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ON A GENERALIZATION OF THE LEBESGUE INTEGRAL IN E_m

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A generalization γ of the integral defined in [4] and a simultaneous generalization σ of γ and of the Lebesgue integral are investigated. The well-known transformation formula with respect to a biunique regular mapping is proved for the integral σ and, with the help of γ, the Gauss' theorem on the representation of a surface integral by means of a volume integral is generalized.

1. Throughout this paper let m be an integer greater than 1. The meaning of the symbols $|A|$, $\|A\|$, $\bar{A}$, A°, $\dot{A}$, $\mathfrak{A}$, $P(A, v)$, $\mathfrak{Z}$, $\mathfrak{P}$, $Z_n \to Z$ $(Z_n, Z \in \mathfrak{Z})$, $\boldsymbol{u}\mathfrak{R}$, $\mathfrak{R}\mathfrak{T}$, $A\mathfrak{R}$ $(\mathfrak{R}, \mathfrak{T} \subset \mathfrak{Z}$, $A \in \mathfrak{Z})$, the operations in the ring $\mathfrak{Z}$ as well as the continuity and the additivity of a mapping of a set $\mathfrak{R} \subset \mathfrak{Z}$ into E_1 are defined in [4], section 1. Further let $\mathfrak{P}_0$ be the system of all sequences $\{A_n\}_{n=1}^{\infty}$ such that $A_n \in \mathfrak{A}$ $(n = 1, 2, \ldots)$, $\|A_n\| \to \to 0$. (We shall see that $\mathfrak{P}_0 \subset \mathfrak{P}$.)

2. *Let μ be a measure on a σ-algebra $\mathfrak{S}$. Let $f_1, \ldots, f_n$ be non-negative measurable functions on a set $S \in \mathfrak{S}$; suppose that $q_i > 1$ $(i = 1, \ldots, n)$ and $\sum_{i=1}^{n} 1/q_i = 1$. Then*

$$\int_S \prod_{i=1}^{n} f_i \, \mathrm{d}\mu \leqq \prod_{i=1}^{n} \left(\int_S f_i^{q_i} \, \mathrm{d}\mu \right)^{1/q_i}, \tag{1}$$

$$\int_S \prod_{i=1}^{n} f_i^{1/n} \, \mathrm{d}\mu \leqq \prod_{i=1}^{n} \left(\int_S f_i \, \mathrm{d}\mu \right)^{1/n}. \tag{2}$$

Proof. The relation (1) follows by induction from the Hölder inequality. If we set in (1) $f_i^{1/n}$ in place of f_i and n in place of q_i, we get (2).

3. *Let μ be a measure on a σ-algebra $\mathfrak{S}$. Let $f_1, \ldots, f_n$ $(n > 1)$ be non-negative measurable functions on a set $S \in \mathfrak{S}$ and let κ be a finite non-negative number. Then*

$$\int_S \min\left(\kappa, \prod_{i=1}^{n} f_i^{1/(n-1)} \right) \mathrm{d}\mu \leqq \left(\kappa \prod_{i=1}^{n} \int_S f_i \, \mathrm{d}\mu \right)^{1/n}. \tag{3}$$

Proof. The relation (3) is obvious if $\kappa = 0$; we may therefore assume that $\kappa > 0$. Denote by L the left-hand side of (3). Then $L = \kappa \int_S \min(1, \prod_{i=1}^{n} g_i^{1/(n-1)}) \, d\mu$ with $g_i = \kappa^{(1-n)/n} . f_i$. Since $\min(1, a^{1/(n-1)}) \leqq \min(1, a^{1/n}) \leqq a^{1/n}$ for every $a \geqq 0$, we have by (2) $L \leqq \kappa \int_S \prod_{i=1}^{n} g_i^{1/n} \, d\mu \leqq \kappa \prod_{i=1}^{n} (\int_S g_i \, d\mu)^{1/n} = (\kappa \prod_{i=1}^{n} \int_S f_i \, d\mu)^{1/n}$.

4. Let k, n be integers, $1 \leqq k \leqq n$, $n > 1$. For $x = [x_1, \ldots, x_n] \in E_n$ put $p_k(x) = [x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n]$. For $M \subset E_n$, $y = [y_1, \ldots, y_{n-1}] \in E_{n-1}$, $z \in E_1$ let M_y^k be the set of all $t \in E_1$ such that $[y_1, \ldots, y_{k-1}, t, y_k, \ldots, y_{n-1}] \in M$ and let M_k^z be the set of all $x = [x_1, \ldots, x_{n-1}] \in E_{n-1}$ such that $[x_1, \ldots, x_{k-1}, z, x_k, \ldots, x_{n-1}] \in M$.

