2) \quad \text{for} \quad y^2 > y_0^2 . \tag{6.7}$$

Hence, $\eta(y^2, y_0)$ intersects Φ^- sooner than $\gamma_\varepsilon^+(y^2)$, i.e. if we denote $\bar{y}$ the point of intersection of $\eta(y^2, y_0)$ with Φ^-, then we have

$$\bar{y}^2 < \varepsilon' = y_0^2 . \tag{6.8}$$

Denote $\varepsilon_n = y^2(\tau_{2n})$. Then from (6.5), (6.6), (6.7) and from the similar relation for the solutions of (mR_ε^-) it follows that $\{\varepsilon_n\}$ is a decreasing sequence. Denote its limit by ε^*. Evidently, $\varepsilon \leqq \varepsilon^*$.

Suppose $\varepsilon^* > \varepsilon$. From Corollary 4.3 it follows that the solution of (mR_ε^+) through $y_{\varepsilon*}$ intersects Φ^- in the point $-y_{\varepsilon*}$ which contradicts (6.8). Thus, $\varepsilon^* = \varepsilon$.

From Corollary 4.3 it follows further,

$$\lim_{t \to \infty} \max_{y^2} \varrho(G_\varepsilon, (\eta(y^2, y_{\varepsilon_n}), y^2)) = 0 . \tag{6.9}$$

The function $y^1 = y^1(t^{-1}(y^2))$ is for $t \in \langle \tau_{2n}, t_{2n+1} \rangle$ a solution of (R_ε^+). As $y^1 = \eta(y^2, y_{\varepsilon 2n})$ is the lower boundary of the zone of emission $Z(y^1_{\varepsilon 2n}, \varepsilon_{2n}, y^2(t_{2n+1}))$ of (R_ε^+), $y(t)$ is contained for $t \in \langle \tau_{2n}, t_{2n+1} \rangle$ in a domain, bounded by curves Γ_ε^+, Φ^+, Φ^- and the graph of $y^1 = \eta(y^2, y_{\varepsilon 2n})$. From this it follows that

$$\varrho(G_\varepsilon, y(t)) \leqq \max_{y^2} \varrho(G_\varepsilon, (\eta(y^2, y_{\varepsilon 2n}), y^2) \quad \text{for} \quad t \in \langle \tau_{2n}, t_{2n+1} \rangle .$$

Similarly we have

$$\varrho(G_\varepsilon, y(t)) \leqq \max_{y^2} \varrho(G_\varepsilon) (\eta(y^2) \, y_{\varepsilon 2n-1}), p^2) \text{ for } t \in \langle \tau_{2n-1}, t_{2n} \rangle$$

From this and (6.9) we get

$$\lim_{t \to \infty} \varrho(G_\varepsilon, y(t)) = 0 .$$

7.

Consider now an arbitrary norm $|y|$ in E_2. We shall choose the best ε-stabilizing control in the sense of this norm in the special system (S_ε).

It appears that it is necessary to distinguish two cases according to whether a point of minimal norm on the line $y^2 = 1$ is in the right halfplane or not. In the first case the control u_ε is also the best ε-stabilizing control in the sense of $|y|$ for sufficiently small $\varepsilon > 0$. In the second case the best ε-stabilizing control has to be constructed in another form.

Theorem 7.1. *Let a point $y_0 = (y_0^1, 1)$ exist such that*

(7.1) $$|y_0| = \min_{y^1} |(y^1, 1)| , \quad y_0^1 \geqq 0 .$$

Then u_ε is for sufficiently small $\varepsilon > 0$ the best ε-stabilizing control in the sense of $|y|$ and G_ε is the minimal (u, ε)-invariant region. If a point $y_1 = (y_1^1, 1)$ such that

(7.2) $$|y_1| = \min_{y^1} |(y^1, 1)| , \quad y_1^1 < 0$$

does not exist, then for sufficiently small $\varepsilon > 0$, G_ε is the unique minimal (u, ε)-invariant region.

Remark 7.1. The fact that u_ε is the best ε-stabilizing control in the sense of the Euclidean norm for some ε is not sufficient for u_ε to be the best ε-stabilizing control in the sense of $|y|$; we state this only for ε sufficiently small. The reason lies in the fact that while (5.6) is satisfied provided u_ε and G_ε can be constructed, the equality $|y_\varepsilon| = |G_\varepsilon|$ need not yet be satisfied.

For the proof of Theorem 7.1 we shall need three lemmas.

Lemma 7.1.

$$\lim_{\varepsilon \to 0} \frac{d\gamma_\varepsilon^+(y^2)}{dy^2} = 0 \tag{7.3}$$

uniformly for $|y^2| \leqq \varepsilon$ *and* $y_\varepsilon^1 = o(\varepsilon)$, *i.e.*

$$\lim_{\varepsilon \to 0} \frac{y_\varepsilon^1}{\varepsilon} = 0 . \tag{7.4}$$

Proof. For sufficiently small $\varepsilon > 0$ we have

$$|\alpha\gamma_\varepsilon^+(y^2) + \beta y^2 + 1 + p^2| > \tfrac{1}{2} \quad \text{for} \quad p \in \varepsilon P .$$

Hence,

$$\left|\frac{d\gamma_\varepsilon^+(y^2)}{dy^2}\right| = |m_\varepsilon^+(y)| \leqq 2(|y^2| + \varepsilon) \leqq 4\varepsilon , \tag{7.5}$$

from which we get (7.3). From (7.5) it follows that

$$2y_\varepsilon^1 = -\int_{-\varepsilon}^{+\varepsilon} \frac{d\gamma_\varepsilon^+(y^2)}{dy^2}\, dy^2 \leqq \int_{-\varepsilon}^{+\varepsilon} 4\varepsilon \, dy^2 \leqq 8\varepsilon^2$$

from which we get (7.4).

Lemma 7.2. *Let* $y^2 = \varphi(y^1)$ *be a convex*[1] *continuous function on* $\langle a, b\rangle$ *and let* $a < c < b$. *Then a number* $K > 0$ *exists such that for every* $y_1^1, y_2^1 \in \langle c, b\rangle$, $y_1^1 \leqq y_2^1$ we have

$$\varphi(y_1^1) - \varphi(y_2^1) \leqq K(y_2^1 - y_1^1) .$$

Proof. Suppose the contrary. Then a sequence $\{y_n^1\}$ exists such that $y_{2n}^1 > y_{2n-1}^1$ and

$$\varphi(y_{2n-1}^1) - \varphi(y_{2n}^1) \geqq n(y_{2n}^1 - y_{2n-1}^1)$$

for $n = 1, 2, 3, \ldots$ Denote $\mu = \min\limits_{y^1 \in \langle a,b\rangle} \varphi(y^1)$. From the convexity of $\varphi(y^1)$ we conclude

$$\varphi(a) \geqq \varphi(y_{2n}^1) + \frac{a - y_{2n}^1}{y_{2n-1}^1 - y_{2n}^1}(\varphi(y_{2n-1}^1) - \varphi(y_{2n}^1)) \geqq \mu - n(a - y_{2n}^1) \geqq \mu + n(c - a)$$

for an arbitrary n, which is impossible.

Lemma 7.3. *Let* $|y|$ *be an arbitrary norm in* E_2. *Then for sufficiently small* $\varepsilon > 0$,

$$|G_\varepsilon| = |y_\varepsilon| . \tag{7.6}$$

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[1]) The function $\varphi(y^1)$ is said to be convex on $\langle a, b\rangle$ if $\varphi(\frac{1}{2}(y_1^1 + y_1^2)) \leqq \frac{1}{2}(\varphi(y_1^1) + \varphi(y_1^2))$ for arbitrary $y_1^1, y_2^1 \in \langle a, b\rangle$.

