Czechoslovak Mathematical Journal

Vítězslav Novák

On the lexicographic product of ordered sets

Czechoslovak Mathematical Journal, Vol. 15 (1965), No. 2, 270–282

Persistent URL: http://dml.cz/dmlcz/100669

Чехословацкий математический журнал, т. 15 (90) 1965, Прага

ON THE LEXICOGRAPHIC PRODUCT OF ORDERED SETS

VÍTĚZSLAV NOVÁK, Brno

(Received January 24, 1964)

In this paper there is proved the necessary and sufficient condition for the lexicographic product of ordered sets over an ordered set to be an ordered set. The well extension of an ordered set is defined and the necessary and sufficient condition for the existence of a well extension of an ordered set is proved. A certain calculation of the dimension of the lexicographic product of ordered sets is obtained. Especially the dimension of the ordinal product of ordered sets is determined.

1. THE LEXICOGRAPHIC PRODUCT OF ORDERED SETS

Any set throughout whole this paper is assumed non-empty. card G denotes the cardinality of a set G. A linearly ordered set is called a chain, a set in which every two distinct elements are incomparable is called an antichain. If G is a set in which ordering relations are defined, then by the symbol $G(\leqq)$ the set G together with the relation $\leqq$ is meant. If the ordering one relation $\leqq$ is defined in G we write G instead of $G(\leqq)$.

Definition 1. Let H be an ordered set, let $\{G_\alpha \mid \alpha \in H\}$ be a system of ordered sets. By the lexicographic product[1] $\prod_{\alpha\in H} G_\alpha$ we mean the set of all functions f defined on H and such that $f(\alpha) \in G_\alpha$ for every $\alpha \in H$ with the relation $\leqq$ defined as follows: $f \leqq g \Leftrightarrow$ if there exists $\alpha_0 \in H$ such that $f(\alpha_0) \neq g(\alpha_0)$ then there exists $\alpha_1 \leqq \alpha_0$ in H such that $f(\alpha_1) < g(\alpha_1)$.

If all sets G_α are equal to the same set G the corresponding lexicographic product will be called the ordinal power and denoted ${}^H G$. If $f \in \prod_{\alpha\in H} G_\alpha$, $g \in \prod_{\alpha\in H} G_\alpha$, $f \leqq g$ but $f \neq g$ we write $f < g$.

The relation $\leqq$ in the lexicographic product is in general, however, only reflexive as it is shown in the following examples. Hence $\prod_{\alpha\in H} G_\alpha$ is not generally an ordered set.

Example 1. Let us consider the ordinal power ${}^H G$ where G is a chain of type **2** and H a chain of type ω^*, i.e. $G = \{a, b \mid a < b\}$, $H = \{\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots, |\alpha_0 >$

[1]) see [3], p. 14.

$> \alpha_1 > \ldots > \alpha_n > \ldots\}$. If we define two functions f, g as follows: $f(\alpha_{2n}) = a$, $f(\alpha_{2n+1}) = b$, $n = 0, 1, 2, \ldots$; $g(\alpha_{2n}) = b$, $g(\alpha_{2n+1}) = a$, $n = 0, 1, 2, \ldots$, then $f \in {}^H G$, $g \in {}^H G$, $f \leqq g$, $g \leqq f$, but $f \neq g$.

Example 2. Let us consider the ordinal power ${}^H G$ where $G = \{a, b, c \mid a < b, a \parallel c, b \parallel c\}$, H is a chain of type ω^*, i. e. $H = \{\alpha_0, \alpha_1, \ldots \alpha_n, \ldots \mid \alpha_0 > \alpha_1 > \ldots > \alpha_n > \ldots\}$. If we define three functions f, g, h as follows: $f(\alpha_{3n}) = a$, $f(\alpha_{3n+1}) = b$, $f(\alpha_{3n+2}) = c$, $n = 0, 1, 2, \ldots$ $g(\alpha_{3n}) = b$, $g(\alpha_{3n+1}) = c$, $g(\alpha_{3n+2}) = a$, $n = 0, 1, 2, \ldots$ $h(\alpha_{3n}) = c$, $h(\alpha_{3n+1}) = a$, $h(\alpha_{3n+2}) = b$, $n = 0, 1, 2, \ldots$ then $f \in {}^H G$, $g \in {}^H G$, $h \in {}^H G$, $f \leqq g$, $g \leqq h$ but $f \nleqq h$ (even $f < g$, $g < h$, $f > h$).

If we choose H as a well-ordered set, or as an antichain we obtain special cases of a lexicographic product – the so called ordinal and cardinal product. Let us give definitions of both these operations.

Definition 2. Let H be a well-ordered set, let $\{G_\alpha \mid \alpha \in H\}$ be a system of ordered sets. By the ordinal product $\mathbf{P}_{\alpha \in H} G_\alpha$ the set of all functions f defined on H is meant such that $f(\alpha) \in G_\alpha$ for every $\alpha \in H$ together with the relation $<$ defined as follows: $f < g \Leftrightarrow$ there exists $\alpha_0 \in H$ such that $f(\alpha) = g(\alpha)$ for every $\alpha < \alpha_0$ whereas $f(\alpha_0) < g(\alpha_0)$.

If H is a two-point chain, this definition agrees with the Birkhoff's definition of the ordinal product of two ordered sets ([1], [2]).

Definition 3. Let H be a set, let $\{G_\alpha \mid \alpha \in H\}$ be a system of ordered sets. By the cardinal product $\prod_{\alpha \in H} G_\alpha$ the set of all functions f defined on H is meant such that $f(\alpha) \in G_\alpha$ for every $\alpha \in H$ together with the relation $\leqq$ defined as follows:

$$f \leqq g \Leftrightarrow f(\alpha) \leqq g(\alpha) \quad \text{for every} \quad \alpha \in H .$$

If all sets G_α are equal to the same set G we call the relevant cardinal product by the cardinal power whose basis is G and exponent the antichain H and denote it G^H.

If H is a two-point set, this definition agrees with the Birkhoff's definition of the cardinal product of two ordered sets ([1], [2]). In the following it will be shown that both the ordinal product and the cardinal product of ordered sets are ordered sets. But it is not difficult to prove directly these statements.

Now we give a necessary and sufficient condition for the lexicographic product of ordered sets to be an ordered set. For this reason we need the following definition.

Definition 4. Let G be an ordered set. We say that G satisfies the descendig chain condition if for any element $x_0 \in G$ every chain $C = \{x_0 > x_1 > x_2 > \ldots\}$ in G is finite.

