Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 15 (1965), No. 2, 169–178

Persistent URL: http://dml.cz/dmlcz/100660

ЧЕХОСЛОВАЦКИЙ МАТЕМАТИЧЕСКИЙ ЖУРНАЛ

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Т. *15* (*90*) ПРАГА 18. VI. 1965 г., No 2

CONTINUOUS DEPENDENCE OF EIGENVALUES ON THE DOMAIN

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(Received January 16, 1961 — in revised form August 4, 1964)

1. INTRODUCTION, NOTATION

Recently I. HONG [1], [2], [3] investigated the continuous dependence of eigenfunctions and eigevalues for the Laplace operator on the domain. We employ the variational method, which enables us to prove the continuous dependence of eigenvalues and eigenfunctions on the domain for selfadjoint elliptic operators of higher orders.

We employ the following notation: G will be an open bounded set of the r-dimensional Euclidean space E_r, $\bar{G}$ the closure of G, $\dot{G}$ the boundary of G; $D(G)$ the set of infinitely continuously differentiable functions with compact support in G, for the elements of $D(G)$ small Greek letters will be used. The symbol D^i, where $i = (i_1, i_2, \ldots, i_r)$ (i_s being nonnegative integers), will denote the weak derivative of order $i = (i_1, i_2, \ldots, i_r)$

$$\left(D^i = \frac{\partial^{|i|}}{\partial_{x_1}^{i_1} \partial_{x_2}^{i_2}, \ldots, \partial_{x_r}^{i_r}}\right).$$

Let m and n be nonnegative integers, $m > n$. Let a_{ij}, b_{ij} be bounded measurable function on E_r, $a_{ij} = a_{ji}$, $b_{ij} = b_{ji}$. For $\varphi \in D(G)$, $\psi \in D(G)$ we put

$$(1) \qquad (\varphi, \psi)_m = \sum_{|i|=m} \int_{E_r} D^i\varphi D^i\psi \,\mathrm{d}x ,$$

$$(2) \qquad (\varphi, \psi)_n = \sum_{|i|=n} \int_{E_r} D^i\varphi D^i\psi \,\mathrm{d}x ,$$

$$(3) \qquad \{\varphi, \psi\} = \sum_{|i|=|j|=m} \int_{E_r} a_{ij} D^i\varphi D^j\psi \,\mathrm{d}x ,$$

$$(4) \qquad [\varphi, \psi] = \sum_{|i|=|j|=n} \int_{E_r} b_{ij} D^i\varphi D^j\psi \,\mathrm{d}x .$$

Let us assume that the equations (3) and (4) define scalar products. Completing $D(G)$ under the norms which are associated with scalar products (1)–(4) we obtain the spaces $\mathring{W}_2^m$, $\mathring{W}_2^n$, H_m, H_n respectively. Denoting the norm of an element u belonging to $\mathring{W}_2^m$, $\mathring{W}_2^n$, H_m, H_n by $|u|_{W^m}$, $|u|_{W^n}\|u\|_m$, $\|u\|_n$ respectively, we see immediately that

$$\|u\|_m \leqq C_1|u|_{W^m}\,, \quad \|u\|_n \leqq C_2|u|_{W^n}\,.$$

We shall assume once and for all that the opposite inequalities are also true, i.e. that there are constants C_3, C_4 such that

$$\|u\|_m \geqq C_3|u|_{W^m}\,, \quad \|u\|_n \geqq C_4|u|_{W^n}$$

for any u belonging to H_m, H_n respectively. Under these restrictions H_m and H_n are Hilbert spaces.

We say the function u is the generalized eigenfunction provided, there is a number λ such that for any $\varphi \in D(G)$ the relation

$$\{u, \varphi\} = \lambda[u, \varphi]$$

holds. The number λ is by definition generalized eigenvalue. Sometimes we shall omit the word generalized.