5. *Let M be an open set in E_n $(n > 1)$. Then*

$$|M|^{n-1} \leqq \prod_{i=1}^{n} |p_i(M)| \, . \text{*)}$$

Proof. The case $n = 2$ is obvious. Suppose therefore that $n > 2$ and that the assertion holds for $n - 1$. We may assume that $|p_n(M)| < \infty$. Clearly $|M| = \int_{E_1} |M_n^z| \, dz$; the sets M_n^z are open in E_{n-1} and $M_n^z \subset p_n(M)$. By induction hypothesis, $|M_n^z|^{n-2} \leqq \prod_{i=1}^{n-1} |p_i(M_n^z)|$ for each z. It is easy to see that $p_i(M_n^z) = (p_i(M))_{n-1}^z$ for $i = 1, \ldots, n-1$. Thus we get $|M_n^z| \leqq \min(\kappa, \prod_{i=1}^{n-1} (f_i(z))^{1/(n-2)})$ with $\kappa = |p_n(M)|$, $f_i(z) = |(p_i(M))_{n-1}^z|$. Now the relations $\int_{E_1} f_i(z) \, dz = |p_i(M)|$ $(i = 1, \ldots, n-1)$ and (3) (with $n - 1$ in place of n) imply our assertion.

6. *Let M be a subset of E_m. Then*

$$|M|^{m-1} \leqq \prod_{i=1}^{m} |p_i(M)| \, . \tag{4}$$

Proof. Write $M_i = p_i(M)$. If $|M_i| = 0$ for some i, then $|M| = 0$ and (4) is valid. Hence the inequality (4) holds if $|M_i| = \infty$ for some i. Assume therefore that $\sum_{i=1}^{m} |M_i| < \infty$ and choose a number $\varepsilon > 0$. For every i there exists an open set $U_i \subset E_{m-1}$ such that $M_i \subset U_i$ and $|U_i| < |M_i| + \varepsilon$. Denote by U_i^* the set of all $x \in E_m$ with $p_i(x) \in U_i$; further put $V = \bigcap_{i=1}^{m} U_i^*$, $V_i = p_i(V)$. The set V is clearly open and $M \subset V$, $V_i \subset U_i$ $(i = 1, \ldots, m)$. By 5 we have $|M|^{m-1} \leqq |V|^{m-1} \leqq \prod_{i=1}^{m} |V_i| \leqq \prod_{i=1}^{m} |U_i| \leqq \prod_{i=1}^{m} (|M_i| + \varepsilon)$, whence (4) follows immediately.

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*) See also [5].

7. The meaning of the symbol $\|A\|_k$ for a bounded measurable set $A \subset E_m$ and for $k = 1, \ldots, m$ is defined by [1], 3. According to [1], 4 we have

$$\max_k \|A\|_k \leqq \|A\| \leqq \sum_{k=1}^{m} \|A\|_k . \tag{5}$$

Further put $\mathfrak{A}_k = \{A; \|A\|_k < \infty\}$. Given a bounded Borel function f on the boundary of a set $A \in \mathfrak{A}_k$, we define $P_k(A, f)$ as in [1], 14, remark 1. For $C, D \subset E_1$ we write $C \sim D$ if $|(C \cup D) - (C \cap D)| = 0$. From [1], 33 and 20 we get immediately:

Given a set $A \in \mathfrak{A}_k$, there exists a subset $K(k, A)$ of E_{m-1} with the following properties:

1) $|E_{m-1} - K(k, A)| = 0$;

2) *for each $x \in K(k, A)$ there exist a non-negative integer $r = \varphi_A^k(x)$ and real numbers a_i, b_i such that $a_1 < b_1 < \ldots < a_r < b_r$ and that $A_x^k \sim \bigcup_{i=1}^{r} (a_i, b_i)$*;

3) $2 \int_{E_{m-1}} \varphi_A^k(x) \, dx = \|A\|_k$;

4) *if f is a bounded Borel function on the boundary of A and if we put*
$\Phi_k(f, A, x) = \sum_{i=1}^{r} (f(x_1, \ldots, x_{k-1}, b_i, x_k, \ldots, x_{m-1}) - f(x_1, \ldots, x_{k-1}, a_i, x_k, \ldots, x_{m-1}))$
for each $x = [x_1, \ldots, x_{m-1}] \in K(k, A)$, then $P_k(A, f) = \int_{E_{m-1}} \Phi_k(f, A, x) \, dx$.

Now we can write $Q(k, A) = \{x \in K(k, A); \varphi_A^k(x) > 0\}$, $Z(k, A) = \{z \in A; p_k(z) \in Q(k, A)\}$.