Theorem 1. *Let H be an ordered set, let $\{G_\alpha \mid \alpha \in H\}$ be a system of ordered sets. Then $\prod_{\alpha \in H} G_\alpha$ is an ordered set if and only if the set $H' = \{\alpha \mid \alpha \in H, G_\alpha$ is not an antichain$\}$ satisfies the descending chain condition.*

Proof.[2]) 1. Let the condition of the theorem be satisfied. We shall prove that the relation $<$ in $\prod_{\alpha \in H} G_\alpha$ is nonsymmetric[3]). Hence let $f \in \prod_{\alpha \in H} G_\alpha$, $g \in \prod_{\alpha \in H} G_\alpha$, $f < g$ and assume that at the same time $g < f$. Choose any $\alpha_0 \in H$ such that $f(\alpha_0) \neq g(\alpha_0)$. Such an α_0 exists, since otherwise $f = g$. Then there exists $\alpha_1 \leqq \alpha_0$ in H such that $f(\alpha_1) < g(\alpha_1)$. This implies that G_{α_1} is not an antichain. As $g < f$, there exists $\alpha_2 < \alpha_1$ in H such that $g(\alpha_2) < f(\alpha_2)$ and therefore G_{α_2} is not an antichain. As $f < g$, there exists $\alpha_3 < \alpha_2$ in H such that $f(\alpha_3) < g(\alpha_3)$, which implies that G_{α_3} is not an antichain. By induction we can construct an infinite chain $C = \{\alpha_1 > \alpha_2 > > \alpha_3 > \ldots\}$ in H such that G_{α_n} is not an antichain for every $n = 1, 2, \ldots$. This is a contradiction, so that the relation $g < f$ does not hold and $<$ is nonsymmetric. We shall prove that the relation $\leqq$ is transitive. Let $f, g, h \in \prod_{\alpha \in H} G_\alpha$ and $f \leqq g$, $g \leqq h$. If $f = h$, then $f \leqq h$. In the other case there exists $\alpha_0 \in H$ such that $f(\alpha_0) \neq h(\alpha_0)$. Then either $f(\alpha_0) \neq g(\alpha_0)$ or $g(\alpha_0) \neq h(\alpha_0)$, let us say that $f(\alpha_0) \neq g(\alpha_0)$ (the case $g(\alpha_0) \neq h(\alpha_0)$ would be treated in a similar way). As $f \leqq g$, there exists $\alpha_1 \leqq \alpha_0$ in H such that $f(\alpha_1) < g(\alpha_1)$, from this it follows that G_{α_1} is not an antichain. If $g(\alpha_1) \leqq h(\alpha_1)$, then $f(\alpha_1) < h(\alpha_1)$. In the other case there exists $\alpha_2 < \alpha_1$ in H such that $g(\alpha_2) < h(\alpha_2)$ so that G_{α_2} is not an antichain. If $f(\alpha_2) \leqq g(\alpha_2)$, then $f(\alpha_2) < h(\alpha_2)$. In the other case there exists $\alpha_3 < \alpha_2$ in H such that $f(\alpha_3) < g(\alpha_3)$; therefore G_{α_3} is not an antichain. This construction must finish after a finite number of steps since otherwise we should obtain an infinite chain $C = \{\alpha_1 > \alpha_2 > \alpha_3 > \ldots\}$ in H'. Hence there exists $\alpha_n \leqq \alpha_0$ such that $f(\alpha_n) < h(\alpha_n)$ so that $f \leqq h$. The relation $\leqq$ is therefore transitive and $\prod_{\alpha \in H} G_\alpha$ is an ordered set.

2. Let the condition of the theorem be not satisfied. Then there exists an infinite chain $C = \{\alpha_0 > \alpha_1 > \ldots > \alpha_n > \ldots\}$ in H such that G_{α_n} is not an antichain for every $n = 0, 1, 2, \ldots$. For any $n = 0, 1, \ldots$ choose two elements $a_n \in G_{\alpha_n}$, $b_n \in G_{\alpha_n}$ such that $a_n < b_n$ and define two functions f, g in the following way:

$$f(\alpha_{2n}) = a_n\,, \quad f(_{2n+1}) = b_n\,, \quad n = 0, 1, 2, \ldots\,, \quad f(\alpha) = x_\alpha$$

for $\alpha \in H - C$ where $x_\alpha \in G_\alpha$ is any element;

$$g(\alpha_{2n}) = b_n\,, \quad g(\alpha_{2n+1}) = a_n\,, \quad n = 0, 1, 2, \ldots, g(\alpha) = x_\alpha$$

for $\alpha \in H - C$. Then $f \in \prod_{\alpha \in H} G_\alpha$, $g \in \prod_{\alpha \in H} G_\alpha$, $f < g$, $g < f$ so that the relation $<$ is not nonsymmetric. Hence $\prod_{\alpha \in H} G_\alpha$ is not an ordered set.

Corollary 1. *Let H be an ordered set. Then $\prod_{\alpha \in H} G_\alpha$ is an ordered set for any system $\{G_\alpha \mid \alpha \in H\}$ of ordered sets if and only if H satisfies the descending chain condition.*

---

[2]) Another proof can be found in [3]. The definition of the lexicographic product, however, and hence all other formulations in [3] are dual with respect to our definitions and formulations.

[3]) i.e. $f < g \Rightarrow g \nless f$ for any $f, g \in \prod_{\alpha \in H} G_\alpha$.

Corollary 2. *Let G, H be ordered sets. Then ${}^H G$ is an ordered set if and only if either H satisfies the descending chain condition or G is an antichain.*

Corollary 3. *Let H be a well-ordered set, let $\{G_\alpha \mid \alpha \in H\}$ be a system of ordered sets. Then $\mathbf{P}_{\alpha \in H} G_\alpha$ is an ordered set.*

Corollary 4. *Let H be a set, let $\{G_\alpha \mid \alpha \in H\}$ be a system of ordered sets. Then $\prod_{\alpha \in H} G_\alpha$ is an ordered set.*

Theorem 2. *Let $H(\leqq)$ be an ordered set, let $\{G_\alpha \mid \alpha \in H\}$ be a system of ordered sets such that $\prod_{\alpha \in H(\leqq)} G_\alpha$ is an ordered set. Then it is possible to define an ordering $\preceq$ in H such that $H(\preceq)$ satisfies the descending chain condition and $\prod_{\alpha \in H(\leqq)} G_\alpha = \prod_{\alpha \in H(\preceq)} G_\alpha$.*

Proof. Denote $H_1 = \{\alpha \mid \alpha \in H, G_\alpha$ is not an antichain$\}$, $H_2 = H - H_1$. Then $H_1(\leqq)$ satisfies the descending chain condition. Let us define the ordering $\preceq$ in H in the following way:

$$x, y \in H_1 \Rightarrow x \prec y \quad \text{if and only if} \quad x < y$$

$$x, y \in H_2 \Rightarrow x \prec y \text{ if and only if there exists } z \in H_1 \text{ such that } x < z < y$$

$$x \in H_1,\ y \in H_2 \Rightarrow x \underset{(\succ)}{\prec} y \quad \text{if and only if} \quad x \underset{(>)}{<} y\,.$$