If the coefficients and the domain are suitably regular generalized eigenfunctions satisfy the equation

$$(-1)^m \sum_{|i|=|j|=m} D^i a_{ij} D^j u = (-1)^n \lambda \sum_{|i|=|j|=n} D^i b_{ij} D^j u$$

and the normal derivatives of u up to the order m vanish on the boundary. In this paper we shall avoid the regularity problem of eigenfunctions, because such considerations would involve some restrictions on the open set G and we are going to consider most general regions.

We now summarize some known results, which can be proved using Hilbert space technique [4] [5].

I. The smallest eigenvalue λ_1 is the greatest lower bound of the functional

$$\frac{\{u, u\}}{[u, u]} \tag{1}$$

for $u \in H_m$, $u \neq 0$. The greatest lower bound is attained by the function u_1, which is the eigenfunction.

II. The eigenvalues (if suitably arranged) form a nondecreasing divergent sequence. To every eigenvalue there corresponds an eigenfunction. If $\lambda_1, \ldots, \lambda_{r-1}$, are the first eigenvalues and $u_1, \ldots, u_{r-1}$ the corresponding eigenfunction, then the r^{th} eigenfunction gives the minimum value to the functional (1) between all functions u

satisfying following conditions $[u, u_t] = 0$, $t = 1, 2, \ldots, r - 1$. The eigenfunctions obtained in such a way are orthogonal and obviously, they can be supposed to be orthonormal. The set $\{u_t\}$ is complete in H_m i. e. for every $x \in H_m$ the formula

$$x = \sum_{t=1}^{\infty} c_t u_t$$

is valid (where $c_i = \{x, u_i\}$) the convergence being understood in H_m.

III. The p-th eigenvalue and p-th eigenfunction can be obtained also in the following way. One choose $p - 1$ linear independent function v_t and looks for the minimum of the functional

$$\{u, u\}$$

under the conditions

$$[u, u] = 1\,, \quad [u, v_i] = 0\,, \quad i = 1, \ldots, p - 1\,.$$

This minimum A depends on the choice of v_t; $A = A(v_1, v_2, \ldots, v_{p-1})$. The inequality

$$A(v_1, v_2, \ldots, v_{p-1}) \leqq \lambda_p$$

is always valid and A attains its maximum value λ_p for $u_t = v_t$, $t = 1, 2, \ldots, p - 1$.

An immediate consequence of III. is that the eigenvalues are nonincreasing functions of the domain, i.e. if $\lambda_s(G_k)$ is the s-th eigenvalue for the region G_k, $k = 1, 2$ and $G_1 \subset G_2$, then $\lambda_s(G_1) \geqq \lambda_s(G_2)$.

2. LEMMAS

Lemma 1. *Let p be a positive integer, A positive number. There exists a constant M depending only on A and p (and diameter of G) such that, if*

1) $u_t \in H_m$, $[u_t, u_s] = \delta_{ts}$, $\|u_t\|_m \leqq A$, $t, s = 1, 2, \ldots, p$,

2) *for* $y_s \in H_m$ *and sufficiently small* $\varepsilon > 0$

$$\|u_s - y_s\|_m < \varepsilon\,, \quad s = 1, 2, \ldots, p\,.$$

Then there exists a set of elements $v_s \in H_m$ $s = 1, 2, \ldots, p$ *such that*

I. v_t *are linear combinations of* y_t

II. $[v_t, v_s] = \delta_{ts}$

III. $\|u_t - v_t\|_m < M\varepsilon\,,\ t = 1, 2, \ldots, p.$

Remark. The assertion III. of lemma 1 will be abbreviated by $\|u_t - v_t\|_m = O(\varepsilon)$.

