Czechoslovak Mathematical Journal

Jiří Jarník

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Czechoslovak Mathematical Journal, Vol. 15 (1965), No. 1, 124–128,129,130,131–144,145–152,153–160

Persistent URL: http://dml.cz/dmlcz/100657

Чехословацкий математический журнал, т. 15 (90) 1965, Прага

DEPENDENCE OF SOLUTIONS OF A CLASS OF DIFFERENTIAL EQUATIONS OF THE SECOND ORDER ON A PARAMETER

JIŘÍ JARNÍK, Praha

(Received January 16, 1964)

0. Preliminary. The Soviet physician P. L. KAPICA investigated the motion of a mathematical pendulum the point of hanging of which oscillated in vertical direction with a large frequency and small amplitude. The equation of such a pendulum reads

$$\ddot{\Theta} = (gL^{-1} - AL^{-1}\omega^2 \sin \omega t) \sin \Theta$$

(see fig. 1).

Kapica substituted the solution of this equation when ω is large and A small (more precisely: $\omega \to \infty$ and $A\omega = \text{const.}$) by the solution of

$$\ddot{\Theta} = (gL^{-1} - \tfrac{1}{2}L^{-2}A^2\omega^2 \cos \Theta) \sin \Theta \, .$$

The same equations were investigated by N. N. BOGOLYUBOV and YU. A. MITROPOLSKIJ [1, pp. 344–348] by means of the averaging method; the authors made use of a substitution of a very special form.

S. ŁOJASIEWICZ in his paper [7] formulated and proved a general theorem on equations of the second order which are analogous to those mentioned above. He made use of the following essential facts: The quickly oscillating function $\varphi(\Theta, t, \omega)$ (in the case mentioned above the function $- AL^{-1}\omega^2 \sin \omega t \sin \Theta$) is sufficiently smooth in Θ, periodic in t with a period of order ω^{-1} and has zero mean value:

$$\int_0^{\omega^{-1}} \varphi(\Theta, t, \omega) \, \mathrm{d}t = 0 \, .$$

The aim of the present paper is to prove some similar results. The most important assumption on the quickly oscillating right-hand term $\varphi(\Theta, t, \omega)$ will be the convergence of the second primitive function of φ with respect to the time variable t to zero (when $\omega \to \infty$) and the boundedness of the first primitive function independently of ω.

We will make use of the concepts and methods that have been introduced by J. Kurzweil [3]–[6]. Therefore, let us recollect the definition and fundamental properties of the generalized integral in the required form.

Let $U(\tau, \sigma)$ be a function of two variables defined on a square $\langle T_1, T_2\rangle \times \langle T_1, T_2\rangle$. Let $T_1 \leqq t_1 < t_2 \leqq T_2$. Let us form a sequence of divisions

$$A_n \equiv \{\alpha_0^{(n)}, \tau_0^{(n)}, \alpha_1^{(n)}, \tau_1^{(n)}, \ldots, \tau_{n-1}^{(n)}, \alpha_n^{(n)}\}$$

where $t_1 = \alpha_0^{(n)} \leqq \tau_0^{(n)} \leqq \alpha_1^{(n)} \leqq \tau_1^{(n)} \leqq \ldots \leqq \tau_{n-1}^{(n)} \leqq$ $\leqq \alpha_n^{(n)} = t_2$ such that

$$\lim_{n\to\infty} \max_{i=0,1,\ldots,n-1} |\alpha_{i+1}^{(n)} - \alpha_i^{(n)}| = 0 .$$

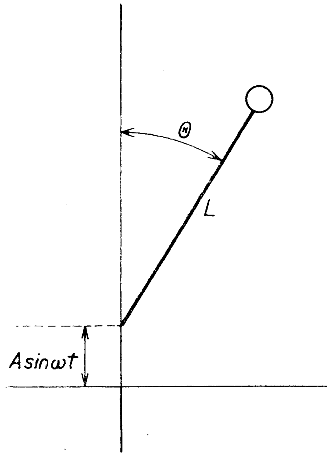

Fig. 1.

Definition. If there exists the limit of the sequence of sums

$$\sum_{i=0}^{n-1} [U(\tau_i^{(n)}, \alpha_{i+1}^{(n)}) - U(\tau_i^{(n)}, \alpha_i^{(n)})]$$

independent of the choice of the sequence of divisions A_n, we call it the generalized (Riemann) integral of the function U with respect σ over the interval $\langle t_1, t_2\rangle$ and write

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} [U(\tau_i^{(n)}, \alpha_{i+1}^{(n)}) - U(\tau_i^{(n)}, \alpha_i^{(n)})] = \int_{t_1}^{t_2} D_\sigma U(\tau, \sigma) .$$

It is evident that the generalized integral – if it exists – can be approximated with an arbitrary accuracy by a sum

$$\sum_{i=0}^{n-1} [U(\alpha_i, \alpha_{i+1}) - U(\alpha_i, \alpha_i)]$$

where $t_1 = \alpha_0 \leqq \alpha_1 \leqq \ldots \leqq \alpha_{n-1} \leqq \alpha_n = t_2$ if only $\max |\alpha_{i+1} - \alpha_i|$ is small enough $(i = 0, 1, 2, \ldots, n-1)$.