First we shall prove that $\prec$ is an ordering, i.e. it is a nonreflexive and transitive relation. The nonreflexivity is clear. We shall prove the transitivity. Hence let $x, y, z \in H$, $x \prec y$, $y \prec z$. If $x, y, z \in H_1$, then $x < y$, $y < z$ so that $x < z$ and from this $x \prec z$. If $x, y, z \in H_2$, then there exist $u, v \in H_1$ such that $x < u < y$, $y < v < z$. Then $x < u < z$ so that $x \prec z$. If $x \in H_2$, $y \in H_1$, $z \in H_2$, then $x < y$, $y < z$ so that $x \prec z$. It is not difficult to prove that in all remaining cases there holds $x \prec z$. Hence $\prec$ is transitive and therefore it is an ordering. Now we shall prove that $H(\preceq)$ satisfies the descending chain condition. Suppose the existence of an infinite descending chain in $H(\preceq) : x_0 \succ x_1 \succ x_2 \succ \ldots \succ x_n \succ \ldots$. As $H_1(\preceq) = H_1(\leqq)$ satisfies the descending chain condition, there is $x_n \in H_1$ only for finitely many n and we may assume $x_n \in H_2$ for every n. As $x_1 \prec x_0$, there exists $y_0 \in H_1$ such that $x_1 < y_0 < x_0$. As $x_2 \prec x_1$, there is $y_1 \in H_1$ such that $x_2 < y_1 < x_1$ and hence $y_1 < y_0$. By induction we can construct $y_n \in H_1$ for every n such that $x_{n+1} < y_n < x_n < y_{n-1}$. Then $\{y_n\}_{n=1}^{\infty}$ is an infinite descending chain in $H_1(\leqq)$ which is a contradiction. It is left to prove $\prod_{\alpha \in H(\leqq)} G_\alpha = \prod_{\alpha \in H(\preceq)} G_\alpha$. Let $f, g \in \prod_{\alpha \in H} G_\alpha$, $f < g$ in $\prod_{\alpha \in H(\leqq)} G_\alpha$. Assume that $f \nless g$ in $\prod_{\alpha \in H(\preceq)} G_\alpha$. Hence either $f > g$ or $f \parallel g$ in $\prod_{\alpha \in H(\preceq)} G_\alpha$. If $f > g$, then there exists $\alpha_0 \in H$ such that $f(\alpha_0) > g(\alpha_0)$ whereas $f(\alpha) = g(\alpha)$ for every $\alpha \in H(\preceq)$ for which $\alpha \prec \alpha_0$. As $\alpha_0 \in H_1$, there is $\alpha < \alpha_0 \Leftrightarrow \alpha \prec \alpha_0$ so that $f(\alpha) = g(\alpha)$ for every $\alpha \in H(\leqq)$ for which $\alpha < \alpha_0$. Hence the relation $f < g$ in $\prod_{\alpha \in H(\leqq)} G_\alpha$ is impossible and this is a contradiction.

If $f \parallel g$, then two cases are possible:

1) either there exists $\alpha_0 \in H(\precsim)$ such that $f(\alpha_0) \parallel g(\alpha_0)$ whereas $f(\alpha) = g(\alpha)$ for every $\alpha \prec \alpha_0$,

2) or there exist $\alpha_1, \alpha_2 \in H(\precsim)$ such that $f(\alpha_1) \prec g(\alpha_1)$, $f(\alpha) = g(\alpha)$ for every $\alpha \prec \alpha_1$; $f(\alpha_2) \succ g(\alpha_2)$, $f(\alpha) = g(\alpha)$ for every $\alpha \prec \alpha_2$.

In case 1) we have $\alpha < \alpha_0$, $\alpha \not\prec \alpha_0 \Rightarrow \alpha \in H_2$ so that $f(\alpha) = g(\alpha)$ or $f(\alpha) \parallel g(\alpha)$ for every $\alpha \in H(\leqq)$, $\alpha < \alpha_0$. Hence the relation $f < g$ in $\prod_{\alpha \in H(\leqq)} G_\alpha$ is impossible and this is a contradiction. In case 2) there is $\alpha_2 \in H_1$ so that $\alpha < \alpha_2 \Leftrightarrow \alpha \prec \alpha_2$. Hence the relation $f < g$ in $\prod_{\alpha \in H(\leqq)} G_\alpha$ is again impossible and this is a contradiction. Therefore $f < g$ in $\prod_{\alpha \in H(\precsim)} G_\alpha$.

As the ordering $\leqq$ is the extension[4]) of the ordering $\precsim$ it is clear that $f \parallel g$ in $\prod_{\alpha \in H(\leqq)} G_\alpha$ implies $f \parallel g$ in $\prod_{\alpha \in H(\precsim)} G_\alpha$ and the proof is completed.

2. THE WELL EXTENSION OF AN ORDERED SET

Definition 5. Let $G(\leqq)$ be an ordered set. An ordered set $H(\precsim)$ is called an extension of the set $G(\leqq)$ if $H = G$ and $x \leqq y \Rightarrow x \precsim y$. The ordering $\precsim$ is called an extension of the ordering $\leqq$.

Definition 6. An extension $\precsim$ of ordering $\leqq$ is called a linear extension if $\precsim$ is a linear ordering of the set G.

Definition 7. A linear extension $\precsim$ of ordering $\leqq$ is called a well extension if G is a well-ordered set with respect to $\precsim$.

It is clear that every ordered set $G(\leqq)$ has at least one extension-for instance $G(\leqq)$. The existence of a linear extension of any ordered set was first proved by E. Szpilrajn-Marczewski in [13]. Other proofs can be found in [5], [8], [11], [12]. On the other hand it is clear that not every ordered set has a well extension. If, for instance, G is a chain of type ω^*, then G has no well extension. We shall give a necessary and sufficient condition for the existence of a well extension of an ordered set G.

Theorem 3. *Let G be an ordered set. Then G has a well extension if and only if G satisfies the descending chain condition.*

Proof. The necessity of this condition is clear. We shall prove the sufficiency. Hence let $G(\leqq)$ satisfy the descending chain condition. Denote G_0 the set of all minimal elements in G (the mentioned assumption guarantees the existence of a minimal element below any element in G). Suppose that we have defined all sets G_λ

4) See Definition 5.

for every ordinal $\lambda < \alpha$. Then we define G_α in this way: $G_\alpha = \{x \mid x \in G - \bigcup_{\lambda<\alpha} G_\lambda;$ for every $y \in G$, $y < x$ there is $y \in \bigcup_{\lambda<\alpha} G_\lambda\}$. Denote β the smallest ordinal λ for which $G_\lambda = \emptyset$. Such an ordinal exists for if card $G \leqq \aleph_i$ then $G_\lambda = \emptyset$ for every $\lambda \geqq \omega_{i+1}$. Then $G_\lambda \neq \emptyset$ for every $\lambda < \beta$ and $G_{\lambda_1} \cap G_{\lambda_2} = \emptyset$ for $\lambda_1 < \beta$, $\lambda_2 < \beta$, $\lambda_1 \neq \lambda_2$. Choose any well ordering $\precsim$ of $G_\lambda(\lambda < \beta)$ and put $G(\precsim) = \sum_{\lambda<\beta} G_\lambda(\precsim)$ [5]. $G(\precsim)$ as a lexicographic sum of well ordered sets over a well ordered set is a well ordered set. We shall show that $G(\precsim)$ is an extension of $G(\leqq)$. Hence let $x, y \in G$, $x < y$. Then there exist $\lambda_1, \lambda_2 < \beta$ such that $x \in G_{\lambda_1}$, $y \in G_{\lambda_2}$. The assumption $x < y$ implies $\lambda_1 < \lambda_2$ and from this $x \prec y$ in $G(\precsim)$. Hence $G(\precsim)$ is a well extension of $G(\leqq)$. Now we shall describe a certain construction which will be called the construction (K).