Proof. Denote

$$S_\eta = \{y : |y| \leqq \eta\},$$

$$s_\eta^+(y^1) = \max\{y^2 : |(y^1, y^2)| = 1\}, \quad s_\eta^-(y^1) = \min\{y^2 : |(y^1, y^2)| = 1\}.$$

The set S_1 is convex ([11], II. 4.1) and contains the origin in its interior. From this it follows that a $h > 0$ exists such that the function $y^2 = s_1^-(y^1)$ is defined, continuous and convex on $\langle -2h, 2h\rangle$. Due to Lemma 7.2 a $K > 0$ exists such that for every $y_1^1, y_2^1 \in \langle -h, h\rangle$, $y_2^1 \geqq y_1^1$,

$$s_1^-(y_1^1) - s_1^-(y_2^1) \leqq K(y_2^1 - y_1^1). \tag{7.7}$$

From the homogenity of the norm it follows that $s_\eta^-(y^1)$ is defined on $\langle -2\eta h, 2\eta h\rangle$ and

$$s_\eta^-(y^1) = \eta s_1^-(\eta^{-1}y^1).$$

From this and (7.7) we have for every $y_1^1, y_2^1 \in \langle -\eta h, \eta h\rangle$, $y_2^1 \geqq y_1^1$,

(7.8)

$$s_\eta^-(y_1^1) - s_\eta^-(y_2^1) = \eta s_1^-(\eta^{-1}y_1^1) - \eta s_1^-(\eta^{-1}y_2^1) \leqq \eta K(\eta^{-1}y_2^1 - \eta^{-1}y_1^1) = K(y_2^1 - y_1^1)$$

with $K > 0$ independent on η.

From Lemma 7.1 it follows that

$$\varepsilon|(0,1)| = |(0,\varepsilon)| = |(-y_\varepsilon^1, \varepsilon) - (-y_\varepsilon^1, 0)| \leqq |y_\varepsilon| + |(-y_\varepsilon^1, 0)| = |y_\varepsilon| + o(\varepsilon),$$

$$|y_\varepsilon| = \varepsilon|(0,1)| + o(\varepsilon),$$

$$|y_\varepsilon|^{-1}\, y_\varepsilon^1 = \frac{y_\varepsilon^1}{\varepsilon|(0,1)| + o(\varepsilon)} \to 0 \quad \text{for} \quad \varepsilon \to 0. \tag{7.9}$$

Denote $\eta = |y_\varepsilon|$. From (7.9) we conclude that for $\varepsilon > 0$ sufficiently small it holds

$$y_\varepsilon^1 < \eta h \tag{7.10}$$

and

$$\min_{|y^2| \leqq \varepsilon} \frac{\mathrm{d}\gamma_\varepsilon^+(y^2)}{\mathrm{d}y^2} > -\frac{1}{K} \tag{7.11}$$

From (7.11) it follows that

$$\gamma_\varepsilon^+(y^2) - y_\varepsilon^1 = \int_{-\varepsilon}^{y^2} \frac{\mathrm{d}\gamma_\varepsilon^+(y^2)}{\mathrm{d}y^2}\,\mathrm{d}y^2 \geqq -\int_{-\varepsilon}^{y^2} K^{-1}\,\mathrm{d}y^2 = -K^{-1}(y^2 + \varepsilon). \tag{7.12}$$

Due to Lemma 4.4 we have

$$\frac{\mathrm{d}\gamma_\varepsilon^+(y^2)}{\mathrm{d}y^2} = m_\varepsilon^+(\gamma_\varepsilon^+(y^2), y^2) < 0 \quad \text{for} \quad y^2 < \varepsilon.$$

From this it follows that on $\langle -y_\varepsilon^1, y_\varepsilon^1 \rangle$ there exists a function $y^2 = (\gamma_\varepsilon^+)^{-1}(y^1)$ inverse to the function $y^1 = \gamma_\varepsilon^+(y^2)$, for which due to (7.12) we have

$$(\gamma_\varepsilon^+)^{-1}(y^1) + \varepsilon \geqq K(y_\varepsilon^1 - y^1).$$

From (7.8) and (7.10) we conclude that for $y^1 \in \langle -y_\varepsilon^1, y_\varepsilon^1 \rangle$,

$$s_\eta^-(y^1) - s_\eta^-(y_\varepsilon^1) \geqq K(y_\varepsilon^1 - y^1).$$

As $|y_\varepsilon| = \eta$, we have $-\varepsilon \geqq s_\eta^-(y_\varepsilon^1)$ and therefore $s_\eta^-(y^1) + \varepsilon \leqq K(y_\varepsilon^1 - y^1)$, from which we get

$$(\gamma_\varepsilon^+)^{-1}(y^1) \geqq s_\eta^-(y^1). \tag{7.13}$$

Similarly, it may be proved that

$$(\gamma_\varepsilon^-)^{-1}(y^1) \leqq s_\eta^+(y^1). \tag{7.14}$$

Let now $y \in G_\varepsilon$. Then $y^1 \in \langle -y_\varepsilon^1, y_\varepsilon^1 \rangle$, $(\gamma_\varepsilon^+)^{-1}(y^1) \leqq y^2 \leqq (\gamma_\varepsilon^-)^{-1}(y^1)$ from which due to (7.14), (7.15) we have $y \in S_\eta$. Thus $|y| \leqq \eta = |y_\varepsilon|$, from which we get (7.6).

Proof of Theorem 7.1. Let $\varepsilon > 0$ be so small that (7.6) is valid. From the homogenity of the norm we have

$$|\varepsilon y_0| = \min_{y^1} |(y^1, \varepsilon)|. \tag{7.15}$$

Let $y_1 = (y_1^1, \varepsilon)$ be an arbitrary point such that $y_1^1 \leqq -y_\varepsilon^1$. Then $-y_\varepsilon$ belongs to the segment with the endpoints y_0, y_1 and, therefore,

$$-y_\varepsilon = \lambda y_1 + (1 - \lambda)\, y_0, \quad 0 < \lambda \leqq 1.$$

Hence

$$|y_\varepsilon| = |-y_\varepsilon| \leqq \lambda|y_1| + (1 - \lambda)\,|y_0| \leqq \lambda|y_1| + (1 - \lambda)\,|y_1| = |y_1|. \tag{7.16}$$

Similarly, for an arbitrary point $y_2 = (y_\varepsilon^1, -\varepsilon)$, $y_\varepsilon^1 \geqq y_\varepsilon^1$ we have

$$|y_\varepsilon| \leqq |y_2|. \tag{7.17}$$

Due to Lemma 5.2 every (u, ε)-invariant region G contains either a point $y_1 = (y_1^1, \varepsilon)$ such that $y_1^1 \leqq -y_\varepsilon^1$ or a point $y_2 = (y_2^1, -\varepsilon)$ such that $y_2^1 \geqq y_\varepsilon^1$. In both cases we have from (7.6), (7.16) and (7.17), $|G| \geqq |G_\varepsilon|$ i.e. G is the minimal (u, ε)-invariant region.

If (7.2) is valid, then for $y_1^1 < -y_\varepsilon^1$, $y_2^1 > y_\varepsilon^1$ the strong inequalities (7.16), (7.17) hold and thus the uniqueness of the minimal invariant region G_ε follows from the fact that it is the only region having no other points than $\pm y_\varepsilon$ in common with the lines $y^2 = \pm\varepsilon$ (see the proof of Theorem (5.3)).