8. *If $A \in \mathfrak{A}_k$, then*

$$2|Q(k, A)| \leqq \|A\|_k , \tag{6}$$

$$|A - Z(k, A)| = 0 . \tag{7}$$

Proof. The relation (6) is an immediate consequence of 7, 3). The set $(A - Z(k, A))_x^k$ is empty for $x \in Q(k, A)$ and has measure zero for $x \in K(k, A) - Q(k, A)$; hence (7) follows at once.

9. *If $A \in \mathfrak{A}$, then*

$$|A|^{m-1} \leqq \prod_{k=1}^{m} |Q(k, A)| . \tag{8}$$

Proof. Write $B = \bigcap_{k=1}^{m} Z(k, A)$. From (7) we obtain $|A| = |B|$; clearly $p_k(B) \subset Q(k, A)$. Now we apply (4).

10. *If A is a bounded measurable subset of E_m, then*

$$2^m |A|^{m-1} \leqq \prod_{k=1}^{m} \|A\|_k . \tag{9}$$

Proof. If $\|A\|_k = 0$ for some k, then, by (6), $|Q(k, A)| = 0$. In this case we have obviously $|Z(k, A)| = 0$ and, by (7), $|A| = 0$, so that (9) holds. Hence it follows that (9) holds if $\|A\|_k = \infty$ for some k. If $A \in \mathfrak{A}$, we obtain (9) by (8) and (6).

11. *We have* $\mathfrak{P}_0 \subset \mathfrak{P}$.

(This follows from (5) and (9).)

12. *Let* M_1, M_2 *be subsets of* E_1; *let* M_i *have* r_i *components* $(r_i < \infty, i = 1, 2)$. *Then the set* $M_1 - M_2$ *has at most* $r_1 + r_2$ *components.*

Proof. It is easy to see that the assertion holds for $r_2 = 1$. Now we proceed by induction.

13. *If* $\{A_n\} \in \mathfrak{P}_0$, $A \in \mathfrak{A}$, *then* $\{A_n \cap A\} \in \mathfrak{P}_0$, $\{A_n - A\} \in \mathfrak{P}_0$.

Proof. We use the notation of 7. Let k, n be natural numbers, $k \leqq m$. Write $B = A_n$, $C = A_n - A$, $K = K(k, A) \cap K(k, B) \cap K(k, C)$ and choose an $x \in K$. There exist numbers $a_1 < b_1 < \ldots < a_r < b_r$ $(r = \varphi_A^k(x))$ such that the set $J_A = \bigcup_{i=1}^{r} (a_i, b_i)$ fulfils the condition $J_A \sim A_x^k$; let J_B, J_C have analogous meaning and put $J = J_B - J_A$. Clearly $J \sim B_x^k - A_x^k = C_x^k$ and so $J \sim J_C$. According to 12 the set J has at most $\varphi_A^k(x) + \varphi_B^k(x)$ components and the number of the components of J_C is at most equal to that of J. Thus it is proved that $\varphi_C^k(x) \leqq \varphi_A^k(x) + \varphi_B^k(x)$. For $x \in K - Q(k, B)$ evidently $\varphi_C^k(x) = 0$. If we put $Q_n = Q(k, A_n)$, we have therefore $\frac{1}{2}\|A_n - A\|_k = \frac{1}{2}\|C\|_k = \int_{Q_n} \varphi_C^k(x)\,dx \leqq \int_{Q_n} \varphi_A^k(x)\,dx + \int_{Q_n} \varphi_B^k(x)\,dx \leqq \int_{Q_n} \varphi_A^k(x)\,dx + \frac{1}{2}\|A_n\|_k$. The relation $|Q_n| \leqq \frac{1}{2}\|A_n\|_k$ (see (6)) implies $|Q_n| \to 0$. Now it is easy to see that $\|A_n - A\|_k \to 0$. Hence it follows by (5) that $\|A_n - A\| \to 0$ and by [1], 35 we have $\|A_n \cap A\| \leqq \|A_n\| + \|A_n - A\| \to 0$.

14. We define a convergence $\overset{0}{\to}$ on the set $\mathfrak{Z}$ in the following way: $P_n \overset{0}{\to} P$ means that $P_n \subset P$ $(n = 1, 2, \ldots)$, $\{P - P_n\} \in \mathfrak{P}_0$. According to 13 and [2], 4, the convergence $\overset{0}{\to}$ satisfies the conditions 1), 2) of [2], 3 (with $A = \mathfrak{A}$, $Z = \mathfrak{Z}$). The closure of a set $\mathfrak{R} \subset \mathfrak{Z}$ with respect to this convergence is defined by [2], 1 and we denote it by $\boldsymbol{u}_0\mathfrak{R}$. The continuity of a mapping of a set $\mathfrak{R} \subset \mathfrak{Z}$ into E_1 with respect to the convergence $\overset{0}{\to}$ is defined in an obvious manner (see [2], 1). By 11, the relation $P_n \overset{0}{\to} P$ implies $P_n \to P$; therefore $\boldsymbol{u}_0\mathfrak{R} \subset \boldsymbol{u}\mathfrak{R}$ for each $\mathfrak{R} \subset \mathfrak{Z}$. If a mapping of a set $\mathfrak{R} \subset \mathfrak{Z}$ into E_1 is continuous with respect to $\to$, then it is continous with respect to $\overset{0}{\to}$ as well.