On the existence of the generalized integral and on its relation to the usual Riemann integral there holds (cf. [6], Lemma 1):

0.1. If in an interval $\langle T_1, T_2\rangle$ there exists a continuous partial derivative of $U(\tau, \sigma)$ with respect to σ: $\partial U/\partial\sigma = u(\tau, \sigma)$, then $\int_{t_1}^{t_2} D_\sigma U(\tau, \sigma)$ exists and

$$\int_{t_1}^{t_2} D_\sigma U(\tau, \sigma) = \int_{t_1}^{t_2} u(\sigma, \sigma)\, d\sigma .$$

(On the right-hand side is a usual Riemann integral.)

Further, a theorem on integration by parts holds (see [4]):

0.2. Let two of integrals

$$\int_{t_1}^{t_2} \mathrm{D}_\sigma\, U(\tau, \sigma)\,, \quad \int_{t_1}^{t_2} \mathrm{D}_\tau\, U(\tau, \sigma)\,, \quad \int_{t_1}^{t_2} \mathrm{D}\, V(\tau, \sigma)$$

where

$$V(\tau, \sigma) = U(\tau, \tau) - U(\tau, \sigma) - U(\sigma, \tau) + U(\sigma, \sigma)\,,$$

$$\int_{t_1}^{t_2} \mathrm{D}_\tau\, U(\tau, \sigma) = \int_{t_1}^{t_2} \mathrm{D}_\sigma\, U^*(\tau, \sigma)\,, \quad U^*(\tau, \sigma) = U(\sigma, \tau)$$

exist. (As $\int \mathrm{D}_\sigma V = \int \mathrm{D}_\tau V$ evidently, we may write simply $\int \mathrm{D} V$.) Then the third integral exists, too, and the following equation holds:

$$\int_{t_1}^{t_2} \mathrm{D}_\sigma\, U(\tau, \sigma) + \int_{t_1}^{t_2} \mathrm{D}_\tau\, U(\tau, \sigma) = U(t_2, t_2) - U(t_1, t_1) - \int_{t_1}^{t_2} \mathrm{D}\, V(\tau, \sigma)\,.$$

Note. If the both partial derivatives of $U(\tau, \sigma)$ exist and are continuous then $\int_{t_1}^{t_2} \mathrm{D} V = 0$ (cf. 0.1).

Finally, let us mention an important approximation of a generalized integral by means of the values of an integrand at the end-points of the interval:

0.3. For a function $U(\tau, \sigma)$ defined on a square $\langle T_1, T_2 \rangle \times \langle T_1, T_2 \rangle$ let hold

$$|U(\tau + \eta, \sigma + \eta) - U(\tau + \eta, \sigma) - U(\tau, \sigma + \eta) + U(\tau, \sigma)| \leqq C\eta^2$$

for $0 < \eta < \eta_0$. Then $\int_{t_1}^{t_2} \mathrm{D}_\sigma\, U(\tau, \sigma)$, $T_1 \leqq t_1 \leqq t_2 \leqq T_2$ exists and

$$\left| \int_{t_1}^{t_2} \mathrm{D}_\sigma\, U(\tau, \sigma) - U(t_1, t_2) + U(t_1, t_1) \right| \leqq C(t_2 - t_1)^2$$

holds.

This theorem is a direct consequence of Theorem 3,1 [3, p. 432] or of Theorem 1 [6, p. 565].

1. We shall investigate a differential equation of the second order

$$(1) \qquad \ddot{x} = f(x, t, \lambda) + \varphi(x, t, \lambda)$$

the functions f, φ being defined for $x \in G$ (G an open subset of E_n), $t \in \langle s, T \rangle$, $0 < \lambda \leqq \lambda_0$ and continuous in (x, t) on $G \times \langle s, T \rangle$.*)

Function $f(x, t, \lambda)$ let be bounded in all its definition domain by a constant K_1 independent of λ:

$$(2) \qquad |f(x, t, \lambda)| \leqq K_1\,.$$

The following assumptions will be put on the function $\varphi(x, t, \lambda)$:

i) There exist functions Φ, P, Φ_x, P_x of (x, t, λ) defined on $G \times \langle s, T \rangle \times (0, \lambda_0\rangle$