Corollary 7.1. Let ε_0 satisfy the assumptions of Theorem 6.1 and let (7.6) hold for $\varepsilon \leqq \varepsilon_1$. Denote $\varepsilon_2 = \min\{\varepsilon_0, \varepsilon_1\}$. Then the control $\tilde{u}(y)$, defined by (6.2) is the ε_2-universal best ε-stabilizing control in the sense of the norm $|y|$.

Let us now suppose that the given norm does not satisfy (7.1), i.e. there exists a point $y_0 = (y_0^1, 1)$ such that

$$(7.18) \qquad |y_0| = \min_{y^1} |(y^1, 1)|, \quad y_0^1 < 0,$$

and that for every $y^1 > y_0^1$

$$(7.19) \qquad |(y^1, 1)| > |y_0|$$

(by y_0 we denote the point with the minimal norm and maximal first coordinate on the line $y^2 = 1$).

From (7.18), (7.19) due to the homogenity of the norm we have

$$(7.20) \qquad |\varepsilon y_0| = \min_{y^1} |(y^1, \varepsilon)|, \; |(y^1, \varepsilon)| > \varepsilon |y_0| \quad \text{for} \quad y^1 > \varepsilon y_0^1 .$$

For the proof of the minimality of G_ε we have essentially used the fact that for sufficiently small $\varepsilon > 0$ (7.6), (7.16) and (7.17) was valid.

For sufficiently small $\varepsilon > 0$ we have, due to (7.4),

$$(7.21) \qquad -y_\varepsilon^1 > \varepsilon y_0^1 .$$

From (7.20) it follows that for $\varepsilon > 0$ satisfying (7.21) the inequalities (7.16) and (7.17) will not be valid and thus the region G_ε need not be minimal. However, we shall prove that there exists an ε-stabilizing control u'_ε with an $(u'_\varepsilon, \varepsilon)$-invariant region G'_ε having exactly the points $\pm\varepsilon y_0$ in common with the lines $y^2 = \pm\varepsilon$ and satisfying for sufficiently small $\varepsilon > 0$ the relation

$$|G'_\varepsilon| = |\varepsilon y_0| .$$

From Lemma 5.2 and (7.20) it follows that G'_ε will be the minimal (u, ε)-invariant region.

Theorem 7.2. *Let $y_0 = (y_0^1, 1)$ satisfy the condition* (7.18). *Then*

$$(7.22) \qquad u'(y) = \begin{cases} +1 & \text{for} \quad y^1 < y_0^1 y^2 , \\ -1 & \text{for} \quad y^1 > y_0^1 y^2 , \end{cases}$$

is the best ε-stabilizing control in the sense of $|y|$ for $\varepsilon > 0$ sufficiently small.

Proof. Let $\varepsilon < \varepsilon(\delta)$ for $\delta = 1 + \alpha\varepsilon y_0^1$ (see Lemma 4.1). Denote Θ the segment with the endpoints $-\varepsilon y_0$, $+\varepsilon y_0$. We shall distinguish two cases according to the validity of the inequalities

$$(7.23) \qquad m_\varepsilon^+(-\varepsilon y_0) \geqq y_0^1 ,$$

$$(7.24) \qquad m_\varepsilon^+(-\varepsilon y_0) < y_0^1 .$$

If (7.23) is valid, then at every interior point y of Θ the inequality

$$m_\varepsilon^+(y) > y_0^1 \tag{7.25}$$

is true. To prove this observe that

$$m_\varepsilon^+(\varepsilon y_0) = \min_{p\in\varepsilon P} \frac{\varepsilon + p^1}{\alpha\varepsilon y_0^1 + \beta\varepsilon + 1 + p^2} = 0 > y_0^1 \tag{7.26}$$

as the denominator of the minimized fraction is positive and $p^1 \geqq -\varepsilon$. If at some interior point of Θ (7.25) would fail to hold, then due to the continuity of $m_\varepsilon^+(y)$ there would exist an interior point y' of Θ such that

$$m_\varepsilon^+(y') = y_0^1 \,. \tag{7.27}$$

From the continuity of $m_\varepsilon^+(y)$, (7.23), (7.26) and (7.27) it follows that there exist at least two points of Θ with the same value of $m_\varepsilon^+(y)$ and thus, due to Lemma 4.5, $m_\varepsilon^+(y)$ is constant on Θ. This contradicts (7.23) and (7.27).

From the symmetry of $m_\varepsilon^+(y)$, $m_\varepsilon^-(y)$ we have

$$m_\varepsilon^-(y) > y_0^1 \tag{7.28}$$

at every interior point y of Θ.

From (7.25), (7.28) due to Theorem 2.6 it follows that no solution of (S_ε) can leave Θ in its interior point. Similarly as in Theorem 5.1 it may be proved that any solution of (S_ε) cannot leave Θ even through the points $\pm\varepsilon y_0$. Hence, under the assumption (7.23), Θ is a (u', ε)-invariant region. In this case, we denote $G_\varepsilon' = \Theta$. (Fig. 3a).)

Suppose now, that (7.24) holds. Then for y^2 sufficiently near to $-\varepsilon$ we have $\gamma_\varepsilon^+(y^2, y_0^1) < y^2$. However, there exists an y^2, $|y^2| < \varepsilon$ such that $\gamma_\varepsilon^+(y^2, y_0^1) = y_0^1 y^2$ (i.e. $\Gamma_\varepsilon^+(-y_0^1)$ crosses Θ). This follows from the fact that due to Corollary 4.2 we have $\gamma_\varepsilon^+(\varepsilon, -\varepsilon y_0^1) > -y_\varepsilon^1$.

Denote y_1 the common point of $\Gamma_\varepsilon^+(-y_0^1)$ and Θ with the least second coordinate. At y_1 we have evidently $m_\varepsilon^+(y^1) \geqq y_0^1$. Hence, from (7.24) it follows that between points $-\varepsilon y_0$ and y_1 there exists a point $y' \in \Theta$ at which (7.27) is true. From Lemma 4.5 and (7.26) it follows that (7.27) may be true at most at one point of Θ. From this we conclude that between points y_1 and εy_0 there cannot exist any point in common of $\Gamma_\varepsilon^+(-y_0^1)$ and Θ. Further we get that in the interior of the segment with endpoints y_1, εy_0 (7.25) is valid. From this it follows that a solution $y(t)$ of (S_ε) cannot leave the segment with endpoints y_1, εy_0 into the halfplane $y^1 < y_0^1 y^2$. From the symmetry of the curves $\Gamma_\varepsilon^+(-y_0^1)$, $\Gamma_\varepsilon^-(y_0^1)$ and of the functions $m_\varepsilon^+(y)$, $m_\varepsilon^-(y)$ it follows that $\Gamma_\varepsilon^-(y_0^1)$ crosses Θ at a unique point $-y_1$ and no solution of (S_ε^-) can leave the segment with the endpoints $-y_1$, $-\varepsilon y_0$ at its interior point into the halfplane $y^1 > y_0^1 y^2$. As $\Gamma_\varepsilon^+(-y_0^1)$, $\Gamma_\varepsilon^-(y_0^1)$ are the lower boundaries of the zones of emission $Z(-y_0^1, -\varepsilon, \varepsilon)$ of (R_ε^+) and $Z(y_0^1, -\varepsilon, \varepsilon)$ of (R_ε^-) respectively, we conclude from this that the region bounded by the curve $\Gamma_\varepsilon^+(-y_0^1)$ between points $-\varepsilon y_0$, y_1, the curve

$\Gamma_\varepsilon^-(y_0^1)$ between points εy_0, $-y_1$ and the segment Θ is $(u'_\varepsilon, \varepsilon)$-invariant. This region we denote by G'_ε. That $y(t)$ cannot leave G'_ε at points $\pm\varepsilon y_0$ may be proved similarly as in Theorem 5.1. (Fig. 3b),c).)