Let Ψ_0 be the set of all mappings ψ with the following property: The domain of definition, Dom ψ, of the mapping ψ is a subring of $\mathfrak{Z}$, $\mathfrak{A}$ Dom $\psi \subset$ Dom ψ, ψ is an additive mapping into E_1 and is continuous with respect to the convergence $\overset{0}{\to}$. Further let Ψ be the set of all mappings $\psi \in \Psi_0$ continuous with respect to the convergence $\to$.

With each $\psi \in \Psi$ let us associate a mapping $\beta(\psi)$ in the same way as in [2], 19 β was associated with μ. (We put, of course, $Z = \mathfrak{Z}$, $A = \mathfrak{A}$, $\mathfrak{G} = E_1$ and take the closure and the continuity with respect to $\rightarrow$.) By [2], 22 we have $\beta(\psi) \in \Psi$. (See also [2], 24.)

Replacing in the foregoing consideration the convergence $\rightarrow$ by the convergence $\overset{0}{\rightarrow}$ we obtain a transformation β_0 associating a mapping $\beta_0(\psi)$ with each $\psi \in \Psi_0$.

Now put $\gamma(\psi) = \beta_0(\beta(\psi))$ for each $\psi \in \Psi$. (We have $\gamma(\psi) \in \Psi_0$.) If $A \in \operatorname{Dom} \gamma(\psi)$, we write $(\gamma(\psi))(A) = \gamma(\psi, A)$; the symbols $\beta(\psi, A)$, $\beta_0(\psi, A)$ have an obvious meaning. Instead of "$A \in \operatorname{Dom} \gamma(\psi)$" we shall usually write "$\gamma(\psi, A)$ exists" etc.

15. *For each $\psi \in \Psi$ the following statements hold*:

1) $\operatorname{Dom} \beta(\psi)$, $\operatorname{Dom} \gamma(\psi)$ *are ideals in* $\mathfrak{A}$;

2) $\operatorname{Dom} \beta(\psi) \subset \operatorname{Dom} \gamma(\psi) \subset \boldsymbol{u}_0(\operatorname{Dom} \beta(\psi)) \subset \boldsymbol{u}(\operatorname{Dom} \psi)$;

3) $\beta(\psi, A) = \psi(A)$ *for each* $A \in \mathfrak{A} \cap \operatorname{Dom} \psi$ *and* $\gamma(\psi, A) = \beta(\psi, A)$ *for each* $A \in \operatorname{Dom} \beta(\psi)$.

Proof. The statement 1) holds according to [2], 22. From [2], 19 we obtain $\operatorname{Dom} \beta(\psi) \subset \boldsymbol{u}(\operatorname{Dom} \psi)$, $\operatorname{Dom} \gamma(\psi) \subset \boldsymbol{u}_0(\operatorname{Dom} \beta(\psi))$, whence, by [2], 20, we get 2) and 3).

16. a) *Suppose that* $\psi, \psi_1, \psi_2 \in \Psi$. *Let* $s = \gamma(\psi_1, A) + \gamma(\psi_2, A)$ *and let* $\psi(V) = \psi_1(V) + \psi_2(V)$ *hold for each* $V \in A\mathfrak{A} \cap \operatorname{Dom} \psi_1 \cap \operatorname{Dom} \psi_2$. *Then* $\gamma(\psi, A) = s$.

b) *Suppose that* $\chi, \psi \in \Psi$, $c \in E_1$. *Let* $\gamma(\psi, A)$ *exist and let* $\chi(V) = c\,\psi(V)$ *hold for each* $V \in A\mathfrak{A} \cap \operatorname{Dom} \psi$. *Then* $\gamma(\chi, A) = c\,\gamma(\psi, A)$.

c) *If* $\psi \in \Psi$, $c \in E_1$, $c \neq 0$, *then* $\operatorname{Dom} \gamma(\psi) = \operatorname{Dom} \gamma(c\psi)$.