Proof. The case $p = 1$ is obvious. Assuming the theorem is true for p we shall prove it for $p + 1$. First of all it follows from the assumption 2) that $\|u_i - y_i\|_n =$

$= O(\varepsilon)$ and hence $\|y_t\|_n = 1 + O(\varepsilon)$, $t = 1, 2, \ldots, p, p + 1$. By induction hypothesis we have $\|u_t - v_t\|_m = O(\varepsilon)$, $t = 1, 2, \ldots, p$ and hence $\|u_t - v_t\|_n = O(\varepsilon)$. Therefore

$$[y_{p+1}, v_t] = [u_{p+1}, v_t] + [y_{p+1} - u_{p+1}, v_t] = \\ = [u_{p+1}, v_t] + O(\varepsilon) = [u_{p+1}, u_t] + [u_{p+1}, v_t - u_t] + O(\varepsilon) = O(\varepsilon).$$

Putting

$$\bar{v}_{p+1} = y_{p+1} - \sum_{t=1}^{p-1} [y_{p+1}, v_t]\, v_t$$

we obtain successively

$$\|\bar{v}_{p+1} - y_{p+1}\|_n = O(\varepsilon), \quad \|\bar{v}_{p+1}\|_n = 1 + O(\varepsilon), \quad \|\bar{v}_{p+1}\|_n \neq 0 .$$

Now, we are allowed to put

$$v_{p+1} = \frac{\bar{v}_{p+1}}{\|\bar{v}_{p+1}\|_n} .$$

The set of functions $v_1, v_2, \ldots, v_p, v_{p+1}$ is the desired one. As a matter of fact $\|u_t - v_t\|_m = O(\varepsilon)$, $t = 1, \ldots, p$, by induction hypothesis and

$$\|u_{p+1} - v_{p+1}\|_m \leqq \|u_{p+1} - y_{p+1}\|_m + \|y_{p+1} - \bar{v}_{p+1}\|_m + \|\bar{v}_{p+1} - v_{p+1}\|_m = O(\varepsilon) .$$

Lemma 2. *Let us assume the hypothesis 1) from lemma 1. Then there is a system $\psi_1, \ldots, \psi_p \in D(G)$ such, that*

$$\|u_t - \psi_t\|_m < \varepsilon \tag{1}$$

$$[\psi_t, \psi_s] = \delta_{ts} \tag{2}$$

$t, s = 1, 2, \ldots, p.$

Lemma 2 follows immediately from lemma 1 using the definition of the space H_m.

Lemma 3. *Let u_t $(t = 1, 2, \ldots, p)$ be a set of elements of H_m such that*

$$[u_t, u_s] = \delta_{ts} \quad t, s = 1, 2, \ldots, p . \tag{3}$$

If for any $\varphi \in D(G)$

$$\{u_t, \varphi\} = \lambda_t [u_t, \varphi] , \tag{4}$$

then to every $\varepsilon > 0$ there exists a system of functions $\psi_1, \psi_2, \ldots, \psi_t \in D(G)$ such that in addition to (1) and (2) the relation

$$\{\psi_t, \psi_s\} = \delta_{ts}\lambda_s + O(\varepsilon) \tag{5}$$

is valid.

Proof. The existence of ψ_t satisfying (1) and (2) is insured by lemma 2. Since obviously $\{u_t, u_s\} = \delta_{ts}\lambda_s$, the desired inequality (5) follows from

$$\{\psi_t, \psi_s\} = \{\psi_t - u_t, \psi_s\} + \{u_t, \psi_s\} = \{u_t, u_s\} + \{u_t, \psi_s - u_s\} + O(\varepsilon) = \lambda_s\delta_{ts} + O(\varepsilon).$$

In the sequel we shall keep the following notation; s is positive integer, p is non-negative integer, K_p is the subspace of H_m spanned on the first p eigenfunctions $u_1, u_2, \ldots, u_p$ (for convenience we put $K_0 = 0$ and $\lambda_0 = 0$), $\overline{K}_p$ denotes orthogonal complement of K_p in H_m.