The construction (K): Let $G(\leqq)$ be an ordered set. Let β be an initial ordinal of cardinality card G. Let us form a simple sequence[6] of type β containing all elements of G: $G = \{g_0, g_1, \ldots, g_\lambda, \ldots \mid \lambda < \beta\}$. Put $G_\alpha = \{g_0, g_1, \ldots, g_\lambda, \ldots \mid \lambda < \alpha\}$. for every $\alpha \leqq \beta$. Then $G_\beta = G$. Now let us define the ordering $\precsim$ in every $G_\alpha(\alpha \leqq \beta)$ by the transfinite induction in the following manner: G_0 is an empty set, therefore G_0 can be assumed as ordered. Assume that we have defined the ordering $\precsim$ in each set G_λ for each $\lambda < \alpha$. Then we define the ordering $\precsim$ in G_α so:

1) if α is an isolated ordinal then

   $g_{\alpha-1} \prec g_\lambda$ $(\lambda < \alpha - 1) \Leftrightarrow$ there exists $\lambda_{\alpha-1} < \alpha - 1$ such that
   $g_{\alpha-1} < g_{\lambda_{\alpha-1}}$ (in $G(\leqq)$) and $g_{\lambda_{\alpha-1}} \precsim g_\lambda$ in $G_{\alpha-1}(\precsim)$,
   $g_{\alpha-1} \succ g_\lambda$ $(\lambda < \alpha - 1)$ in all remaining cases,
   $g_\lambda \prec g_\mu$ $(\lambda, \mu < \alpha - 1) \Leftrightarrow g_\lambda \prec g_\mu$ in $G_{\alpha-1}(\precsim)$.

2) If α is a limit ordinal then we put for $g_\lambda \in G_\alpha$, $g_\mu \in G_\alpha$ $g_\lambda \prec g_\mu \Leftrightarrow$ there exists $\nu < \alpha$ such that $g_\lambda \prec g_\mu$ in $G_\nu(\precsim)$.

It is clear that each G_α is a chain with respect to $\precsim$.

Theorem 4. *Let $G(\leqq)$ be an ordered set, $G_\beta(\precsim)$ a chain obtained from $G(\leqq)$ by the construction (K). Then $G_\beta(\precsim)$ is a linear extension of $G(\leqq)$. If, moreover, $G(\leqq)$ satisfies the descending chain condition then $G_\beta(\precsim)$ is a well extension of $G(\leqq)$.*

Proof. $G_\beta(\precsim)$ is a chain so that it is sufficient to show that $G_\beta(\precsim)$ is an extension of $G(\leqq)$. Let $x, y \in G$, $x < y$. Then there exist ordinals $\lambda < \beta$, $\mu < \beta$ such that $x = g_\lambda$, $y = g_\mu$. If $\lambda > \mu$ then clearly $g_\lambda \prec g_\mu$ in $G_{\lambda+1}(\precsim)$ and from this $g_\lambda \prec g_\mu$ in $G_\beta(\precsim)$. Assume therefore that there are ordinals μ with this property: there is $\lambda < \mu$ such that $g_\lambda < g_\mu$ in $G(\leqq)$ but $g_\lambda \succ g_\mu$ in $G_{\mu+1}$. Let μ_0 be the smallest

[5]) $\sum_{\lambda<\beta} G_\lambda(\precsim)$ denotes the lexicographic sum of sets $G_\lambda(\precsim)$ over the set of all ordinals $\lambda < \beta$.

[6]) i.e. $g_{\lambda_1} \neq g_{\lambda_2}$ for $\lambda_1 \neq \lambda_2$.

ordinal with this property. Thus $g_{\mu_0} = x$, $g_\lambda = y$ where $\lambda < \mu_0$, $y < x$ in $G(\leqq)$ but $g_{\mu_0} \prec g_\lambda$ in G_{μ_0+1}. Then there exists $\nu < \mu_0$ such that $g_{\mu_0} < g_\nu$ in $G(\leqq)$ and $g_\nu \prec g_\lambda$ in $G_{\mu_0}(\preceq)$. As $g_\nu > g_{\mu_0} = x > y$ (in $G(\leqq)$), there is $g_\nu > y = g_\lambda$ (in $G(\leqq)$) but $g_\nu \prec g_\lambda$ in $G_{\mu_0}(\preceq)$. But this is impossible for $\nu < \lambda$ and for $\nu > \lambda$ we have obtained a contradiction with the minimality of μ_0. Hence $G_\beta(\preceq)$ is an extension of $G(\leqq)$.

Assume now that $G(\leqq)$ satisfies the descending chain condition. We shall prove that in this case $G_\beta(\preceq)$ is a well extension of $G(\leqq)$, i.e. $G_\beta(\preceq)$ also satisfies the descending chain condition. Proof will be made by transfinite induction. Assume that $G_\beta(\preceq)$ contains an infinite descending chain with the greatest element g_0: $g_0 \succ \succ g_{\lambda_1} \succ g_{\lambda_2} \succ \ldots \succ g_{\lambda_n} \succ \ldots$. The ordinal λ_1 can be chosen such that it is the smallest ordinal with the property: $G_\beta(\preceq)$ contains an infinite descending chain whose first and second elements are g_0 and g_{λ_1}. Analogously λ_2 can be chosen as the smallest ordinal with the property: $G_\beta(\preceq)$ contains an infinite descending chain whose first, second and third elements are $g_0, g_{\lambda_1}, g_{\lambda_2}$. Assume that we have defined the ordinals $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$. Then λ_n can be chosen as the smallest ordinal with the property: $G_\beta(\preceq)$ contains an infinite descending chain with the first, second, ..., $(n+1)^{\text{th}}$ elements $g_0, g_{\lambda_1}, \ldots, g_{\lambda_n}$. Then $\{\lambda_n\}_{n=1}^{\infty}$ is a strictly ascending sequence of ordinals for if there exists n_0 such that $\lambda_{n_0} > \lambda_{n_0+1}$, then $g_0 \succ g_{\lambda_1} \succ \ldots \succ \succ g_{\lambda_{n_0-1}} \succ g_{\lambda_{n_0+1}} \succ g_{\lambda_{n_0+2}} \ldots$ is also an infinite descending chain in $G_\beta(\preceq)$ whose $(n+1)^{\text{th}}$ element has index $\lambda_{n_0+1} < \lambda_{n_0}$ which contradicts the choice of the ordinal λ_{n_0}. Now there are two possibilities:

1. Either $g_0 > g_{\lambda_1} > g_{\lambda_2} > \ldots > g_{\lambda_n} > \ldots$ in $G(\leqq)$. This is, however, a contradiction because $G(\leqq)$ contains no infinite descending chain.