Proof. By theorem 25 of [2] we have $\beta(\psi, B) = \beta(\psi_1, B) + \beta(\psi_2, B)$ for $B \in A\mathfrak{A} \cap \operatorname{Dom} \beta(\psi_1) \cap \operatorname{Dom} \beta(\psi_2)$ and from the same theorem we get $\gamma(\psi, A) = \beta_0(\beta(\psi), A) = \beta_0(\beta(\psi_1), A) + \beta_0(\beta(\psi_2), A) = s$. Using theorems 26 and 29 of [2], we can prove b) and c), respectively, in a similar way.

17. The meaning of the symbols $\mathscr{F}$, $\lambda(f)$, $\mathfrak{M}(f)$ is defined in [4], 1. Further let Λ be the set of all mappings $\lambda(f)$ $(f \in \mathscr{F})$. By [4], 1 and 5 we have $\Lambda \subset \Psi$. Instead of $\gamma(\lambda(f))$ we write $\gamma(f)$. For $A \in \operatorname{Dom} \gamma(f)$ we put $(\gamma(f))(A) = \gamma(f, A)$; instead of "$A \in \operatorname{Dom} \gamma(f)$" we say "$\gamma(f, A)$ exists" etc. If we write $\beta(f) = \beta(\lambda(f))$ (as in [4], 6), then obviously $\gamma(f) = \beta_0(\beta(f))$.

If α is a mapping of a set $\mathfrak{R} \subset \mathfrak{Z}$ and if $Z \in \mathfrak{Z}$, we define mappings α_Z, α'_Z by setting $\alpha_Z(C) = \alpha(C \cap Z)$ for every C with $C \cap Z \in \mathfrak{R}$ and $\alpha'_Z(C) = \alpha(C - Z)$ for every C with $C - Z \in \mathfrak{R}$. (This is consistent with [3], 1.) If $f \in \mathscr{F}$, $Z \in \mathfrak{Z}$ and if $M = \operatorname{Dom} f$, we put $f_Z(x) = f(x)$ for $x \in Z \cap M$, $f_Z(x) = 0$ for $x \in E_m - Z$ (so that $\operatorname{Dom} f_Z = M \cup (E_m - Z)$). If either $(\lambda(f))_Z(C)$ or $\lambda(f_Z, C)$ exists, then obviously $(\lambda(f))_Z(C) = \lambda(f, Z \cap C) = \lambda(f_Z, C)$; hence

$$(\lambda(f))_Z = \lambda(f_Z)\,. \tag{10}$$

We see that $\mu_Z \in \Lambda$ for each $\mu \in \Lambda$ and for each $Z \in \mathfrak{Z}$. Since $\mu'_Z = \mu_V$ with $V = E_m - \acute{Z}$, we have $\mu'_Z \in \Lambda$ too. Choose a $c \in E_1$ and put $Z = \mathfrak{Z}$, $A = \mathfrak{A}$, $\mathfrak{G} = E_1$, $\Theta = \Psi_0$, $\omega(t) = ct$ $(t \in E_1)$ in [3], 1 and 2. Then the set Λ and the transformation $\mu \to \gamma(\mu)$ $(\mu \in \Lambda)$ fulfil the condition R1) of [3], 2. The obvious relation $\lambda(-f) = -\lambda(f)$, 16, c) and 15, 1) imply R2); 15, 3) implies R3); 16, a) implies R4) and 16, b) implies R5) ([3], 2). Hence by [3], 8 we can associate a mapping $\sigma(\mu, .)$ with each $\mu \in \Lambda$. If $\mu = \lambda(f)$, we write $\sigma(\mu, .) = \sigma(f, .)$.

18. *Suppose $f \in \mathscr{F}$. Then $\sigma(f, S)$ exists if and only if there is an $A \in \mathfrak{A}$ such that the sum*

$$s = \gamma(f_S, A) + \lambda(f, S - A) \tag{11}$$

is meaningful; in this case $\sigma(f, S) = s$.

(This follows from (10) and [3], 8.)

19. *Let $\sigma(f, S)$ exist. Then f is measurable on S, $f(x) \in E_1$ for almost all $x \in S$ and there are $A_n \in \mathfrak{A}$ such that $|A_n| \to 0$, $S - A_n \in \mathfrak{M}(f)$ $(n = 1, 2, \ldots)$.*

Proof. Choose an $A \in \mathfrak{A}$ such that the sum (11) has a meaning. By 15, 2) we have $A \in \mathbf{u}(\mathfrak{M}(f_S))$; by [4], 2 there exist $A_n \in \mathfrak{A}$ such that $A_n \subset A$, $|A_n| \to 0$, $A - A_n \in \mathfrak{M}(f_S)$. Since $S - A_n = (S - A) \cup (S \cap (A - A_n))$, we have $S - A_n \in \mathfrak{M}(f)$. Hence it follows that f is measurable on S and that $f(x) \in E_1$ almost everywhere on S.