Lemma 4. *If* $w \in H_m$, $\|w\|_n = 1$, $w = x + y + z$, $x \in K_{s-1}$, $y \in K_{s+p} \cap \overline{K}_{s-1}$, $z \in \overline{K}_{s+p}$, $\|w\|_m^2 = \lambda^*$, $\|x\|_n \leqq \delta$ *and if*

$$\lambda_s \leqq \lambda_{s+1} \leqq \ldots \leqq \lambda_{s+p} = \lambda_{s+p+1} - \alpha\,, \quad \alpha > 0$$

(λ_t *being the t-th eigenvalue), then*

$$\|z\|_m^2 \leqq \frac{1}{\alpha}\left[\lambda^* - \lambda_s + \delta^2\lambda_s\right].$$

Proof. Obviously

$$\lambda^* = \{w, w\} \geqq \|y\|_m^2 + \|z\|_m^2\,. \tag{6}$$

By the definition of y and z we have

$$\|y\|_m^2 \geqq \lambda_s\|y\|_n^2\,, \tag{7}$$

$$\|z\|_m^2 \geqq \lambda_{s+p+1}\|z\|_n^2\,. \tag{8}$$

In view of the evident equation

$$1 = \|w\|_n^2 = \|x\|_n^2 + \|y\|_n^2 + \|z\|_n^2 \tag{9}$$

and the assumption $\|x\|_n < \delta$ we obtain from (6), (7) (8)

$$\lambda^* \geqq \lambda_s(1 - \delta^2) + \alpha\|z\|_n^2\,.$$

The lemma has been proved.

Lemma 5. *Let us suppose in addition to the assumptions of lemma* 4 $\|z\|_n \leqq \delta$. *Then*

$$\|x\|_m^2 \leqq \lambda_{s-1}\delta^2 \tag{10}$$

$$\|z\|_m^2 \leqq \lambda^* - \lambda_s + 2\delta^2\lambda_s\,. \tag{11}$$

Proof. The inequality (10) follows from the definition of x. Further, we have

$$\lambda^* = \{w, w\} \geqq \lambda_s\|y\|_n^2 + \|z\|_m^2\,,$$

and since $\|y\|_n^2 \geqq 1 - 2\delta^2$, we obtain

$$\lambda^* \geqq \lambda_s(1 - 2\delta^2) + \|z\|_m^2 ,$$

which is the inequality we wanted to prove.

Remark. In the proof of lemma 5 we did not make use of the assumption $\alpha > 0$ from lemma 4 and this assumption may be omitted in lemma 5.

3. APPROXIMATION OF THE DOMAIN FROM INTERIOR

Let G^k be a sequence of open set. Let us denote by λ_p^k, λ_p the p-th generalized eigenvalue and u_p^k, u_p the p-th eigenfunction for the set G^k, G respectively.

Theorem 1. *If $G^k \subset G^{k+1} \subset G$ and if $G \subset \bigcup_{k=1}^{\infty} G^k$ then*

$$\lim_{k \to \infty} \lambda_p^k = \lambda_p . \tag{1}$$

Proof. First of all $\lambda_p \leqq \lambda_p^{k+1} \leqq \lambda_p^k$, hence there exists $\lim_{k \to \infty} \lambda_p^k = \tilde{\lambda}$

$$\lambda_p \leqq \tilde{\lambda} . \tag{2}$$

Let us consider the function ψ_i $(t = 1, 2, \ldots, p)$ from lemma 3

$$A(\psi_1, \psi_2, \ldots, \psi_{p-1}) \leqq \{\psi_p, \psi_p\} \leqq \lambda_p + O(\varepsilon) .$$

If k is large enough, i.e. if G^k contains supports of all ψ_i $(t = 1, 2, \ldots, p)$ then by III. section 1

$$\lambda_p^k \leqq \lambda_p + O(\varepsilon)$$

and hence

$$\tilde{\lambda} \leqq \lambda_p + O(\varepsilon) . \tag{3}$$

The inequalities (2) and (3) prove the theorem.

We say $\lim_{k \to \infty} G^k = G$ provided that

i) to every compact set $F \subset G$ there is a number k_0 such that $F \subset G^k$ for $k > k_0$.
ii) to every open set $0 \supset \bar{G}$ there exists a number k_0 such that $G^k \subset 0$ for $k > k_0$.