*) Their moduli of continuity may in general depend on λ.

and continuous on $G \times \langle s, T \rangle$ such that

$$(3) \qquad \varphi(x, t, \lambda) = \frac{\partial \Phi}{\partial t},$$

$$\Phi(x, t, \lambda) = \frac{\partial P}{\partial t}, \quad \Phi_x(x, t, \lambda) = \frac{\partial P_x}{\partial t}$$

where an index x denotes the partial derivative with respect to the variable x.

ii) The functions Φ and Φ_x are bounded in all their definition domain by a constant K_1 independent of λ:

$$(4) \qquad |\Phi(x, t, \lambda)| \leqq K_1 , \quad |\Phi_x(x, t, \lambda)| \leqq K_1 .$$

iii) There holds

$$(5) \qquad \lim_{\lambda \to 0+} P(x, t, \lambda) = 0 , \quad \lim_{\lambda \to 0+} P_x(x, t, \lambda) = 0$$

uniformly on $G \times \langle s, T \rangle$.

iv) There exist $\eta_0 > 0$ and a continuous nondecreasing function $\omega(\eta)$ on $\langle 0, \eta_0 \rangle$, $\omega(0) = 0$ such that for $|t_2 - t_1| \leqq \eta_0$, $|x_2 - x_1| \leqq \eta_0$ and for all $\lambda \in (0, \lambda_0\rangle$

$$(6) \qquad |P(x_1, t_1, \lambda) - P(x_1, t_2, \lambda) - P(x_2, t_1, \lambda) + P(x_2, t_2, \lambda)| \leqq \leqq |t_2 - t_1| \, \omega(|x_2 - x_1|),$$

$$|P_x(x_1, t_1, \lambda) - P_x(x_1, t_2, \lambda) - P_x(x_2, t_1, \lambda) + P_x(x_2, t_2, \lambda)| \leqq \leqq |t_2 - t_1| \, \omega(|x_2 - x_1|).$$

Equation (1) the right-hand side of which fulfils all the conditions mentioned above will be noted by $(\mathfrak{E})$.

Note. Assumption iv) may be replaced evidently by an assumption of continuity of Φ_x in x with a modulus of continuity ω independent of λ. Validity of the first inequality (6) with $\omega(\eta) = K_1 \eta$ follows from the assumptions i) and ii) by using the mean-value theorem.

Theorem 1. *Let $x(t, \lambda)$ be a solution of equation $(\mathfrak{E})$ in the interval $\langle s, T \rangle$ with initial conditions $x(s, \lambda) = \tilde{x}_1(\lambda)$, $\dot{x}(s, \lambda) = \tilde{x}_2(\lambda)$ which are bounded (for $0 < \lambda \leqq \lambda_0$) by a constant K_2 independent of λ:*

$$(7) \qquad |\tilde{x}_1(\lambda)| \leqq K_2 , \quad |\tilde{x}_2(\lambda)| \leqq K_2 .$$

Then it holds

$$(8) \qquad x(t, \lambda) = \tilde{x}_1(\lambda) + [\tilde{x}_2(\lambda) - \Phi(\tilde{x}_1(\lambda), s, \lambda)] (t - s) + + \int_s^t \int_s^\tau f(x(\sigma, \lambda), \sigma, \lambda) \, d\sigma \, d\tau - - \int_s^t \int_s^\tau \Phi_x(x(\sigma, \lambda), \sigma, \lambda) \, \Phi(x(\sigma, \lambda), \sigma, \lambda) \, d\sigma \, d\tau + o(1) .$$

Note. Symbol $o(\lambda^\alpha)$ means any function $v(x, t, \lambda)$ such that $\lim_{\lambda \to 0+} \lambda^{-\alpha} v(x, t, \lambda) = 0$ uniformly on $G \times \langle s, T \rangle$; $o(1) = o(\lambda^0)$.

Proof. According to 0.1 there is

$$\int_s^t \mathrm{D}_\sigma \Phi(x(\tau, \lambda), \sigma, \lambda) = \int_s^t \varphi(x(\tau, \lambda), \tau, \lambda) \, \mathrm{d}\tau .$$

Therefore we may write

$$\dot{x}(t, \lambda) = \tilde{x}_2(\lambda) + \int_s^t f(x(\tau, \lambda), \tau, \lambda) \, \mathrm{d}\tau + \int_s^t \mathrm{D}_\sigma \Phi(x(\tau, \lambda), \sigma, \lambda) . \tag{9}$$