Remark. The following assertions 20–27 follow easily from [3], sections 10, 11, 17, 21, 15, 13, 9, 22 and 16.

20. *Suppose that $f, g, h \in \mathscr{F}$. If $s = \sigma(f, S) + \sigma(g, S)$ and if $h(x) = f(x) + g(x)$ for almost all $x \in S$, then $\sigma(h, S) = s$.*

21. *Suppose that $f, g \in \mathscr{F}$, $c \in E_1$. If $\sigma(f, S)$ exists and if $g(x) = c\, f(x)$ for almost all $x \in S$, then $\sigma(g, S) = c\, \sigma(f, S)$.*

22. *If $S_1 \subset S_2$, $S_3 \cap S_4 = \emptyset$, then $\sigma(f, S_2 - S_1) = \sigma(f, S_2) - \sigma(f, S_1)$, $\sigma(f, S_3 \cup S_4) = \sigma(f, S_3) + \sigma(f, S_4)$, whenever the corresponding right-hand side has a meaning.*

23. *If $A \in \mathfrak{A}$ and if $\sigma(f, S)$ exists, then $\sigma(f, S \cap A)$ exists.*

24. *If $A \in \mathfrak{A}$, $A \subset S$ and if $\sigma(f, S)$ exists, then $\gamma(f, A)$ exists.*

25. *If $S, T \in \mathfrak{Z}$, $f \in \mathscr{F}$, then $\sigma(f_S, T) = \sigma(f, S \cap T)$, whenever at least one side of this equality has a meaning.*

26. *For each $f \in \mathscr{F}$, the mapping $\sigma(f, .)$ is an extension of both mappings $\lambda(f)$, $\gamma(f)$ and is continuous with respect to the convergence $\overset{0}{\to}$.*

27. *If $f \in \mathscr{F}$, $A \in \mathfrak{A}$, then $\sigma(f, A) = \gamma(f, A)$ whenever at least one side of this equality has a meaning.*

28. *Let ζ be a biunique regular mapping of an open set $G \subset E_m$ into E_m. If S is a bounded set such that $\bar{S} \subset G$ and if $S_n \overset{0}{\to} S$, then $\zeta(S_n) \overset{0}{\to} \zeta(S)$.*

(This follows from [4], 9.)

29. Theorem. *Let ζ be a biunique regular mapping of an open set $G \subset E_m$ into E_m; let D be the functional determinant of ζ and let $f \in \mathscr{F}$. Put $g(x) = f(\zeta(x)) |D(x)|$ for all $x \in G$ with $\zeta(x) \in \operatorname{Dom} f$. Suppose that $S \subset G$, $\bar{T} \subset G$. Then the following assertions hold:*

a) *$\gamma(g, T) = \gamma(f, \zeta(T))$, whenever at least one side of this equality has a meaning;*

b) *if $\lambda(g, S - T)$ and $\sigma(g, S)$ exist, then $\sigma(g, S) = \sigma(f, \zeta(S))$.*

Proof. Since $\gamma(g) = \beta_0(\beta(g))$, the assertion a) can be proved in a similar way as theorem 11 in [4] (with the help of this theorem and of lemma 28). Now let $\lambda(g, S - T)$ and $\sigma(g, S)$ exist. Put $R = \zeta(S)$ and $g^*(x) = f_R(\zeta(x)) |D(x)|$ for all $x \in G$ with $\zeta(x) \in \operatorname{Dom} f_R$. Clearly $g_S(x) = g^*(x)$ for all $x \in G \cap \operatorname{Dom} g_S$. According to 18 there is an $A \in \mathfrak{A}$ such that $S - A \in \mathfrak{M}(g)$. The set $V = A \cap T$ is bounded and $\bar{V} \subset G$; hence there is a compact set $K \in \mathfrak{A}$ with $V \subset K \subset G$. On account of 23, $\sigma(g, S \cap K)$ exists and by 25 we have $\sigma(g, S \cap K) = \sigma(g_S, K)$. From 19 and 21 (with $c = 1$) we obtain $\sigma(g_S, K) = \sigma(g^*, K)$; by 27, $\sigma(g^*, K) = \gamma(g^*, K)$; by a), $\gamma(g^*, K) = \gamma(f_R, \zeta(K))$; by 27 and 25, $\gamma(f_R, \zeta(K)) = \sigma(f, R \cap \zeta(K))$. Hence

$$\sigma(g, S \cap K) = \sigma(f, R \cap \zeta(K)) . \tag{12}$$

As $S - K \subset (S - A) \cup (S - T)$, we have $S - K \in \mathfrak{M}(g)$; by 26 and by the transformation theorem for the Lebesgue integral we get

$$\sigma(g, S - K) = \lambda(g, S - K) = \lambda(f, R - \zeta(K)) = \sigma(f, R - \zeta(K)) . \tag{13}$$

The relations (12) and (13) imply b).