Theorem 2. *If $G^k \subset G$ and if $\lim_{k \to \infty} G^k = G$ then (1) holds.*

Theorem 2 follows immediately from theorem 1.

Let L_1 and L_2 be s-dimensional subspaces of H_m and $w_1, w_2, \ldots, w_s, v_1, v_2, \ldots, v_s$ orthonormal bases in L_1, L_2 respectively. We put

$$\tau(L_1, L_2) = \inf \sum_{i=1}^{s} \|w_i - v_i\|_m ,$$

where the greatest lower bound is taken over all bases of L_1 and L_2.

The following assertion is quite an obvious one. If $w_1, \ldots, w_p$ is a bases of L_1 and if $\tau(L_1, L_2) < \varepsilon$ then there exists a basis $v_1, v_2, \ldots, v_p$ of L_2 such that

$$\sum_{t=1}^{p} \|w_t - v_t\| < p\varepsilon .$$

Definition. Let L_k, L be p-dimensional subspaces of H_m. We say $\lim_{k\to\infty} L_k = L$, if $\lim_{k\to\infty} \tau(L_k, L) = 0$.

Theorem 3. *If $G^k \subset G$, $G = \lim_{k\to\infty} G^k$ and if*

$$\lambda_{s-1} < \lambda_s = \lambda_{s+1} = \ldots = \lambda_{s+p} < \lambda_{s+p+1}$$

then

$$\lim K_{s+p}^k \cap \overline{K}_{s-1}^k = K_{s+p} \cap \overline{K}_{s-1}$$

where we have denoted by K_t^k, $\overline{K}_t^k$ the space K_t, $\overline{K}_t$ respectively for the set G^k.

As a corollary of theorem 3 we obtain Theorem 4.

Theorem 4. *Let $u_s, u_{s+1}, \ldots, u_{s+p}$ be the system of eigenfunctions associated with the eigenvalue λ_s of multiplicity $p + 1$. If $\lambda_{s-1} < \lambda_s = \lambda_{s+p} < \lambda_{s+p+1}$ and if $G^k \subset G, \lim_{k\to\infty} G^k = G$ then there exists a sequence of sets of eigenfunctions $u_s^k, u_{s+1}^k, \ldots, \ldots, u_{s+p}^k$ such that*

$$\lim_{k\to\infty} \|u_{s+t}^k - u_{s+t}\| = 0$$

$t = 0, 1, 2, \ldots, p$.

Proof of theorem 3. We decompose u_t^k as follows (u_t^k being t-th eigenfunction for the set G^k)

$$u_t^k = x_t^k + y_t^k + z_t^k ,$$

$$x_t^k \in K_{s-1} , \quad y_t^k \in K_{s+p} \cap \overline{K}_{s-1} , \quad z_t^k \in \overline{K}_{s+p} , \quad t = s , \quad s + 1, \ldots, s + p .$$

Let us consider first the case $s = 1$. By lemma 4

$$\|z_t^k\|_n^2 \leqq \frac{1}{\alpha} (\lambda_t^k - \lambda_1) .$$

By lemma 5

$$\|u_t^k - y_t^k\|_m^2 \leqq \lambda_t^k - \lambda_1 + \frac{2}{\alpha} (\lambda_t^k - \lambda_1) \lambda_1 = \eta_k .$$

Using lemma 1 we find functions $v_t \in K_{p+q}$ such that $[v_t, v_q] = \delta_{tq}$ and

$$\|u_t^k - v_t\|_m^2 = O(\eta_k)$$

$t = s, s + 1, \ldots, s + p$. Clearly

$$\left\| \frac{u_t^k}{\lambda_t^k} - \frac{v_t}{\lambda_1} \right\|_m^2 = O(\eta_k) + O(\lambda_t^k - \lambda_1) .$$