2. Or there exists the smallest integer n_0 such that $g_{\lambda_{n_0}} < g_{\lambda_{n_0-1}}$ in $G(\leqq)$ but $g_{\lambda_{n_0+1}} \parallel g_{\lambda_{n_0}}$ in $G(\leqq)$. As $g_{\lambda_{n_0+1}} \prec g_{\lambda_{n_0}}$ in $G_{\lambda_{n_0+1}+1}$, there exists an ordinal $\mu < \lambda_{n_0+1}$ such that $g_{\lambda_{n_0+1}} < g_\mu$ in $G(\leqq)$ and $g_\mu \prec g_{\lambda_{n_0}}$ in $G_{\lambda_{n_0+1}}$. Then $g_0 \succ g_{\lambda_1} \succ \ldots \succ g_{\lambda_{n_0}} \succ \succ g_\mu \succ g_{\lambda_{n_0+1}} \succ \ldots$ is also an infinite descending chain in $G_\beta(\preceq)$ whose $(n+2)^{\text{th}}$ element has index $\mu < \lambda_{n_0+1}$ which is a contradiction with the choice of the ordinal λ_{n_0+1}. Assume now that we have proved that $G_\beta(\preceq)$ does not contain any infinite descending chain with the greatest element g_λ for every $\lambda < \alpha$. Assume that $G_\beta(\preceq)$ contains an infinite descending chain with the greatest element $g_\alpha : g_\alpha \succ \succ g_{\lambda_1} \succ g_{\lambda_2} \succ \ldots \succ g_{\lambda_n} \succ \ldots$. Then $\lambda_n > \alpha$ for every $n = 1, 2, \ldots$ for if n_0 existed such that $\lambda_{n_0} < \alpha$ then $g_{\lambda_{n_0}} \succ g_{\lambda_{n_0+1}} \succ \ldots$ would be an infinite descending chain in $G_\beta(\preceq)$ with the greatest element $g_{\lambda_{n_0}}$ where $\lambda_{n_0} < \alpha$ which contradicts the induction assumption. In a similar way as in the first induction step λ_1 can be chosen as the smallest ordinal with the property: $G_\beta(\preceq)$ contains an infinite descending chain with the first and second elements g_α, g_{λ_1} and in general, λ_n can be chosen as the smallest ordinal with the property: $G_\beta(\preceq)$ contains an infinite descending chain with the first, second, ..., $(n+1)^{\text{th}}$ elements $g_\alpha, g_{\lambda_1}, \ldots, g_{\lambda_n}$. It is easy to see that $\{\lambda_n\}_{n=1}^{\infty}$ is again a strictly ascending sequence of ordinals. Two cases are now possible:

1. Either $g_\alpha > g_{\lambda_1} > g_{\lambda_2} > \ldots > g_{\lambda_n} > \ldots$ in $G(\leqq)$. This is a contradiction with the assumption of the theorem.

2. Or there exists the smallest integer n_0 such that $g_{\lambda_{n0}} < g_{\lambda_{n0-1}}$ in $G(\leqq)$ but $g_{\lambda_{n0+1}} \parallel g_{\lambda_{n0}}$ in $G(\leqq)$. As $g_{\lambda_{n0+1}} \prec g_{\lambda_{n0}}$ in $G_{\lambda_{n0+1}+1}(\preceq)$, there exists an ordinal $\mu < \lambda_{n_0+1}$ such that $g_{\lambda_{n0+1}} < g_\mu$ in $G(\leqq)$, $g_\mu \prec g_{\lambda_{n0}}$ in $G_{\lambda_{n0+1}}(\preceq)$. But then $g_\alpha \succ \succ g_{\lambda_1} \succ g_{\lambda_2} \succ \ldots \succ g_{\lambda_{n0}} \succ g_\mu \succ g_{\lambda_{n0+1}} \succ \ldots$ is also an infinite descending chain in $G_\beta(\preceq)$ whose $(n + 2)^{\text{th}}$ element has index $\mu < \lambda_{n_0+1}$ and this is a contradiction with the choice of the ordinal λ_{n_0+1}. Hence $G_\beta(\preceq)$ contains no infinite descending chain and is therefore a well extension of $G(\leqq)$.

3. THE DIMENSION OF A LEXICOGRAPHIC PRODUCT

Definition 8. Let G be an ordered set, let $\{L_\kappa \mid \kappa \in K\}$ be a system of chains, let f_κ be an one-one isoton mapping of G into L_κ. If, for any two elements $x, y \in G$, $x \leqq y \Leftrightarrow f_\kappa(x) \leqq f_\kappa(y)$ for every $\kappa \in K$, we shall say that the system $\{L_\kappa, f_\kappa \mid \kappa \in K\}$ is a realizer of the set G. By the cardinality of this realizer we mean the cardinality card K.

It is known ([4]) that every ordered set has at least one realizer. Hence we can define:

Definition 9. The minimum of cardinalities of realizers of the set G will be called the dimension of the ordered set G:

$$\dim G = \min\left(\operatorname{card} K \mid \{L_\kappa, f_\kappa \mid \kappa \in K\} \text{ is a realizer of } G\right)^{7)} .$$

The following results are known:

(1) If card $G \geqq 4$ then $\dim G \leqq [\frac{1}{2} \operatorname{card} G]$ ([5]),

(2) $\dim P(m) = m$ [8] ([7]).

(3) If G_α is a chain with card $G_\alpha \geqq 2$ for every $\alpha \in H$ then $\dim \prod_{\alpha \in H} G_\alpha = \operatorname{card} H$ ([6, [10]),

(4) $\dim \sum_{\alpha \in N} M_\alpha = \sup \{\dim N, \dim M_\alpha(\alpha \in N)\}$ ([9]),

(5) $\dim \prod_{\alpha \in H} G_\alpha \leqq \sum_{\alpha \in H} \dim G_\alpha$ ([6]).

We shall give a certain calculation for the dimension of the lexicographic product of ordered sets. According to Theorem 2, we can limit ourselves to the case that the set H satisfies the descending chain condition.

Theorem 5. *Let H be an ordered set satisfying the descending chain condition, let $\{G_\alpha \mid \alpha \in H\}$ be a system of ordered sets. Let $\mathscr{K}$ be a system of chains in H whose union is H. Then* $\dim \prod_{\alpha \in H} G_\alpha \leqq \sum_{K \in \mathscr{K}} \sup \{\dim G_\alpha \mid \alpha \in K\}$.

---

[7]) See [4], Definition 2.2.

[8]) $P(m)$ denotes the set of all subsets of any set of cardinality m ordered by a set inclusion.