30. In the rest of this paper, the symbol H denotes the outer $(m - 1)$-dimensional Hausdorff measure in E_m. The term "vector" is used for a mapping into E_m. The meaning of the expression "continuous vector" etc. is obvious.

31. *Suppose $A \in \mathfrak{A}$. Let v, w be bounded Borel vectors on $\dot{A}$ such that $v(z) = w(z)$ for H – almost all $z \in \dot{A}$. Then $P(A, v) = P(A, w)$.*

Proof. Put $v = [v_1, \ldots, v_m]$, $w = [w_1, \ldots, w_m]$. It is easy to see that, with the notation of 7, $\Phi_k(A, v_k, x) = \Phi_k(A, w_k, x)$ for almost all $x \in E_{m-1}$; hence $P_k(A, v_k) = P_k(A, w_k)$ for $k = 1, \ldots, m$. By [1], 15 we have $P(A, v) = P(A, w)$.

32. Suppose $A \in \mathfrak{A}$, $D \subset E_m$, $H(D) = 0$ and let v be a bounded continuous vector on $\dot{A} - D$. It is easy to see that there exists a bounded Borel vector w on $\dot{A}$ such that

$w(z) = v(z)$ for H – almost all $z \in \dot{A}$. According to 31 the number $P(A, w)$ does not depend on the choice of w so that we can define $P(A, v) = P(A, w)$. If $v = [v_1, \ldots, v_m]$ and if $(\sum_{i=1}^{m} (v(x))^2)^{\frac{1}{2}} \leqq c$ for $x \in \dot{A} - D$, we can choose w in such a way that $(\sum_{i=1}^{m} (w_i(x))^2)^{\frac{1}{2}} \leqq c$ for each $x \in \dot{A}$; then, by [1], 16, c),

$$(14) \qquad |P(A, v)| = |P(A, w)| \leqq c\|A\| .$$

33. Theorem. *Let a $D \subset E_m$, an $A \in \mathfrak{A}$ and an open set $G \subset E_m$ be given such that $H(D) = 0$ and $\bar{A} - G = \bigcup_{n=1}^{\infty} M_n$ with $H(M_n) < \infty$ $(n = 1, 2, \ldots)$. Let v be a bounded continuous vector on $(\bar{A} - D) \cup G$; let f be a function on G such that $\lambda(f, K)$ exists and is equal to $P(K, v)$ for each cube $K \subset G$. Then $\gamma(f, A)$ exists and is equal to $P(A, v)$.*

Proof. According to [4], 21, there exist open sets U_n such that $D \subset U_n \in \mathfrak{A}$, $\|U_n\| \to 0$. Put $A_n = A - U_n$. Then $\|A - A_n\| = \|A \cap U_n\|$ and by 13 we have $A_n \overset{0}{\to} A$. The relation $A_n \subset \bar{A} - U_n$ implies $\bar{A}_n \subset \bar{A} - U_n \subset \bar{A} - D$. Let us denote by $\mathfrak{R}$ the system of all $B \in \mathfrak{A}$ with $\bar{B} \subset \bar{A} - D$. If $B \in \mathfrak{R}$, then $\bar{B} - G \subset \bar{A} - G = \bigcup_{n=1}^{\infty} M_n$ and v is continuous on $\bar{B} \cup G$. According to theorems 23 and 14 of [4], $\beta(f, B)$ exists and is equal to $P(B, v)$. Since $A_n \in \mathfrak{R}$, we have $A \in \mathbf{u}_0\mathfrak{R}$. Put $\varphi(C) = P(C, v)$ for each $C \in A\mathfrak{A}$. The relation (14) implies easily that φ is continuous with respect to the convergence $\overset{0}{\to}$. Since φ and β coincide on $\mathfrak{R} \cap A\mathfrak{A}$, it follows from [2], 21 that $\gamma(f, A) = \beta_0(\beta(f), A) = \varphi(A) = P(A, v)$.