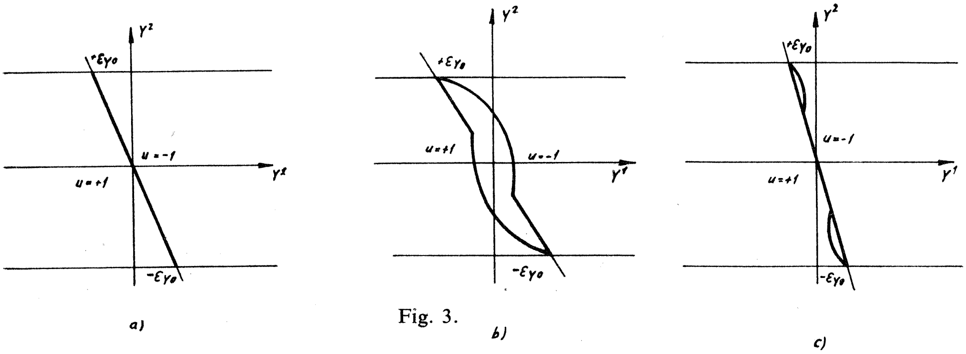

Fig. 3.

From Lemma 7.1 it follows that for sufficiently small $\varepsilon > 0$ we have (7.23), and therefore, G'_ε will be identical with Θ. Then evidently $|G'_\varepsilon| = |\varepsilon y_0|$, which due to Lemma 5.1 and (7.18) implies the minimality of the region G'_ε. Thus, the ε-stabilizing control u' is best in the sense of the norm $|y|$.

Remark 7.2. For the proof of Theorem 7.3 alone it was not necessary to prove that u' is ε-stabilizing provided (7.24) is valid, as for sufficiently small $\varepsilon > 0$ (7.23) holds. However, a following proposition, which we shall need in Theorems 7.3, 7.4, is true:

If $|G'_{\varepsilon_1}| = |\varepsilon_1 y_0|$, then

$$(7.29)\qquad |G'_\varepsilon| = |\varepsilon y_0|\,, \quad \text{for} \quad \varepsilon \leqq \varepsilon_1\,.$$

In order to prove this observe that for $\varepsilon \leqq \varepsilon_1$ G'_ε is contained in the region bounded by the segment Θ, the solutions of the equations (mR_ε^+), (mR_ε^-) starting at $-\varepsilon_1 y_0$, $+\varepsilon_1 y_0$, respectively, and that this region is contained in G'_{ε_1}. Hence

$$\varepsilon^{-1}\varepsilon_1 G'_\varepsilon \subset G'_{\varepsilon_1}\,, \quad |\varepsilon^{-1}\varepsilon_1 G'_\varepsilon| \leqq |G_{\varepsilon_1}| = |\varepsilon_1 y_0|\,, \quad |G'_\varepsilon| \leqq |\varepsilon y_0|\,.$$

As $\varepsilon y_0 \in G'_\varepsilon$, we have (7.29).

Remark 7.3. Similarly as the control u_ε also u' might be arbitrary except for a set of exterior points of G'_ε from an arbitrary small neighbourhood of it.

Remark 7.4. Evidently, the minimal (u, ε)-invariant region G'_ε is not unique.

Theorem 7.3. *Let $|y|$ satisfy (7.18), (7.19). Let u' be ε-stabilizing for $\varepsilon \leqq \varepsilon_0$ and $|G_{\varepsilon_0}| = |\varepsilon_0 y_0|$. Then, u' is a ε_0-universal best ε-stabilizing control in the sense of $|y|$, and for every solution of (S_ε) starting at G'_{ε_0} we have*

$$\lim_{t\to\infty} \varrho(G'_\varepsilon, y(t)) = 0\,.$$

Proof. The first part of the theorem is evident. The second one may be proved similarly as in Theorem 6.1.

Theorem 7.4. Let $|y|$ satisfy (7.18), (7.19). Let u_ε be ε-stabilizing with the $(u_\varepsilon, \varepsilon)$-invariant region G_ε for $\varepsilon \leqq \varepsilon_0$. Let ε_1 be such a number that

$$-y_{\varepsilon_1} = \varepsilon_1 y_0^1, \; -y_\varepsilon^1 \geqq \varepsilon y_0 \text{ for } \varepsilon \leqq \varepsilon_1. \tag{7.30}$$

Let

$$|y_\varepsilon| = |G_\varepsilon| \quad \text{for} \quad \varepsilon \in \langle \varepsilon_1, \varepsilon_0 \rangle\,, \tag{7.31}$$

$$-y_{\varepsilon 0}^1 \leqq \varepsilon_0 y_0\,, \quad G_{\varepsilon 0} \subset D_{\varepsilon 0}\,, \tag{7.32}$$

$$y_{\varepsilon'} \in G_\varepsilon \quad \text{for} \quad \varepsilon_1 \leqq \varepsilon' \leqq \varepsilon \leqq \varepsilon_0\,. \tag{7.33}$$

Denote

$$\vartheta(\varepsilon) = \begin{cases} -y_\varepsilon^1 & \text{if} \quad -y_\varepsilon^1 \leqq \varepsilon y_0^1\,, \\ \varepsilon y_0^1 & \text{if} \quad -y_\varepsilon^1 \geqq \varepsilon y_0^1\,. \end{cases}$$

$$\vartheta(-\varepsilon) = -\vartheta(\varepsilon) \quad \text{for} \quad \varepsilon > 0\,.$$

Then the control

$$u^*(y) = \begin{cases} +1 & \text{for} \quad y^1 < \vartheta(y^2)\,, \\ -1 & \text{for} \quad y^1 > \vartheta(y^2)\,, \end{cases}$$

is an ε_0-universal best ε-stabilizing control in the sense of $|y|$ with the minimal (u, ε)-invariant region

$$G_\varepsilon^* = \begin{cases} G_\varepsilon & \text{if} \quad -y_\varepsilon^1 \leqq \varepsilon y_0^1\,, \\ G'_\varepsilon & \text{if} \quad -y_\varepsilon^1 > \varepsilon y_0^1\,, \end{cases}$$

and for every solution $y(t)$ of (S_ε) starting in $G_{\varepsilon 0}$ it holds

$$\lim_{t \to \infty} \varrho(G_\varepsilon^*, y(t)) = 0\,.$$

Proof. From Theorems 5.1, 5.2 it follows that u_ε is the best ε-stabilizing control for ε satisfying the inequality

$$-y_\varepsilon^1 \leqq \varepsilon y_0^1\,, \tag{7.34}$$

as $|y_\varepsilon| = |G_\varepsilon|$ and $\vartheta(y^2)$ satisfies the conditions imposed on $\psi^-(y^2)$ in the neighbourhood of every point $-y_\varepsilon$ satisfying (7.34). From (7.30) and (7.33) it follows that the graph of the function $y^1 = \vartheta(y^2)$, $|y^2| \leqq \varepsilon$ is contained in G_ε^*. From (7.32) it follows that for every $\varepsilon \in (0, \varepsilon_0)$ satisfying $-y_\varepsilon^1 \geqq \varepsilon y_0^1$ there exists such an $\varepsilon' \geqq \max\{\varepsilon, \varepsilon_1\}$ that $-y_{\varepsilon''}^1 \geqq \varepsilon'' y_0^1$ for $\varepsilon'' \in \langle \varepsilon, \varepsilon' \rangle$ and $-y_{\varepsilon'}^1 = \varepsilon' y_0$. As $|G_{\varepsilon'}^*| = |G'_{\varepsilon'}| = |G_{\varepsilon'}|$ we have also $|G'_\varepsilon| = |\varepsilon' y_0|$ and due to Remark 7.2,

$$|G'_\varepsilon| = |\varepsilon y_0| \quad \text{for} \quad \varepsilon \leqq \varepsilon'\,. \tag{7.35}$$

From Theorem 6.1 and Remark 7.3 it follows that u^* is ε-stabilizing and (7.31), (7.35) imply that it is the best for every $\varepsilon \leqq \varepsilon_0$. The remaining part of the theorem may again be proved similarly as in Theorem 6.1.