Proof. Let $K \in \mathscr{K}$ be any chain in H. Denote $\sup \{\dim G_\alpha \mid \alpha \in K\} = m_K$. Let M_K be a set with card $M_K = m_K$ and let $\{L^\kappa_\alpha, f^\kappa_\alpha \mid \kappa \in M_K\}$ be a realizer of the set G_α for every $\alpha \in K$. Such a realizer can be constructed. For $\alpha \in H - K$ choose any chain L^α_α and any one-one isoton mapping f^α_α of the set G_α into L^α_α. Further, let $N_K = M_K \cup \cup (H - K)$ and let $\bar{H}(\precsim)$ be any well extension of the set H. Let φ be any mapping of the set H into the set N_K with the property $\varphi(K) \subseteq M_K$, φ is an identical mapping on $H - K$. Let us put $S_\varphi = \mathbf{P}_{\alpha \in \bar{H}} L^{\varphi(\alpha)}_\alpha$. S_φ is a chain for every mapping φ and every well extension $\bar{H}(\precsim)$. Let us define the mapping h_φ of the set $\prod_{\alpha \in H} G_\alpha$ into the set S_φ in the following way: $h_\varphi(g)[\alpha] = f^{\varphi(\alpha)}_\alpha[g(\alpha)]$. We shall prove that h_φ is an one-one isoton mapping of $\prod_{\alpha \in H} G_\alpha$ into S_φ for every mapping φ of H into N_K with the above mentioned property and for every well extension $\bar{H}(\precsim)$ of $H(\leqq)$. First we prove that h_φ is isoton. Hence let $g_1, g_2 \in \prod_{\alpha \in H} G_\alpha$, $g_1 < g_2$. Denote $\bar{H}_{(g_1,g_2)} = \{\alpha \mid \alpha \in \bar{H}, g_1(\alpha) \neq g_2(\alpha)\}$. Then $\bar{H}_{(g_1,g_2)}$ is non-empty and hence it has the smallest element α_0. The necessarily $g_1(\alpha_0) < g_2(\alpha_0)$ because $g_1(\alpha_0) \nleq g_2(\alpha_0)$ implies the existence of an element $\alpha_1 < \alpha_0$ in $H(\leqq)$ such that $g_1(\alpha_1) < g_2(\alpha_1)$ so that $\alpha_1 \prec \alpha_0$ in $\bar{H}(\precsim)$, $\alpha_1 \in \bar{H}_{(g_1,g_2)}$ which is a contradiction. Hence $g_1(\alpha_0) < g_2(\alpha_0)$ so that $f^\kappa_{\alpha_0}[g_1(\alpha_0)] < < f^\kappa_{\alpha_0}[g_2(\alpha_0)]$ for every κ while $f^\kappa_\alpha[g_1(\alpha)] = f^\kappa_\alpha[g_2(\alpha)]$ for every $\alpha \prec \alpha_0$ and every κ. Hence we have: $h_\varphi(g_1)[\alpha] = f^{\varphi(\alpha)}_\alpha[g_1(\alpha)] = f^{\varphi(\alpha)}_\alpha[g_2(\alpha)] = h_\varphi(g_2)[\alpha]$ for every $\alpha \prec \alpha_0$ whereas $h_\varphi(g_1)[\alpha_0] = f^{\varphi(\alpha_0)}_{\alpha_0}[g_1(\alpha_0)] < f^{\varphi(\alpha_0)}_{\alpha_0}[g_2(\alpha_0)] = h_\varphi(g_2)[\alpha_0]$ so that $h_\varphi(g_1) < h_\varphi(g_2)$ in S_φ. This implies that h_φ is isoton. Now we shall prove that h_φ is one-one. Let $g_1, g_2 \in \prod_{\alpha \in H} G_\alpha$, $g_1 \neq g_2$. Then $\bar{H}_{(g_1,g_2)} \neq \emptyset$ so that if $\alpha_0 \in \bar{H}_{(g_1,g_2)}$, there is $f^\kappa_{\alpha_0}[g_1(\alpha_0)] \neq f^\kappa_{\alpha_0}[g_2(\alpha_0)]$ for every κ so that $h_\varphi(g_1)[\alpha_0] = f^{\varphi(\alpha_0)}_{\alpha_0}[g_1(\alpha_0)] \neq \neq f^{\varphi(\alpha_0)}_{\alpha_0}[g_2(\alpha_0)] = h_\varphi(g_2)[\alpha_0]$, i.e. $h_\varphi(g_1) \neq h_\varphi(g_2)$; h_φ is therefore one-one.

Now choose any chain $K \in \mathscr{K}$ in $H(\leqq)$. As $H(\leqq)$ satisfies the descending chain condition this chain is well-ordered and we can write

$$K = \{k_0, k_1, \ldots, k_\lambda, \ldots, | \lambda < \beta_K\} \quad \text{where} \quad k_0 < k_1 < \ldots < k_\lambda < \ldots .$$

Choose any well extension $\bar{H}_K(\precsim)$ of the set $H(\leqq)$ obtained by the construction (K) such that we put $g_\lambda = k_\lambda$ for $\lambda < \beta_K$. Denote Φ_K the set of all mappings φ of the set $\bar{H}_K$ into the set N_K with the above mentioned property and with the property $\varphi(\alpha_1) = \varphi(\alpha_2)$ for any $\alpha_1, \alpha_2 \in K$ (i.e. φ is constant on K). Then clearly card $\Phi_K = m_K$. Further, let $\Phi = \bigcup_{K \in \mathscr{K}} \Phi_K$. There is card $\Phi \leqq \sum_{K \in \mathscr{K}} m_K$. We shall show that $\{S_\varphi, h_\varphi \mid \varphi \in \Phi\}$ is a realizer of $\prod_{\alpha \in H} G_\alpha$. As we have shown that each h_φ is an one-one isoton mapping, it is sufficient to show that for any two incomparable elements $g_1, g_2 \in \prod_{\alpha \in H} G_\alpha$ there are $\varphi_1, \varphi_2 \in \Phi$ such that $h_{\varphi_1}(g_1) < h_{\varphi_1}(g_2)$, $h_{\varphi_2}(g_1) > h_{\varphi_2}(g_2)$. Hence let $g_1, g_2 \in \prod_{\alpha \in H} G_\alpha$, $g_1 \parallel g_2$. There are two possibilities:

1) either $\alpha_0 \in H$ exists such that $g_1(\alpha_0) \parallel g_2(\alpha_0)$ whereas $g_1(\alpha) = g_2(\alpha)$ for every $\alpha < \alpha_0$,

(2) or $\alpha_1, \alpha_2 \in H$ exist such that $g_1(\alpha_1) < g_2(\alpha_1)$ whereas $g_1(\alpha) = g_2(\alpha)$ for every $\alpha < \alpha_1$ and $g_1(\alpha_2) > g_2(\alpha_2)$ whereas $g_1(\alpha) = g_2(\alpha)$ for every $\alpha < \alpha_2$.