34. Example 1. Put $f(x, y) = x^{-2} \sin x^{-1}$ for $x > 0$, $y \in E_1$. Further define $a_n = ((2n + 1)\pi)^{-1}$, $b_n = (2n\pi)^{-1}$, $T_n = \{[x, y];\ 0 < y < x < b_n\}$, $A_n = \{[x, y];\ a_n < x < b_n,\ 0 < y < a_n\}$, $S_n = \bigcup_{k=n}^{2n} A_k$. Obviously $\int_{A_n} f(x, y)\,dx\,dy = a_n(\cos 2n\pi - \cos(2n + 1)\pi) = 2a_n$, $\|A_n\| = 2b_n$, $S_n \subset T_n \subset T_1$, $|S_n| < |T_n| = \frac{1}{2}b_n^2$, $\|S_n\| = \sum_{k=n}^{2n} \|A_k\| = 2\sum_{k=n}^{2n} b_k = (1/\pi)\sum_{k=n}^{2n} k^{-1} \to (\log 2)/\pi$, $\int_{S_n} f(x, y)\,dx\,dy = 2\sum_{k=n}^{2n} a_k > 2\sum_{k=n}^{2n} b_{k+1} \to (\log 2)/\pi$. It follows that $\{S_n\} \in \mathfrak{P}$ and that $\beta(f, T_1)$ does not exist. But if we set in 33 $G = \{[x, y];\ x > 0\}$, $v(x, y) = [\cos x^{-1}, 0]$, we see that $\gamma(f, T_1)$ exists.

Example 2. Write $C = (0, 1) \times (0, 1)$ and $f(x, y) = x^{-1} \sin x^{-1}$ for $[x, y] \in C$. For $\varepsilon > 0$ put further $M_\varepsilon = \{[x, y] \in C;\ f(x, y) > 1/\varepsilon\}$, $P_\varepsilon = (0, \varepsilon) \times (0, 1)$. Let us denote by $\mathfrak{V}$ the system of all measurable sets $V \subset E_2$ with $\lim_{\varepsilon \to 0+} |V \cap M_\varepsilon|/\varepsilon = 0$. If $B \in \mathfrak{M}(f)$, then $|B \cap M_\varepsilon|/\varepsilon \leqq \lambda(f, B \cap M_\varepsilon)$ and so $B \in \mathfrak{V}$; thus we see that $\mathfrak{M}(f) \subset \mathfrak{V}$.

Now suppose $V_n \in \mathfrak{V}$, $V_n \overset{0}{\to} V$ and put $S_n = V - V_n$. By (8) and (6) we have $|S_n \cap M_\varepsilon| \leqq |S_n \cap P_\varepsilon| \leqq |Q(2, P_\varepsilon)| \cdot |Q(1, S_n)| \leqq \varepsilon \cdot \frac{1}{2}\|S_n\|$; since $\|S_n\| \to 0$, $V_n \in \mathfrak{V}$ and $|V \cap M_\varepsilon| \leqq |V_n \cap M_\varepsilon| + |S_n \cap M_\varepsilon|$, we have $V \in \mathfrak{V}$. This implies $\boldsymbol{u}_0(\mathfrak{M}(f)) \subset \subset \boldsymbol{u}_0\mathfrak{V} = \mathfrak{V}$. As, evidently, C does not belong to $\mathfrak{V}$, $\beta_0(\lambda(f), C)$ does not exist; but, according to [4], 27, $\beta(\lambda(f), C)$ exists (and so $\gamma(f, C) = \beta_0(\beta(\lambda(f)), C)$ exists as well).

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Резюме

ОБ ОДНОМ ОБОБЩЕНИИ ИНТЕГРАЛА ЛЕБЕГА В E_m

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Пусть f — функция, определенная в некоторой части пространства E_m. В статье вводится интеграл $\gamma(f, .)$, который является расшиярением интеграла $\beta(f, .)$ из статьи [4]. Далее вводится интеграл $\sigma(f, .)$, который является одновременным расщирением интеграла $\gamma(f, .)$ и интеграла Лебега от функции f. Отображение $\sigma(f, S)$ аддитивно по отношении к S и линейно по отношении к f. Пусть $\|A\|$ означает периметр ограниченного измеримого множества $A \subset E_m$. Если $\sigma(f, S)$ существует и если $A_n \subset S$ $(n = 1, 2, \ldots)$, $\|A_n\| \to 0$, то $\sigma(f, A_n) \to 0$. Если $\sigma(f, S)$ существует и если $\|A\| < \infty$, то $\sigma(f, S \cap A)$ существует тоже. При взаимно однозначном регулярном отображении σ изменяется по известной формуле.

Пусть, далее, H — $(m - 1)$-мерная хаусдогфова мера в E_m. Пусть A — ограниченное множество в E_m и пусть $H(\dot{A}) < \infty$, где $\dot{A}$ — граница A; пусть v — ограниченный вектор, непрерывный H — почти всюду на $\bar{A}$, для которого существуют непрерывные частные производные первого порядка внутри множества A. Тогда существует γ (div v, A) и равняется поверхностному интегралу вектора v через границу множества A.