According to 0.2 and the following note there is

$$\int_s^t \mathrm{D}_\sigma \Phi(x(\tau, \lambda), \sigma, \lambda) = \Phi(x(t, \lambda), t, \lambda) - \Phi(\tilde{x}_1(\lambda), s, \lambda) - \int_s^t \mathrm{D}_\tau \Phi(x(\tau, \lambda), \sigma, \lambda)$$

which substituted into (9) gives

$$\dot{x}(t, \lambda) = \tilde{x}_2(\lambda) + \int_s^t f(x(\tau, \lambda), \tau, \lambda) \, \mathrm{d}\tau + \Phi(x(t, \lambda), t, \lambda) - \Phi(\tilde{x}_1(\lambda), s, \lambda) - \\ - \int_s^t \mathrm{D}_\tau \Phi(x(\tau, \lambda), \sigma, \lambda) .$$

If we make use of the identity

$$\int_s^t \mathrm{D}_\tau \Phi(x(\tau, \lambda), \sigma, \lambda) = \int_s^t \Phi_x(x(\tau, \lambda), \tau, \lambda) \, \dot{x}(\tau, \lambda) \, \mathrm{d}\tau$$

(see 0.1) we get

$$\dot{x}(t, \lambda) = \tilde{x}_2(\lambda) - \Phi(\tilde{x}_1(\lambda), s, \lambda) + \int_s^t f(x(\tau, \lambda), \tau, \lambda) \, \mathrm{d}\tau + \Phi(x(t, \lambda), t, \lambda) - \\ - \int_s^t \Phi_x(x(\tau, \lambda), \tau, \lambda) \, \dot{x}(\tau, \lambda) \, \mathrm{d}\tau . \tag{10}$$

Let us substitute the right-hand side into the last integral for $\dot{x}(t, \lambda)$ again:

$$\dot{x}(t, \lambda) = \tilde{x}_2(\lambda) - \Phi(\tilde{x}_1(\lambda), s, \lambda) + \int_s^t f(x(\tau, \lambda), \tau, \lambda) \, \mathrm{d}\tau + \Phi(x(t, \lambda), t, \lambda) - \\ - [\tilde{x}_2(\lambda) - \Phi(\tilde{x}_1(\lambda), s, \lambda)] \int_s^t \Phi_x(x(\tau, \lambda), \tau, \lambda) \, \mathrm{d}\tau - \int_s^t \Phi_x(x(\tau, \lambda), \tau, \lambda) \, . \\ . \int_s^\tau [f(x(\sigma, \lambda), \sigma, \lambda) - \Phi_x(x(\sigma, \lambda), \sigma, \lambda) \, \dot{x}(\sigma, \lambda)] \, \mathrm{d}\sigma \, \mathrm{d}\tau - \\ - \int_s^t \Phi_x(x(\tau, \lambda), \tau, \lambda) \, \Phi(x(\tau, \lambda), \tau, \lambda) \, \mathrm{d}\tau . \tag{11}$$

By integration of (11) we get the final formula for the solution of (1):

$$(12)\quad x(t, \lambda) = \tilde{x}_1(\lambda) + [\tilde{x}_2(\lambda) - \Phi(\tilde{x}_1(\lambda), s, \lambda)](t - s) + \int_s^t \int_s^\tau f(x(\sigma, \lambda), \sigma, \lambda)\, d\sigma\, d\tau +$$

$$+ \int_s^t \Phi(x(\tau, \lambda), \tau, \lambda)\, d\tau - [\tilde{x}_2(\lambda) - \Phi(\tilde{x}_1(\lambda), s, \lambda)] \int_s^t \int_s^\tau \Phi_x(x(\sigma, \lambda), \sigma, \lambda)\, d\sigma\, d\tau -$$

$$- \int_s^t \int_s^\tau \Phi_x(x(\sigma, \lambda), \sigma, \lambda) \int_s^\sigma [f(x(\xi, \lambda), \xi, \lambda) - \Phi_x(x(\xi, \lambda), \xi, \lambda)\, \dot{x}(\xi, \lambda)]\, d\xi\, d\sigma\, d\tau -$$

$$- \int_s^t \int_s^\tau \Phi_x(x(\sigma, \lambda), \sigma, \lambda)\, \Phi(x(\sigma, \lambda), \sigma, \lambda)\, d\sigma\, d\tau .$$

In what follows we shall make use of two lemmas.

Lemma 1. *Let $u(t)$ be a bounded non-negative function in an interval $\langle s, T\rangle$, $c_i \geqq 0$ $(i = 1, 2, 3)$, $c_3 \neq 0$. Let*

$$u(t) \leqq c_1 + c_2(t - s) + c_3 \int_s^t u(\tau)\, d\tau$$

for all $t \in \langle s, T\rangle$. Then

$$u(t) \leqq c_1 e^{c_3(t-s)} + \frac{c_2}{c_3} [e^{c_3(t-s)} - 1] .$$

Proof. We can easily assure that

$$u(t) \leqq c_1 \sum_{k=0}^{2^n - 1} \frac{c_3^k (t - s)^k}{k!} + c_2 \sum_{k=0}^{2^n - 1} \frac{c_3^k (t - s)^{k+1}}{(k + 1)!} + c_3^{2^n} \int_s^t \int_2^{t_{2^n}} \dots \int_s^{t_2} u(t_1)\, dt_1 \dots dt_{2^n}$$

holds for all positive integers n. As the function u is bounded, the last right-hand side term converges to zero with $n \to \infty$ and by the limiting process we get after evident transformations the assertion of Lemma 1.