8.

Before proceeding to the general system we shall investigate the system (s_ε) if (4.1) is not satisfied.

Theorem 8.1. *Let in the system (s_ε) be $P = \{p : p^1 = 0, |p^2| \leqq l\}$. Then a control $u(y)$ exists such that for sufficiently small $\varepsilon > 0$ we have:*

1. *The only solution starting at $y(t_0) = 0$ is the trivial solution.*
2. *There exists such a neighbourhood $V(0, \delta)$, $\delta > 0$ of the origin that for every solution $y(t)$ starting in $V(0, \delta)$ we have $\lim_{t\to\infty} (t) = 0$.*

Proof. Denote

$$u(y) = \begin{cases} +1 & \text{for} \quad y^1 < -y^2 \,, \\ -1 & \text{for} \quad y^1 > -y^2 \,. \end{cases}$$

We shall prove that $u(y)$ has the desired properties.

In Lemmas 4.1–4.5 (with the exception of point 4 of Lemma 4.3 which will not be used) only the inequality $\max_{p\in P} |p^1| \leqq 1$ from the assumption (4.1) was used. Hence, they are valid also under the assumptions of this theorem.

From Lemmas 4.1, 4.2 and 4.3 it follows that for sufficiently small $\varepsilon > 0$ an $\eta > 0$ exists such that if $y(t)$ is a solution of (S_ε^+) and $y(t) \in V(0, \eta)$, then $y^1(t^{-1}(y^2))$ is a solution of (R_ε^+) and the lower boundary of the zone of emission $Z(y_0^1, y_0^2, y_0^2 + h)$, $h > 0$ of (R_ε^+) is a solution of (mR_ε^+) for $y_0 \in V(0, \eta)$ and $h > 0$ sufficiently small.

We have $m_\varepsilon^+(0) = m_\varepsilon^-(0) = 0$; as $m_\varepsilon^+(y)$ is continuous, in a sufficiently small neighbourhood of the origin we have

$$m_\varepsilon^+(y) > -1 \,, \quad m_\varepsilon^-(y) > -1 \,. \tag{8.1}$$

Suppose that $\eta > 0$ is so small that in $V(0, \eta)$ (8.1) is valid. Denote T the line $y^1 = -y^2$, T_η the part of T belonging to $V(0, \delta)$. From (8.1) due to Theorem 2.6 it follows that $y(t)$ cannot leave T at any point from T_η.

Let $y(t_0) \in T_\eta$. Then $\dot{y}^1(t) = y^2(t) = -y^1(t)$ as far as $y(t) \in T_\eta$.

From this it follows that

$$y^1(t) = y^1(t_0)\, e^{-(t-t_0)} \quad \text{as far as} \quad y(t) \in T_\eta \,. \tag{8.2}$$

From (8.2) it follows that if $y^1(t_0) = 0$ then $y^1(t) = 0$ for $t \geqq t_0$, i.e. also $y^2(t) = 0$ for $t \geqq t_0$. Hence, the first part of the theorem is proved.

As $y(t)$ cannot leave T in any point from T_η, it follows from (8.2) that if $y(t) \in T_\eta$, $t \geqq t_0$, then $y^1(t)$ is decreasing as far as $y^1(t) > 0$ and increasing as far as $y^1(t) < 0$. Hence, (8.2) is valid for every $t \geqq t_0$ and thus $y^1(t) \to 0$. From this it follows that $y^2(t) = -y^1(t) \to 0$ and, hence, $y(t) \to 0$.

Let $\delta \in (0, \frac{1}{2}\eta)$ be so small that $\delta \leqq 4^{-1}(1 - \varepsilon l)$ and $|\alpha y^1 + \beta y^2| \leqq 2^{-1}(1 - \varepsilon l)$ for $|y^1| \leqq 2\delta$, $|y^2| \leqq 2\delta$. Consider a solution $y(t)$ of (S_ε) with $y(t_0) \in V(0, \delta)$, $y^2 < -y^1$. Then $y(t)$ is a solution of (S_ε^+) in the region

$$L = \{y : |y^1| < 2\delta\,, \quad -2\delta < y^2 < -y^1\}\,,$$

and therefore,

$$\dot{y}^1(t) = y^2(t) \geqq -\delta\,, \tag{8.3}$$

$$\dot{y}^2(t) = \alpha y^1(t) + \beta y^2(t) + 1 + p^2 \geqq 2^{-1}(1 - \varepsilon l)\,. \tag{8.4}$$

Suppose that $y(t)$ does not intersect T_η. From (8.4) it follows that $y^2(t)$ is increasing in L and, therefore, a $t_1 > t_0$ exists such that $y^1(t_1) = -2\delta$, $y^2(t_1) < 2\delta$ fir $y(t) \in L$. From (8.3) it follows that

$$\delta \leqq y^1(t_0) - y^1(t_1) = -\int_{t_0}^{t_1} \dot{y}^1(t)\,dt \leqq \delta(t_1 - t_0)\,.$$

Hence $t_1 - t_0 \geqq 1$. From this and (8.4) we get

$$y^2(t_1) = y^2(t_0) + \int_{t_0}^{t_1} \dot{y}^2(t)\,dt \geqq -\delta + \int_{t_0}^{t_1} 2^{-1}(1 - \varepsilon l)\,dt \geqq -\delta + 2^{-1}(1 - \varepsilon l) \geqq 2\delta$$

contrary to the assumption.

Hence, every solution of (S_ε), starting in a point $y_0 \in V(0, \delta)$, $y_0^1 < -y_0^2$ intersects T_η. From the symmetry of the equations (S_ε^+), (S_ε^-) we conclude that every solution of (S_ε) starting in a point $y_0 \in V(0, \delta)$, $y_0^1 < -y_0^2$ intersects T_η. As for every solution starting in T_η we have $y(t) \to 0$, we have $y(t) \to 0$ also for every solution starting in $V(0, \delta)$.

Consider now the control system (σ_ε) or the corresponding contingens-equation (Σ_ε). Denote

$$C = \begin{pmatrix} c_1^1, c_2^1 \\ c_1^2, c_2^2 \end{pmatrix},$$

$$c_1^1 = a_2^1 b^2 - a_2^2 b^1\,, \quad c_1^2 = -a_1^1 b^2 + a_1^2 b^1\,, \quad c_2^1 = b^1\,, \quad c_2^2 = b^2\,.$$

We have

$$\begin{aligned} -\det C &= (-a_2^1 b^2 + a_2^2 b^1)\, b^2 - (a_1^1 b^2 + a_1^2 b^1)\, b^1 = \\ &= b^1(a_1^2 b^1 + a_2^2 b^2) - b^2(a_1^1 b^1 + a_2^1 b^2) \end{aligned}$$

which is the value of the determinant with the columns b, Ab, which is non-zero according to assumption 4, § 1. Hence, C is nonsingular.

It may easily be verified that

$$C^{-1}AC = M = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}, \quad C^{-1}b = e_2 ,$$

where $\alpha = -\det A$, $\beta = \operatorname{sp} A = a_1^1 + a_2^2$.