The theorem has been proved for $s = 1$. We proceed by induction. Using induction hypothesis we find functions $w_1, w_2, \ldots, w_{s-1} \in K_{s-1}$ such that

$$\{w_t, w_q\} = \delta_{tq}$$

and

$$\left\| \frac{u_t^k}{\lambda_t^k} - w_t \right\|_m < \varepsilon ,$$

$t, q = 1, 2, \ldots, s - 1$. Let us choose k_0 such that for $k > k_0$ the inequality

$$\lambda_t^k - \lambda_t < \varepsilon , \quad t = 1, 2, \ldots, s + p$$

holds. The elements $w_1, w_2, \ldots, w_{s-1}$ form a orthonormal basis of K_{s-1}, hence

$$x_t^k = \sum_{q=1}^{s-1} c_{tq} w_q$$

where $c_{tq} = \{u_t^k, w_q\}$. Since $c_{tq} = O(\varepsilon)$, it is $\|x_t^k\|_m^2 = O(\varepsilon)$ and also $\|x_t^k\|_n^2 = O(\varepsilon)$, $t = s, s + 1, \ldots, s + p$. Applying lemma 4 one obtains $\|z_t^k\|_n^2 = O(\varepsilon)$ and making use of lemma 5 $\|z_t^k\|_m^2 = O(\varepsilon)$. Hence

$$\|u_t^k - y_t\|^2 = O(\varepsilon) , \quad t = s , \quad s + 1, \ldots, s + p .$$

Having functions y_t one can complete the proof in the same way as in the case $s = 1$.

4. APPROXIMATION OF THE DOMAIN FROM OUTSIDE

Throughout this section we shall assume that the boundary of G has no inner boundary points i.e. the sets G and $\bar{G}$ have the same boundary. Further, we shall assume that the n-dimensional measure of $\dot{G}$ is zero.

Let G^k be a sequence of open sets satisfying $G \subset \bar{G}^{k+1} \subset G^k$ and $\lim_{k \to \infty} G^k = G$. We say that the set G is stable provided $H_m = \bigcap_{k=1}^{\infty} H_m^k$, where H_m^k is the space H_m for the set G^k.

It was proved by I. Babuška [6] that the concept of stability does not depend on the choice of the sequence G^k. It was also shown in that paper that (for certain class of elliptic operators) the stability of the domain is a necessary and sufficient condition for the continuous dependence of the solution of the Dirichlet problem on the domain G.

Let us denote by $H^* = \bigcap_{k=1}^{\infty} H_m^k$ and by λ_1^* the minimum of the functional $\{u, u\}$ under the conditions $[u, u] = 1$, $u \in H^*$. Let us denote the function, which gives the minimum by u_1^*. As soon as $\lambda_1^*, \lambda_2^*, \ldots, \lambda_p^*$ and $u_1^*, u_2^*, \ldots, u_p^*$ are defined, we define λ_{p+1}^* by the relations

$$\lambda_{p+1}^* = \operatorname{Min} \{u, u\}$$

for $u \in H^*$, $[u, u] = 1$, $[u, u_t^*] = 0$, $t = 1, 2, \ldots, p$. As soon as λ_{p+1}^* is defined, we define u_{p+1}^* as a function which gives minimum to the functional $\{u, u\}$ and which satisfies $[u_{p+1}^*, u_{p+1}^*] = 1$, $[u_{p+1}^*, u_t] = 0$, $t = 1, 2, \ldots, p$. The usual argument used in the proof of existence of eigenvalues shows, that λ_t^* and u_t^* are well defined.