Lemma 2. *Let $s \leqq a < b \leqq T$. If we use the notation of Theorem 1, then*

$$(13)\qquad \lim_{\lambda \to 0+} \int_a^b \Phi(x(t, \lambda), t, \lambda)\, dt = 0 ,$$

$$\lim_{\lambda \to 0+} \int_a^b \Phi_x(x(t, \lambda), t, \lambda)\, dt = 0 .$$

Proof. Let us prove e.g. the first inequality; the proof of (13) for Φ_x is quite analogous.

According to (3) and 0.1, there is

$$\int_a^b \Phi(x(t, \lambda), t, \lambda)\, dt = \int_a^b D_\sigma\, P(x(\tau, \lambda), \sigma, \lambda) .$$

Let us choose a positive integer n and form an equidistant division of the interval $\langle a, b\rangle$:

$$a = \alpha_0 < \alpha_1 < \ldots < \alpha_{n-1} < \alpha_n = b\,,$$

$$\alpha_{i+1} - \alpha_i = \frac{1}{n}(b - a)\,, \quad i = 0, 1, 2, \ldots, n - 1\,.$$

According to the definition of the generalized integral, there is*)

$$\int_{\alpha_i}^{\alpha_{i+1}} D_\sigma P(x(\tau), \sigma) = \lim_{m\to\infty} \sum_{j=0}^{m-1} [P(x(\tau_j), \beta_{j+1}) - P(x(\tau_j), \beta_j)] = P(x(\alpha_i), \alpha_{i+1}) -$$

$$- P(x(\alpha_i), \alpha_i) + \lim_{m\to\infty} \sum_{j=0}^{m-1} [P(x(\tau_j), \beta_{j+1}) - P(x(\tau_j), \beta_j) - P(x(\alpha_i), \beta_{j+1}) + P(x(\alpha_i), \beta_j)]$$

where

(14) $$\alpha_i = \beta_0 \leqq \tau_0 \leqq \beta_1 \leqq \tau_1 \leqq \ldots \leqq \tau_{m-1} \leqq \beta_m = \alpha_{i+1}$$

is a division of interval $\langle \alpha_i, \alpha_{i+1}\rangle$ the norm of which (i.e. $\max |\beta_{j+1} - \beta_j|$) converges to zero when $m \to \infty$. Thus,

$$\int_{\alpha_i}^{\alpha_{i+1}} D_\sigma P(x(\tau), \sigma) = P(x(\alpha_i), \alpha_{i+1}) - P(x(\alpha_i), \alpha_i) + Z_i\,,$$

$$|Z_i| \leqq \sup_{\Delta} \sum_{j} |P(x(\tau_j), \beta_{j+1}) - P(x(\tau_j), \beta_j) - P(x(\alpha_i), \beta_{j+1}) + P(x(\alpha_i), \beta_j)|$$

where Δ is an arbitrary division (14) of $\langle \alpha_i, \alpha_{i+1}\rangle$.

According to (6), we get immediately an estimate of Z_i:

$$|Z_i| \leqq \sup_{\Delta} \sum_{j} |\beta_{j+1} - \beta_j|\, \omega(|x(\tau_j) - x(\alpha_i)|) \leqq$$

$$\leqq |\alpha_{i+1} - \alpha_i| \sup_{|\eta_2 - \eta_1| \leqq (b-a)/n} \omega(|x(\eta_2) - x(\eta_1)|)\,.$$

Thus we can write

(15) $$\left|\int_a^b \Phi(x(t, \lambda), t, \lambda)\,\mathrm{d}t\right| = \left|\sum_{i=0}^{n-1} \int_{\alpha_i}^{\alpha_{i+1}} D_\sigma P(x(\tau, \lambda), \sigma, \lambda)\right| \leqq$$

$$\leqq \sum_{i=0}^{n-1} |P(x(\alpha_i, \lambda), \alpha_{i+1}, \lambda) - P(x(\alpha_i, \lambda), \alpha_i, \lambda)| +$$

$$(b - a) \sup_{|\eta_2 - \eta_1| \leqq (b-a)/n} \omega(|x(\eta_2, \lambda) - x(\eta_1, \lambda)|)\,.$$

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*) In what follows, we shall not mark the dependence on λ explicitly provided no misunderstanding can appear.