Suppose that there exists such a $r \in R$ for which

$$(8.5) \qquad r^2 b^1 - r^1 b^2 \neq 0 .$$

Denote $p = C^{-1}r$. From (8.5) it follows that

$$(8.6) \qquad p^1 = \frac{r^1 c_2^2 - r^2 c_2^1}{\det C} = \frac{r^1 b^2 - r^2 b^1}{\det C} \neq 0 .$$

As R is symmetrical with respect to the origin, the same is true about $C^{-1}R$. From (8.6) we have

$$\max_{p \in C^{-1}R} |p^1| = \lambda > 0 .$$

Denote

$$P = \lambda^{-1} C^{-1} R , \quad \varepsilon' = \lambda \varepsilon .$$

Then $\max_{p \in P} |p^1| = 1$ and the linear transformation $x = Cy$ transforms the equation (Σ_ε) into the equation $(S_{\varepsilon'})$ satisfying all assumptions of § 4, i.e. $y(t)$ is a solution of $(S_{\varepsilon'})$ if and only if $x(t) = Cy(t)$ is a solution of (Σ_ε) (see [5], Theorem 7).

Let $|x|$ be a given norm in the x-space. Then in the y-space we may define a norm $|y|$ as follows

$$|y| = |Cy| .$$

The transformation $x = Cy$ is then a norm-preserving one and hence, if $u(y)$ is the best ε-stabilizing control with the minimal (u, ε)-invariant region G in the sense of the norm $|y|$ in the y-space, then $\bar{u}(x) = u(C^{-1}x)$ is the best (u, ε)-stabilizing control with the minimal (u, ε)-invariant region $G_0 = CG$ in the sense of the norm $|x|$ in the x-space.

It remains to investigate the case that for every $r \in R$ we have

$$r^2 b^1 - r^1 b^2 = 0 .$$

Then from (8.6) it follows that $p^1 = 0$ for $p \in C^{-1}R$ and thus the assumptions of Theorem 8.1 are satisfied.

By this also the case of a general linear control system with constant coefficients is completely solved.

Example 9.1. Consider the system

$$\dot{y}^1 = \quad y^2 + p^1\,, \qquad P = \{p : (p^1)^2 + (p^2)^2 \leqq 1\}\,.$$
$$\dot{y}^2 = -y^1 + p^2 + u\,,$$

Let us choose the control u_ε and the region G_ε.

The system (S_ε^-) will be as follows:

$$\dot{y}^1 = \quad y^2 + p^1\,, \qquad p \in \varepsilon P\,.$$
$$\dot{y}^2 = -y^1 + p^2 - 1\,,$$

The equation (R_ε^-):

(9.1) $$\frac{\mathrm{d}y^1}{\mathrm{d}y^2} = \frac{y^2 + p^1}{-y^1 + p^2 - 1}\,, \quad p \in \varepsilon P\,.$$

The right-hand side of (9.1) is minimal for such p for which the vector $(y^2 + p^1, -y^1 + p^2 - 1)$ is tangent to the circle $(y^2, -y^1 - 1) + \varepsilon P$, i.e. if the vectors $(y^2 + p^1, -y^1 - 1 + p^2)$, (p^1, p^2) are orthogonal and the point (p^1, p^2) belongs to the boundary of εP, i.e. if

(9.2) $$p^1(y^2 + p^1) + p^2(-y^1 - 1 + p^2) = 0\,,$$
$$(p^1)^2 + (p^2)^2 = \varepsilon^2$$

(see Fig. 4).

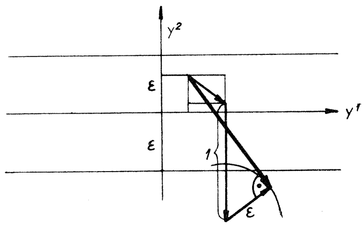

Fig. 4.

By the transformation

$$y^1 = -1 + r\cos\Theta\,, \quad y^2 = r\sin\Theta$$

we get from (9.1) a system

(9.3) $$\dot{r} = p^1\cos\Theta + p^2\sin\Theta, \quad \dot{\Theta} = -1 + r^{-1}(p^2\cos\Theta - p^1\sin\Theta)$$

and from (9.2) the condition

(9.4) $$p^1(r\sin\Theta + p^1) + p^2(-r\cos\Theta + p^2) = 0\,, \quad (p^1)^2 + (p^2)^2 = \varepsilon^2\,.$$

From (9.2) it follows that

$$p^1\sin\Theta - p^2\cos\Theta = -r^{-1}\varepsilon^2, \quad p^1\cos\Theta + p^2\sin\Theta = \varepsilon(1 - \varepsilon^2 r^{-2})^{\frac{1}{2}}\,.$$

Hence, Γ_ε^- satisfies the differential system

$$\dot{r} = \varepsilon(1 - \varepsilon^2 r^{-2})^{\frac{1}{2}}\,, \quad \dot{\Theta} = -1 + \varepsilon^2 r^{-2}\,.$$

As $\dot{r} > 0$, Γ_ε^- satisfies the differential equation

$$\frac{d\Theta}{dr} = -\frac{1}{\varepsilon}(1 - \varepsilon^2 r^{-2})^{\frac{1}{2}}.$$

Integrating it we obtain

$$\Theta = -\varepsilon^{-1}(r^2 - \varepsilon^2)^{\frac{1}{2}} - \arcsin \varepsilon r^{-1} + c .$$

Now, we have to find such $r_1, \Theta_1, r_2, \Theta_2, c$ that

$$\Theta_i = -\varepsilon^{-1}(r_i^2 - \varepsilon^2)^{\frac{1}{2}} - \arcsin \varepsilon r_i^{-1} + c, \quad i = 1, 2,$$
$$r_1 \sin \Theta_1 = \varepsilon, \quad r_2 \sin \Theta_2 = -\varepsilon, \quad (r_1^2 - \varepsilon^2)^{\frac{1}{2}} + (r_2^2 - \varepsilon^2)^{\frac{1}{2}} = 2$$

is satisfied. From this we get an equation which r_1 has to satisfy;

(9.5) $$r_1 \sin \varepsilon^{-1}[1 - (r_1^2 - \varepsilon^2)^{\frac{1}{2}}] = \varepsilon .$$

By the coordinates $y^2 = \varepsilon$, $r = r_1$ the point $-y_\varepsilon$ is determined. From this we get the equation of Γ_ε^-:

(9.6) $$\Theta = -\varepsilon^{-1}[(r^2 - \varepsilon^2)^{\frac{1}{2}} + (r_1^2 - \varepsilon^2)^{\frac{1}{2}} - 2] - \arcsin \varepsilon r^{-1} .$$

From (9.5) we obtain the equation of the curve Φ in the coordinates r, Θ:

(9.7) $$r\Theta \sin \Theta + r \cos \Theta - 1 = 0 .$$

Now we are going to show that it is possible to construct the ε_0-universal ε-stabilizing control $\tilde{u}$. For this it suffices to show that the assumptions of Theorem 6.1 are satisfied.

For Γ_ε^- we have at $-y_\varepsilon$,

$$\frac{dr}{d\Theta} = -\frac{\varepsilon}{(1 - \varepsilon^2 r^{-2})^{\frac{1}{2}}} = -\frac{1}{r}\frac{\sin \Theta}{\cos \Theta}$$

and for Φ at the same point,

$$\frac{dr}{d\Theta} = -\frac{1}{r}\frac{\Theta \cos \Theta}{\Theta \sin \Theta + \cos \Theta} .$$

It is easy to verify that for $0 < \Theta < \frac{1}{2}\pi$ it holds

$$-\frac{\sin \Theta}{\cos \Theta} < -\frac{\Theta \cos \Theta}{\Theta \sin \Theta + \cos \Theta},$$

which implies that for sufficiently small $\varepsilon_2 \geqq \varepsilon_1 > 0$ we have $G_{\varepsilon_1} \subset G_{\varepsilon_2}$.