If (1) holds, let $K \in \mathscr{K}$ be any chain containing α_0. Then clearly $\alpha \parallel \alpha_0$ in $H(\leqq)$ implies $\alpha \succ \alpha_0$ in $\overline{H}_K(\preceq)$ so that $g_1(\alpha) = g_2(\alpha)$ for every $\alpha \prec \alpha_0$ in $\overline{H}_K(\preceq)$. As $g_1(\alpha_0) \parallel g_2(\alpha_0)$ in G_{α_0}, there are $\kappa_1, \kappa_2 \in M_K$ such that $f^{\kappa_1}_{\alpha_0}[g_1(\alpha_0)] < f^{\kappa_1}_{\alpha_0}[g_2(\alpha_0)]$ and $f^{\kappa_2}_{\alpha_0}[g_1(\alpha_0)] > f^{\kappa_2}_{\alpha_0}[g_2(\alpha_0)]$. If $\varphi_1 \in \Phi_K$ is such a mapping that $\varphi_1(\alpha) = \kappa_1$ for every $\alpha \in K$ and $\varphi_2 \in \Phi_K$ is such a mapping that $\varphi_2(\alpha) = \kappa_2$ for every $\alpha \in K$, we have $h_{\varphi_1}(g_1)[\alpha] = f^{\varphi_1(\alpha)}_{\alpha}[g_1(\alpha)] = f^{\varphi_1(\alpha)}_{\alpha}[g_2(\alpha)] = h_{\varphi_1}(g_2)[\alpha]$ for every $\alpha \prec \alpha_0$ in $\overline{H}_K(\preceq)$ whereas $h_{\varphi_1}(g_1)[\alpha_0] = f^{\varphi_1(\alpha_0)}_{\alpha_0}[g_1(\alpha_0)] = f^{\kappa_1}_{\alpha_0}[g_1(\alpha_0)] < f^{\kappa_1}_{\alpha_0}[g_2(\alpha_0)] = f^{\varphi_1(\alpha_0)}_{\alpha_0}[g_2(\alpha_0)] = h_{\varphi_1}(g_2)[\alpha_0]$ which implies $h_{\varphi_1}(g_1) < h_{\varphi_1}(g_2)$ in S_{φ_1}, but $h_{\varphi_2}(g_1)[\alpha] = f^{\varphi_2(\alpha)}_{\alpha}[g_1(\alpha)] = f^{\varphi_2(\alpha)}_{\alpha}[g_2(\alpha)] = h_{\varphi_2}(g_2)[\alpha]$ for every $\alpha \prec \alpha_0$ in $\overline{H}_K(\preceq)$ whereas $h_{\varphi_2}(g_1)[\alpha_0] = f^{\varphi_2(\alpha_0)}_{\alpha_0}[g_1(\alpha_0)] = f^{\kappa_2}_{\alpha_0}[g_1(\alpha_0)] > f^{\kappa_2}_{\alpha_0}[g_2(\alpha_0)] = f^{\varphi_2(\alpha_0)}_{\alpha_0}[g_2(\alpha_0)] = h_{\varphi_2}(g_2)[\alpha_0]$ which implies $h_{\varphi_2}(g_1) > h_{\varphi_2}(g_2)$ in S_{φ_2}.

If (2) holds, let $K_1 \in \mathscr{K}$ be any chain containing α_1, $K_2 \in \mathscr{K}$ any chain containing α_2. As $\alpha \parallel \alpha_1$ in $H(\leqq)$ implies $\alpha \succ \alpha_1$ in $\overline{H}_{K_1}(\preceq)$, we have $g_1(\alpha) = g_2(\alpha)$ for every $\alpha \prec \alpha_1$ in $\overline{H}_{K_1}(\preceq)$ and $f^{\kappa}_{\alpha_1}[g_1(\alpha_1)] < f^{\kappa}_{\alpha_1}[g_2(\alpha_1)]$ for every κ. Choose any $\varphi_1 \in \Phi_{K_1}$, then $h_{\varphi_1}(g_1)[\alpha] = f^{\varphi_1(\alpha)}_{\alpha}[g_1(\alpha)] = f^{\varphi_1(\alpha)}_{\alpha}[g_2(\alpha)] = h_{\varphi_1}(g_2)[\alpha]$ for every $\alpha \prec \alpha_1$ in $\overline{H}_{K_1}(\preceq)$ whereas $h_{\varphi_1}(g_1)[\alpha_1] = f^{\varphi_1(\alpha_1)}_{\alpha_1}[g_1(\alpha_1)] < f^{\varphi_1(\alpha_1)}_{\alpha_1}[g_2(\alpha_1)] = h_{\varphi_1}(g_2)[\alpha_1]$ so that $h_{\varphi_1}(g_1) < h_{\varphi_1}(g_2)$ in S_{φ_1}. Analogously $g_1(\alpha) = g_2(\alpha)$ for every $\alpha \prec \alpha_2$ in $\overline{H}_{K_2}(\preceq)$ and $f^{\kappa}_{\alpha_2}[g_1(\alpha_2)] > f^{\kappa}_{\alpha_2}[g_2(\alpha_2)]$ for every κ so that if we choose any $\varphi_2 \in \Phi_{K_2}$, we have $h_{\varphi_2}(g_1) \,.$ $.\, [\alpha] = f^{\varphi_2(\alpha)}_{\alpha}[g_1(\alpha)] = f^{\varphi_2(\alpha)}_{\alpha}[g_2(\alpha)] = h_{\varphi_2}(g_2)[\alpha]$ for every $\alpha \prec \alpha_2$ in $\overline{H}_{K_2}(\preceq)$ whereas $h_{\varphi_2}(g_1)[\alpha_2] = f^{\varphi_2(\alpha_2)}_{\alpha_2}[g_1(\alpha_2)] > f^{\varphi_2(\alpha_2)}_{\alpha_2}[g_2(\alpha_2)] = h_{\varphi_2}(g_2)[\alpha_2]$ which implies $h_{\varphi_2}(g_1) > h_{\varphi_2}(g_2)$ in S_{φ_2} and the proof is completed.

4. SOME SPECIAL CASES

Theorem 6. *Let H be a well-ordered set, let $\{G_\alpha \mid \alpha \in H\}$ be a system of ordered sets. Then* $\dim \mathbf{P}_{\alpha \in H} G_\alpha = \sup \{\dim G_\alpha \mid \alpha \in H\}$.

Proof. Choose the system $\mathscr{K}$ as the one-point set $\mathscr{K} = \{H\}$. According to Theorem 5 we have $\dim \mathbf{P}_{\alpha \in H} G_\alpha \leqq \sup \{\dim G_\alpha \mid \alpha \in H\}$. On the other hand it is not difficult to construct a subset $G'_{\alpha_0} \subseteq \mathbf{P}_{\alpha \in H} G_\alpha$ isomorphic with G_{α_0} for every $\alpha_0 \in H : G'_{\alpha_0} = \{f \mid f \in \mathbf{P}_{\alpha \in H} G_\alpha,\ f(\alpha) = \text{const. for } \alpha \neq \alpha_0\}$. This implies $\dim G_{\alpha_0} = \dim G'_{\alpha_0} \leqq \dim \mathbf{P}_{\alpha \in H} G_\alpha$ for every $\alpha_0 \in H$ so that $\sup \{\dim G_\alpha \mid \alpha \in H\} \leqq \dim \mathbf{P}_{\alpha \in H} G_\alpha$ and we have $\dim \mathbf{P}_{\alpha \in H} G_\alpha = \sup \{\dim G_\alpha \mid \alpha \in H\}$.

Corollary 5. *Let G, H be ordered sets. Then* $\dim (G \odot H) = \max \{\dim G, \dim H\}$.

Proof follows from Theorem 6 if we put H equal to the two-point chain.

Corollary 6. *Let H be a set, let $\{G_\alpha \mid \alpha \in H\}$ be a system of ordered sets. Then* $\dim \prod_{\alpha \in H} G_\alpha \leqq \sum_{\alpha \in H} \dim G_\alpha$. [9])

Proof follows from Theorem 5 if we choose H as an antichain.

Corollary 7. *Let G, H be ordered sets. Then* $\dim (G \,.\, H) \leqq \dim G + \dim H$.

Proof follows from Corollary 6, if we choose H as a two-point set.