From (10) it follows by means of (2), (3) and (7)

$$|\dot{x}(t, \lambda)| \leqq 2K_1 + K_2 + K_1(t - s) + K_1 \int_s^t |\dot{x}(\tau, \lambda)| \, d\tau$$

and hence, by Lemma 1,

$$|\dot{x}(t, \lambda)| \leqq (2K_1 + K_2 + 1)\, e^{K_1(t-s)} - 1 \,. \tag{16}$$

By integration we get

$$|x(\eta_2, \lambda) - x(\eta_1, \lambda)| \leqq \int_{\eta_1}^{\eta_2} |\dot{x}(t, \lambda)| \, dt \leqq A\, |\eta_2 - \eta_1| \tag{17}$$

where $A = (2K_1 + K_2 + 1)\, e^{K_1(T-s)} - 1$.

Inequality (15) may be now rewriten in the form

$$\left| \int_a^b \Phi(x(t, \lambda), t, \lambda) \, dt \right| \leqq \sum_{i=0}^{n-1} |P(x(\alpha_i, \lambda), \alpha_{i+1}, \lambda) -$$
$$- P(x(\alpha_i, \lambda), \alpha_i, \lambda)| + (b - a)\, \omega(A(b - a)/n)$$

which holds for an arbitrary positive integer n and $0 < \lambda \leqq \lambda_0$. From this estimate we easily get (13).

In fact, if $\varepsilon > 0$, it is sufficient to choose n_0 large enough that e.g.

$$\omega\left(A\, \frac{b - a}{n_0}\right) \leqq \frac{\varepsilon}{2(b - a)}$$

holds and λ_1 so small that $|P(x, t, \lambda)| < \varepsilon/4n_0$ for $x \in G$, $t \in \langle s, T\rangle$ and $0 < \lambda \leqq \lambda_1$ which is possible according to (5). Then

$$\left| \int_a^b \Phi(x(t, \lambda), t, \lambda) \, dt \right| < \varepsilon$$

for $0 < \lambda \leqq \lambda_1$. The proof of Lemma 2 is completed.

Let us now return to the proof of Theorem 1.

From Lemma 2 we get immediately that all the integrals on the right-hand side of (12) except the first and the last ones converge to zero uniformly in $\langle s, T\rangle$ when $\lambda \to 0$. Let us show that for the relatively most complicated last integral. (We shall change the order of integration and use (4) and (16).

$$\left| \int_s^t \int_s^\tau \Phi_x(x(\sigma, \lambda), \sigma, \lambda) \int_s^\sigma [f(x(\xi, \lambda), \xi, \lambda) - \Phi_x(x(\xi, \lambda), \xi, \lambda)\, \dot{x}(\xi, \lambda)] \, d\xi \, d\sigma \, d\tau \right| =$$
$$= \left| \int_s^t \int_s^\tau [f(x(\xi, \lambda), \xi, \lambda) - \Phi_x(x(\xi, \lambda), \xi, \lambda)\, \dot{x}(\xi, \lambda)] \int_\xi^\tau \Phi_x(x(\sigma, \lambda), \sigma, \lambda) \, d\sigma \, d\xi \, d\tau \right| \leqq$$
$$\leqq (K_1 + K_1 A) \int_s^t \int_s^\tau \left| \int_\xi^\tau \Phi_x(x(\sigma, \lambda), \sigma, \lambda) \, d\sigma \right| d\xi \, d\tau = o(1) \,.$$

Hence (8) holds and the proof of Theorem 1 is finished.

Let us formulate Theorem 1 in the form analogous to that of Theorem 1 [7, p. 398]:

Theorem 1*. *Let $x(t, \lambda)$ be a solution of $(\mathfrak{E})$ in the interval $\langle s, T\rangle$, with the initial conditions $x(s, \lambda) = \tilde{x}_1(\lambda)$, $\dot{x}(s, \lambda) = \tilde{x}_2(\lambda)$. Let (7) hold for $0 < \lambda \leqq \lambda_0$. The functions $f(x, t, \lambda)$ and $\Phi_x(x, t, \lambda)$ let fulfil the Lipschitz condition in x with a constant M independent of λ.*

Then for $\lambda > 0$ sufficiently small there exists the solution $y(t, \lambda)$ of the equation

$$\ddot{y} = f(y, t, \lambda) - \Phi_x(y, t, \lambda)\, \Phi(y, t, \lambda)$$

in the interval $\langle s, T\rangle$ with the initial conditions $y(s, \lambda) = \tilde{x}_1(\lambda)$, $\dot{y}(s, \lambda) = \tilde{x}_2(\lambda) - \Phi(\tilde{x}_1(\lambda), s, \lambda)$ and

$$\lim_{\lambda \to 0+} |x(t, \lambda) - y(t, \lambda)| = 0 \tag{18}$$

uniformly for $t \in \langle s, T\rangle$.