Example 9.2. Consider the system

$$\dot{y}^1 = y^2 + p^1 , \quad \dot{y}^2 = \alpha y^1 + u , \quad |p^1| \leqq \varepsilon ,$$

where $\alpha \neq 0$. The equations (R_ε^+) and (mR_ε^+) are as follows:

$$\frac{\mathrm{d}y^1}{\mathrm{d}y^2} = \frac{y^2 + p^1}{\alpha y^1 + 1}, \quad |p^1| \leqq \varepsilon,$$

(9.8)
$$\frac{\mathrm{d}y^1}{\mathrm{d}y^2} = \frac{y^2 - \varepsilon}{\alpha y^1 + 1}.$$

Integrating (9.8) we obtain

(9.9)
$$\gamma_\varepsilon^+(y^2, y_0^1) = \alpha^{-1}\{-1 + [(1 + \alpha y_0^1)^2 + \alpha((y^2)^2 - 2\varepsilon y^2 - 3\varepsilon^2)]^{\frac{1}{2}}\}.$$

From the equation $\gamma_\varepsilon^+(\varepsilon, y_\varepsilon^1) = -y_\varepsilon^1$ we get

(9.10)
$$y_\varepsilon^1 = \varepsilon^2$$

and from this the equation of the curve Φ,

(9.11)
$$y^1 = \varphi(y^2) = (y^2)^2.$$

Substituing from (9.10) into (9.9) we obtain the equation of Γ_ε^+,

$$y^1 = \gamma_\varepsilon^+(y^2) = \alpha^{-1}\{-1 + [(1 + \alpha\varepsilon^2)^2 + \alpha((y^2)^2 - 2\varepsilon y^2 - 3\varepsilon^2)]^{\frac{1}{2}}\}.$$

From (9.8) we get

$$\frac{\mathrm{d}\gamma_\varepsilon^+(-\varepsilon)}{\mathrm{d}y^2} = \frac{-2\varepsilon}{\alpha\varepsilon^2 + 1},$$

and from (9.11),

$$\frac{\mathrm{d}\varphi(-\varepsilon)}{\mathrm{d}y^2} = -2\varepsilon.$$

For $\alpha > 0$ we have

$$\frac{\mathrm{d}\gamma_\varepsilon^+(-\varepsilon)}{\mathrm{d}y^2} = \frac{-2\varepsilon}{\alpha\varepsilon^2 + 1} > -2\varepsilon > = \frac{\mathrm{d}\varphi(-\varepsilon)}{\mathrm{d}y^2}.$$

Hence, the assumptions of Theorem 6.1 are not satisfied.

10.

The problem discussed in this article may be interpreted as a two-person game (a game against the nature). The aim of the first player is to ensure by an appropriate control $u(y)$ the smallest possible deviation of the solution of (σ_ε) from the origin whatever the action of the second may be; conversely the aim of the second is to enlarge with the aid of the perturbations p according to the given possibilities the deviation of the solution of (σ_ε) from the origin independently on the control selected by the first.

Further we shall consider only the system (s_ε) satisfying (4.1) with the Euclidean norm. However, the considerations may be transferred without difficulty to the other cases.

A two-person zero-sum game is given by a triplet

$$H = \{A, B, h\},$$

where A is the set of the strategies of the first player, B is the set of the strategies of the second player and $h(a, b)$ is a real-valued pay-off function on $A \times B$ (see [9], 2.1).

The game H is said to have a solution in the domain of pure strategies, if

$$\max_{a \in A} \inf_{b \in B} h(a, b) = \min_{b \in B} \sup_{a \in A} h(a, b) = \bar{h}\,; \tag{10.1}$$

$\bar{h}$ is called the value of the game, the strategies $\bar{a}, \bar{b}$ for which

$$\bar{h} = \inf_{b \in B} h(\bar{a}, b) = \sup_{a \in A} h(a, \bar{b}) = h(\bar{a}, \bar{b})$$

are called the optimal strategies. It is easy to see that $\bar{a}, \bar{b}$ are optimal strategies if and only if

$$h(a, \bar{b}) \leqq h(\bar{a}, \bar{b}) \leqq h(\bar{a}, b) \tag{10.2}$$

for every $a \in A$, $b \in B$.

For our case it is convenient to generalize the notion of the game by weakening the assumption on $h(a, b)$ as follows: We shall suppose that $h(a, b)$ is a multi-valued function, i.e. for every $a \in A$, $b \in B$, $h(a, b)$ is a set of real numbers. We shall say that the game H has a solution in the domain of pure strategies, if there exist such strategies $\bar{a}, \bar{b}$ that $h(\bar{a}, \bar{b})$ is a one-point-set and

$$\max_{a \in A_\varepsilon} \inf_{b \in B_\varepsilon} \sup h(a, b) = h(\bar{a}, \bar{b}) = \min_{b \in B_\varepsilon} \sup_{a \in A_\varepsilon} \inf h(a, b)\,;$$

$\bar{h} = \bar{h}(a, b)$ will be called the value of the game and $\bar{a}, \bar{b}$ the optimal strategies.

Similarly as in (10.2) we see that a sufficient condition for $\bar{a}, \bar{b}$ to be optimal is

$$\sup h(a, \bar{b}) \leqq h(\bar{a}, \bar{b}) \leqq \inf h(\bar{a}, b) \tag{10.3}$$

for every $a \in A$, $b \in B$.

Denote A_ε the set of measurable functions $p(y) = (p^1(y), p^2(y))$ defined almost everywhere in a domain from E_2 satisfying $p(y) \in \varepsilon P$; further denote B_ε the set of measurable functions $u(y)$, defined almost everywhere in some domain from E_2 and satisfying $|u| \leqq 1$.

Let the assumptions of Theorem 6.1 be satisfied. Denote $y(t, y_0, p, u)$ the solution of (s_ε) under the control u and the perturbation p passing through y_0 at $t = 0$ and

$$h_\varepsilon(p, u) = \{\limsup_{t\to\infty} \|y(t, y_0, p, u)\| : y_0 \in G_{\varepsilon 0}\} .$$

Theorem 10.1. *Let the assumptions of Theorem* 6.1 *be satisfied and let* $V(0, \|y_\varepsilon\|) \subset D_{\varepsilon_0}$. *Then the generalized game* $H_\varepsilon = \{A_\varepsilon, B_\varepsilon, h_\varepsilon\}$ *has for* $\varepsilon \leqq \varepsilon_0$ *a solution in the domain of pure strategies and* $\bar{h}_\varepsilon = \|y_\varepsilon\|$.

Proof. Denote

$$p_\varepsilon(y) = \begin{cases} p_m(y) & \text{for} \quad y \in V(0, \|y_\varepsilon\|),\ y^1 < \varphi(y^2) , \\ -p_m(y) & \text{for} \quad y \in V(0, \|y_\varepsilon\|),\ y^1 > \varphi(y^2) , \end{cases}$$

where $p_m(y)$ is given by (4.6) and φ by (6.1). We are going to show that $p_\varepsilon(y)$, $\tilde{u}(y)$ (defined by (6.2)) are optimal strategies for every $\varepsilon \leqq \varepsilon_0$.