Theorem 4. *If* $G \subset G^{k+1} \subset G^k$, $\lim_{k \to \infty} G^k = G$ *then* $\lim \lambda_t^k = \lambda_t^*$, $t = 1, 2, \ldots$

Proof. Since $\lambda_t^k \leqq \lambda_t^{k+1} \leqq \lambda_t^* \leqq \lambda_t$ there exists $\lim \lambda_t^k = \tilde{\lambda}_t \leqq \lambda_t^*$. Since every set bounded in the norm of H_m^1 is compact in H_n^1 one can choose a subsequence of $\{u_t^k\}_{k=1,2,\ldots}$ which is convergent in H_m^1. We may assume that $\{u_t^k\}_{k=1,2,\ldots}$ in itself is convergent in H_n^1. By the usual procedure one can show that then $\{u_t^k\}_{k=1,2,\ldots}$ is convergent in H_m^1, $\lim_{k \to \infty} u_t^k = \tilde{u}_t \in H^*$.

Clearly $\lim_{k \to \infty} \|u_t^k\|_m = \|\tilde{u}_t\|_m = \lim_{k \to \infty} \lambda_t^k = \tilde{\lambda}_t$. Since $\|\tilde{u}_t\|_n = 1$, we have $\tilde{\lambda}_1 \geqq \lambda_1^*$ in view of the definition of λ_1^*. Hence

$$\tilde{\lambda}_i = \lambda_t^* \tag{4}$$

and

$$\tilde{u}_t = u_t^* \tag{5}$$

for $t = 1$. The relations (4) and (5) can be now proved by induction.

As a consequence of theorem 4 we obtain Theorem 5.

Theorem 5. *If* $G \subset \bar{G}^{k+1} \subset G^k$, $\lim_{k \to \infty} G^k = G$ *and if* G *is stable, then*

$$\lim_{k \to \infty} \lambda_t^k = \lambda_t \, . \tag{6}$$

for every $t = 1, 2, \ldots$

Combining Theorem 5 and Theorem 2 we have

Theorem 6. *If* $\lim G^k = G$ *and if* G *is stable, then* $\lim_{k \to \infty} \lambda_t^k = \lambda_t$ *for* $t = 1, 2, \ldots$

An analogue of Theorem 3 is the following

Theorem 7. *If* $G \subset \bar{G}^{k+1} \subset G^k$, *if* $\lambda_t = \lambda_t^*$ $(t = 1, 2, \ldots, s + p)$ *and if*

$$\lambda_{s-1} < \lambda_s = \lambda_{s+1} = \ldots = \lambda_{s+p} < \lambda_{s+p+1} \tag{7}$$

then

$$\lim_{k \to \infty} K_{s+p}^k \cap \bar{K}_{s-1}^k = K_{s+p} \cap \bar{K}_{s-1} \, . \tag{8}$$

The proof is very similar to the proof of the theorem 3 and therefore we shall omit it.

As a consequence of Theorem 7 we have Theorem 8.

Theorem 8. *If* $G \subset \bar{G}^{k+1} \subset G^k$, $\lim_{k\to\infty} G^k = G$ *if* (7) *is valid and if* G *is stable then* (8) *holds.*

By Theorem 5 the stability of G is a sufficient condition for (6). The following theorem shows, that this condition is in certain sense also necessary.

Theorem 9. *If* $G \subset \bar{G}^k \subset G^{k+1}$, *and if* (6) *holds for* $t = 1, 2, \ldots$, *then* G *is stable.*

Proof. The functions $\tilde{u}_t$ $(t = 1, 2, \ldots)$ form a complete system in H^*. By Theorem 7 $\tilde{u}_t \in H_m$ for $t = 1, 2, \ldots$. Hence $H^* \subset H_m$. The reversed inclusion is trivial. Hence $H^* = H_m$ and G is stable.

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Резюме

НЕПРЕРЫВНАЯ ЗАВИСИМОСТЬ СОБСТВЕННЫХ ЗНАЧЕНИЙ ОТ ОБЛАСТИ

ИВО БАБУШКА (Ivo Babuška), РУДОЛФ ВЫБОРНЫ (Rudolf Výborný), Прага

В статье исследуется непрерывная зависимость собственных чисел (и в определенном смысле и собственных функций) самосопряженного положительно определенного эллиптического оператора от области. Исследуется также связь с понятием устойчивой области для задачи Дирихле.