Proof. In an arbitrary interval $\langle s, T_1\rangle$ in which the solution $y(t, \lambda)$ exists,

$$|x(t, \lambda) - y(t, \lambda)| \leqq o(1) + \left|\int_s^t \int_s^\tau [f(x(\sigma, \lambda), \sigma, \lambda) - f(y(\sigma, \lambda), \sigma, \lambda)]\, d\sigma\, d\tau\right| +$$
$$+ \left|\int_s^t \int_s^\tau [\Phi_x(x(\sigma, \lambda), \sigma, \lambda)\, \Phi(x(\sigma, \lambda), \sigma, \lambda) - \Phi_x(y(\sigma, \lambda), \sigma, \lambda)\, \Phi(y(\sigma, \lambda), \sigma, \lambda)]\, d\sigma\, d\tau\right|$$

holds according to (12). Functions f and $\Phi_x \Phi$ are Lipschitzian in x with a constant independent of λ (cf. (4) and the assumption of Theorem 1). Therefore

$$|\, x(t, \lambda) - y(t, \lambda)| \leqq 0(1) + M_1 \int_s^t \int_s^\tau |x(\sigma, \lambda) - y(\sigma, \lambda)|\, d\sigma\, d\tau\,, \quad M_1 = \text{const.}$$

The validity of (18) uniformly in $\langle s, T_1\rangle$ is a simple consequence of the following lemma, the proof of which is quite analogous to that of Lemma 1:

Lemma 3. *Let $u(t)$ be a bounded non-negative function in an interval $\langle s, T\rangle$. Let*

$$u(t) \leqq c_0 + c_1(t - s) + c_2(t - s)^2 + c_3(t - s)^3 + \alpha^2 \int_s^t \int_s^\tau u(\sigma)\, d\sigma\, d\tau$$

for all $t \in \langle s, T\rangle$, $c_0, c_1, c_2, c_3, \alpha \neq 0$ being non-negative constants. Then

$$u(t) \leqq (c_0 + c_1\alpha^{-1} + 2c_2\alpha^{-2} + 6c_3\alpha^{-3})\, e^{\alpha(t-s)}\,.$$

It is evident that the validity of (18) in every interval in which $y(t, \lambda)$ exists guarantees the existence of $y(t, \lambda)$ in all the interval $\langle s, T\rangle$ for $\lambda > 0$ sufficiently small. This completes the proof of Theorem 1.

Theorem 2. *Let* $\tilde{x}_1(\lambda), \tilde{x}_1 \in G$, $\tilde{x}_1(\lambda) \to \tilde{x}_1$, $\tilde{x}_2(\lambda) - \Phi(\tilde{x}_1(\lambda), s, \lambda) \to \tilde{x}_2$ *when* $\lambda \to 0_+$. *Let exist functions* $H(x, t)$, $f(x, t)$ *defined and continuous on* $G \times \langle s, T \rangle$ *such that*

$$\lim_{\lambda \to 0_+} f(x, t, \lambda) = f(x, t) \tag{19}$$

$$\lim_{\lambda \to 0_+} \int_s^t \Phi_x(x, \tau, \lambda)\, \Phi(x, \tau, \lambda)\, d\tau = \int_s^t H(x, \tau)\, d\tau$$

uniformly for $x \in G$ *and* $t \in \langle s, T \rangle$. *Functions* f, Φ_x *let fulfil the Lipschitz condition in* x *with a constant* M *independent of* λ. *Let the equation*

$$\ddot{x} = f(x, t) - H(x, t) \tag{20}$$

with initial conditions $x(s) = \tilde{x}_1$, $\dot{x}(s) = \tilde{x}_2$ *have a unique solution* $x(t)$ *which is defined on* $\langle s, T \rangle$.

Then for all $\lambda > 0$ *sufficiently small there exists the solution* $x(t, \lambda)$ *of* ($\mathfrak{E}$) *in* $\langle s, T \rangle$ *with initial conditions* $x(s, \lambda) = \tilde{x}_1(\lambda)$, $\dot{x}(s, \lambda) = \tilde{x}_2(\lambda)$ *and*

$$\lim_{\lambda \to 0_+} x(t, \lambda) = x(t)\,, \tag{21}$$

$$\lim_{\lambda \to 0_+} [\dot{x}(t, \lambda) - \Phi(x(t, \lambda), t, \lambda)] = \dot{x}(t)$$

holds uniformly on $\langle s, T \rangle$.