Remark. The calculation of Corollary 6 can not be improved. There are systems of ordered sets $\{G_\alpha \mid \alpha \in H\}$ for which $\dim \prod_{\alpha \in H} G_\alpha = \sum_{\alpha \in H} \dim G_\alpha$ and on the other hand there are systems $\{G_\alpha \mid \alpha \in H\}$ such that $\dim \prod_{\alpha \in H} G_\alpha < \sum_{\alpha \in H} \dim G_\alpha$. If, for instance G_α is a chain with card $G_\alpha \geqq 2$ for every $\alpha \in H$, then $\dim G_\alpha = 1$ for every $\alpha \in H$ and $\dim \prod_{\alpha \in H} G_\alpha = \text{card } H = \sum_{\alpha \in H} \dim G_\alpha$. If G_α is an antichain with card $G_\alpha \geqq 2$ for every $\alpha \in H$, then $\dim G_\alpha = 2$ for every $\alpha \in H$. But $\prod_{\alpha \in H} G_\alpha$ is also an antichain so that $\dim \prod_{\alpha \in H} G_\alpha = 2$ which is less than $\sum_{\alpha \in H} \dim G_\alpha$ for card $H \geqq 2$.

Corollary 8. *Let G be an ordered set, let H be an ordered set satisfying the descending chain condition. Let $\mathscr{K}$ be a system of chains in H whose union is H. Then* $\dim {}^H G \leqq \text{card } \mathscr{K} \,.\, \dim G$.

Proof follows from Theorem 5 if we put $G_\alpha = G$ for every $\alpha \in H$.

Corollary 9. *Let G be an ordered set, let H be a well-ordered set. Then* $\dim {}^H G = \dim G$.

Proof follows from Theorem 6 or from Corollary 8.

References

[1] *G. Birkhoff:* Lattice Theory, New York 1948.
[2] *G. Birkhoff:* Generalized arithmetic. Duke Math. Journ. *9* (1942), 283—302.
[3] *M. M. Day:* Arithmetic of Ordered Systems. Trans. Amer. Math. Soc. *58* (1945), 1—43.
[4] *B. Dushnik-E. W. Miller:* Partially ordered sets. Am. Journ. Math. *63* (1941), 600—610.
[5] *T. Hiraguchi:* On the dimension of partially ordered sets. Sci. Rep. of the Kanzawa Univ. *1* (1951), 77—94.
[6] *T. Hiraguchi:* A note on a Mr. Komm's Theorems. Sci. Rep. of the Kanazawa Univ. *2* (1953), 1—3.
[7] *H. Komm:* On the dimension of partially ordered sets. Am. Journ. Math. *70* (1948), 507—520.
[8] *J. Łoś-G. Ryll-Nardzewski:* On the application of Tychonoff's theorem in mathematical proofs. Fund. Math. *38* (1951), 233—237.
[9] *V. Novák:* O dimensi lexikografického součtu částečně uspořádaných množin. Čas. pěst. mat. *86* (1961), 385—391.

[9]) See [6], Theorem 1.

[10] *V. Novák:* On the pseudodimension of ordered sets. Czech. Math. Journ. *13* (1963), 587 to 598.

[11] *M. Novotný:* Über isotone Funktionale geordneter Mengen. Zeitschr. f. math. Logik und Grundlagen d. Math. *5* (1959), 9—28.

[12] *V. Sedmak:* Dimenzija djelomično uredenih skupova pridruženih poligonima i poliedrima. Glasnik mat.-fiz. i astr. *7* (1952), 169—182.

[13] *E. Szpilrajn-Marczewski:* Sur l'extension de l'ordre partiel. Fund. Math. *16* (1930), 386—389.

Резюме

О ЛЕКСИКОГРАФИЧЕСКОМ ПРОИЗВЕДЕНИИ УПОРЯДОЧЕННЫХ МНОЖЕСТВ

ВИТЕЗСЛАВ НОВАК (Vítězslav Novák), Брно

Лексикографическое произведение упорядоченных множеств определяется следующим образом ([3]): Пусть H – упорядоченное множество и пусть $\{G_\alpha \mid \alpha \in H\}$ – система упорядоченных множеств. Лексикографическое произведение $\prod_{\alpha \in H} G_\alpha$ есть множество всех функций f, отображающих H в $\bigcup_{\alpha \in H} G_\alpha$ со свойством $f(\alpha) \in G_\alpha$ для всякого $\alpha \in H$ с отношением $\leqq$, определяемым так: $f \leqq g \Leftrightarrow$ если существует $\alpha_0 \in H$ такое, что $f(\alpha_0) \neq g(\alpha_0)$, то существует $\alpha_1 \leqq \alpha_0$ такое, что $f(\alpha_1) < g(\alpha_1)$. Это отношение в общем случае только рефлексивно, так что оно не упорядочивает $\prod_{\alpha \in H} G_\alpha$. В статье доказывается, что $\prod_{\alpha \in H} G_\alpha$ есть упорядоченное множество тогда и только тогда, когда множество $H' = \{\alpha \mid \alpha \in H$, в G_α имеются по крайней мере два сравнимых элемента$\}$ удовлетворяет условию обрыва убывающих цепей. (Теорема 1.) Если это условие выполнено, то на H всегда можно определить новое упорядочение $\precsim$ таким образом, что $H(\precsim)$ удовлетворяет условию обрыва убывающих цепей и $\prod_{\alpha \in H(\leqq)} G_\alpha = \prod_{\alpha \in H(\precsim)} G_\alpha$. (Теорема 2.) Если на множестве G определены два упорядочения $\leqq$, $\precsim$, то упорядочение $\precsim$ называется продолжением упорядочения $\leqq$, если $x \leqq y \Rightarrow x \precsim y$. Продолжение $\precsim$ называется линейным продолжением, если G-цепь по отношению к $\precsim$. Линейное продолжение $\precsim$ называется полным продолжением, если G вполне упорядочено по отношению к $\precsim$. Доказывается: Упорядоченное множество G имеет по крайней мере одно полное продолжение тогда и только тогда, когда G удовлетворяет условию обрыва убывающих цепей. (Теорема 3.) Далее, в статье приведена оценка размерности ([4]) лексикографического произведения упорядоченных множеств. Доказывается: Пусть H – упорядоченное множество, удовлетворяющее условию обрыва убрывающих цепей, и пусть $\{G_\alpha \mid \alpha \in H\}$ – система упорядоченных множеств. Пусть $\mathscr{K}$ –

— система цепей на H, покрывающая H. Тогда $\dim \prod_{\alpha \in H} G_\alpha \leqq \sum_{K \in \mathscr{K}} \sup \{\dim G_\alpha \mid \alpha \in K\}$. (Теорема 5.) Эта теорема имеет ряд следствий. Так, например, размерность так называемого ординального произведения $\mathbf{P}_{\alpha \in H} G_\alpha$, т.е. лексикографического произведения, в котором H вполне упорядочено, равна $\sup \{\dim G_\alpha \mid \alpha \in H\}$. (Теорема 6.) Размерность ординальной степени ${}^H G$, в которой H вполне упорядочено равна размерности основания, т.е. $\dim {}^H G = \dim G$ (Следствие 9.) идр.