From the proof of Theorem 6.1 it follows that for every solution $y(t, y_0, p, \tilde{u})$ with $y_0 \in G_{\varepsilon_0}$ we have $\limsup_{t\to\infty} \|y(t, y_0, p, \tilde{u})\| \leqq \|y_\varepsilon\|$, and hence,

$$\sup_{y_0 \in G_{\varepsilon_0}} \limsup_{t\to\infty} \|y(t, y_0, p, \tilde{u})\| \leqq \|y_\varepsilon\| . \tag{10.4}$$

Consider the solution $y(t) = y(t, y_0, p_\varepsilon, u)$ and suppose that a point $y(t_1) = y_1$ exists such that

$$y_1 \in (V(0, \|y_\varepsilon\|) - G_\varepsilon) \cup (\Gamma_\varepsilon^+ \cup \Gamma_\varepsilon^-) \tag{10.5}$$

i.e. a point y_1 belonging to the $\|y_\varepsilon\|$-neighbourhood of the origin but not to the interior of G_ε. Show that then a $t_2 \geqq t_1$ exists such that

$$y(t_2) \notin V(0, \|y_\varepsilon\|) . \tag{10.6}$$

Due to the symmetry it suffices to consider the case $y^1 \leqq \varphi(y^2)$. In this case it may be seen from the proof of Lemma 5.2 that for $t \in \langle t_1, t_1 + \tau \rangle$, $\tau > 0$ $y(t)$ will be contained in the zone of emission $Z(y_1^2, y_1^1, y^1(t_1 + \tau))$ of (5.4) and $\dot{y}^1(t) < 0$ for $y^2 < \varepsilon$; therefore it either leaves D_ε or crosses $y^2 = \varepsilon$ to the left of $-y_\varepsilon$. In both cases the desired t_2 exists.

From the fact just proved it follows that if $y(t)$ contains a point satisfying (10.5), then

$$\limsup_{t\to\infty} \|y(t)\| = \limsup_{t\to\infty} \|y(t, y_0, p_\varepsilon, u)\| \geqq \|y_\varepsilon\| . \tag{10.7}$$

Hence, it remains to prove (10.7) for the solutions contained in the interior of G_ε for $t \geqq t_0$. Let $y(t_0)$ lie below Φ, i.e. $y^1(t_0) < \varphi(y^2(t_0))$. From Lemma 4.3 it follows that $\dot{y}^1(t) = y^2(t) + p_m^1(y(t)) < 0$ as far as $y^1 < \varphi(y^2)$. Hence, $y(t)$ either tends to $-y_\varepsilon$ (in this case (10.7) is evidently satisfied), or it crosses Φ; from this and from the symmetry it follows that every solution which does not leave the interior of G_ε either tends to one of the points $\pm y_\varepsilon$ or has a point in common with Φ.

Suppose that $y(t_1) \in \Phi^+$. Then an $\varepsilon_1 < \varepsilon$ exists such that $y(t_1) = y_{\varepsilon_1}$. From Theorem 2.6 it follows that for $t \geqq t_1$ sufficiently near to t_1 it holds $y^1(t) \leqq \varphi(y^2(t))$ and, therefore, $y(t)$ is contained in the zone of emission $Z(y^2(t_1), y^1(t_1), y^1(t_1 + \tau))$, $\tau > 0$ of (5.4). From this and from the assumptions of Theorem 6.1 it follows that $y^1(t) < \varphi(y^2(t))$ for $t \in (t_1, t_1 + \tau)$; due to Lemma 4.3 we have either $y(t) \to y_\varepsilon$ or that $y(t)$ crosses Φ^- in a point y_{ε_2}, $\varepsilon_2 > \varepsilon_1$. Continuing in the same way we conclude that if $y(t)$ does not tend to any one of the points $\pm y_\varepsilon$, then a sequence $\{t_n\}$, $t_n \to \infty$ exists such that $y(t_{2n-1}) = y_{\varepsilon_{2n-1}} \in \Phi^+$, $y(t_{2n}) = y_{\varepsilon_{2n}} \in \Phi^-$ and $\varepsilon_n < \varepsilon_{n+1}$, $n = 1, 2, 3, \ldots$ Similarly as in the proof of Lemma 5.2 we conclude that $\varepsilon_n \to \varepsilon$ and thus

$$\limsup_{t \to \infty} \|y(t)\| = \lim_{n \to \infty} \|y(t_n)\| = \lim_{n \to \infty} \|y_{\varepsilon_n}\| = \|y_\varepsilon\| .$$

Hence, also in this case (10.7) holds. From (10.4), (10.7) it follows that p_ε, $\tilde{u}$ satisfy (10.3) which completes the proof.

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References

[1] *Ważewski T.:* Systèmes de commande et équations au contingent. Bull. Acad. Polon. Sci., Sér. Math. Astr. Phys. *9* (1961), 151—155.

[2] *Ważewski T.:* Sur une condition d'existence des fonctions implicites mesurables. Ibid. *9* (1961), 861—863.

[3] *Ważewski T.:* Sur une condition équivalente à l'équation au contingent. Ibid. *9* (1961), 865—867.

[4] *Филиппов А. Ф.:* Дифференциальные уравнения с разрывной правой частью. Математический сборник *51* (1960), 99—128.

[5] *Барбашин Е. А., Алимов Ю. И.:* К теории релейных дифференциальных уравнений. Известия ВУЗ, Математика, 1962, 3—13.

[6] *Zaremba S. Ch.:* Sur les équations au paratingent. Bull. des Sci. Math. *60* (1936), 139—160.

[7] *Понтрягин Л. С., Болтянский В. Г., Гамкрелидзе Р. В., Мищенко Е. Ф.:* Математическая теория отпимальных процессов. Москва 1962.

[8] *La Salle J. P.:* Stability and Control. RIAS Technical Report 61 — 17 (1961).

[9] *Karlin S.:* Mathematical Methods and Theory in Games, Programming and Economics II. Reading—London, 1959.

[10] *Coddington E. A., Levinson N.:* Theory of Ordinary Differential Equations. New York, 1955.

[11] *Dunford N., Schwartz J. T.:* Linear Operators I. New York—London, 1958.

Резюме

О НАИЛУЧШЕМ СТАБИЛИЗИРУЮЩЕМ УПРАВЛЕНИИ ПРИ ДАННОМ КЛАССЕ ВОЗМУЩЕНИЙ

ПАВОЛ БРУНОВСКИ (Pavol Brunovský), Братислава

Рассматривается линейная система управления второго порядка с постоянными коэффициентами

$$(\sigma_\varepsilon) \qquad \dot{x} = Ax + bu + r$$

где $r(t)$ — постоянно действующие возмущения, т.е. измеримые функции t, удовлетворяющие ограничению $r \in \varepsilon R$, где R — выпуклый компакт, симметричный относительно начала координат.

Управление $u(x)$ называется ε-стабилизирующим, если существует такая компактная область G, содержащая начало, что ни одно решение системы (σ_ε) не покидает ее, каков бы ни был вид возмущения $r(t)$. G называется (u, ε)-инвариантной областью.

Пусть $|x|$ — заданная норма в E_2. Положим $|G| = \max_{x \in G} |x|$.

Доказывается, что если система (σ_ε) управляема, т.е. векторы b, Ab линейно независимы, то для достаточно малого $\varepsilon > 0$ существует наилучшее ε-стабилизирующее управление $\bar{u}$ с $(\bar{u}, \varepsilon)$-инвариантной областью G_0 в том смысле, что для любой (u, ε)-инвариантной области G выполняется $|G_0| \leqq |G|$.

При некоторых дополнительных условиях существует управление $\tilde{u}(x)$, являющееся наилучшим ε-стабилизирующим для всех $\varepsilon \leqq \varepsilon_0$.

Так как эта задача ведет к разрывному управлению, то решение понимается в обобщенном смысле Филиппова [4].