Proof. If we prove that (21) holds uniformly in an arbitrary interval $\langle s, T_1 \rangle$ such that for $\lambda > 0$ sufficiently small the solution $x(t, \lambda)$ with the given initial conditions exists (and belongs to G) for all $t \in \langle s, T_1 \rangle$, then it is evident that $x(t, \lambda)$ exists and (21) holds in all the interval $\langle s, T \rangle$.*) Consequently, it is sufficient to prove (21) under the assumption that $x(t, \lambda)$ exists in the whole interval $\langle s, T \rangle$.

From (17) and from the assumption of the convergence of the initial conditions it follows that the set of all $x(t, \lambda)$ on the interval $\langle s, T \rangle$, $0 < \lambda \leqq \lambda_0$ is uniformly bounded and equicontinuous. According to the Arzela's lemma it is thus possible to choose from it a convergent sequence of functions. We shall prove that the function $x(t)$ given by

$$x(t) = \tilde{x}_1 + \tilde{x}_2(t - s) + \int_s^t \int_s^\tau f(x(\sigma), \sigma)\, d\sigma\, d\tau - \int_s^t \int_s^\tau H(x(\sigma), \sigma)\, d\sigma\, d\tau$$

is a (uniform) limit of every convergent sequence of functions $x(t, \lambda_n)$ with $\lambda_n \to 0_+$. From this and from the unicity of the solution of equation (20) the assertion of Theorem 2 follows.

For the sake of simplicity let us denote $\Phi_x \Phi = \Psi$ and write λ instead of λ_n.

) Cf. proof of Theorem 1.

Obviously,

$$(23)\quad |x(t,\lambda) - x(t)| \leqq o(1) + \left|\int_s^t \int_s^\tau [f(x(\sigma,\lambda),\sigma,\lambda) - f(x(\sigma),\sigma)]\,d\sigma\,d\tau\right| + \\ + \left|\int_s^t \int_s^\tau [\Psi(x(\sigma,\lambda),\sigma,\lambda) - H(x(\sigma),\sigma)]\,d\sigma\,d\tau\right|$$

according to the convergence of the initial conditions, (8) and (22). As $f(x, t, \lambda)$ fulfils the Lipschitz condition in x,

$$|f(x(\sigma,\lambda),\sigma,\lambda) - f(x(\sigma),\sigma)| \leqq |f(x(\sigma,\lambda),\sigma,\lambda) - f(x(\sigma),\sigma,\lambda)| + \\ + |f(x(\sigma),\sigma,\lambda) - f(x(\sigma),\sigma)| \leqq M|x(\sigma,\lambda) - x(\sigma)| + o(1)$$

and hence

$$(24)\quad \left|\int_s^t \int_s^\tau [f(x(\sigma,\lambda),\sigma,\lambda) - f(x(\sigma),\sigma)]\,d\sigma\,d\tau\right| \leqq o(1) + \\ + M\int_s^t \int_s^\tau |x(\sigma,\lambda) - x(\sigma)|\,d\sigma\,d\tau\,.$$

Further,

$$(25)\quad \left|\int_s^t \int_s^\tau [\Psi(x(\sigma,\lambda),\sigma,\lambda) - H(x(\sigma),\sigma)]\,d\sigma\,d\tau\right| \leqq \\ \leqq \int_s^t \int_s^\tau |\Psi(x(\sigma,\lambda),\sigma,\lambda) - \Psi(x(\sigma),\sigma,\lambda)|\,d\sigma\,d\tau + \\ + \int_s^t \left|\int_s^\tau [\Psi(x(\sigma),\sigma,\lambda) - H(x(\sigma),\sigma)]\,d\sigma\right| d\tau\,.$$

The first right-hand integral can be estimated analogously as in the case of function f according to the fact that Ψ fulfils the Lipschitz condition in x. (The functions Φ and Φ_x are Lipschitzian and bounded; without any loss of generality we may assume that the Lipschitz constant of Ψ is M again.) To get a suitable estimate for the second right-hand integral in (25), we shall proceed in the following way:

Choose an arbitrary positive integer n and construct an equidistant division of the interval $\langle s, \tau\rangle$:

$$s = t_0 < t_1 < \ldots < t_{n-1} < t_n = \tau,\ t_{i+1} - t_i = \frac{\tau - s}{n} \quad (i = 1, 2, \ldots, n-1)\,.$$

Then

$$(26)\quad \left|\int_s^t [\Psi(x(\sigma),\sigma,\lambda) - H(x(\sigma),\sigma)]\,d\sigma\right| \leqq \sum_{i=0}^{n-1} \left|\int_{t_i}^{t_{i+1}} [\times]\,d\sigma\right| \leqq \\ \leqq \sum_{i=0}^{n-1} \left\{\int_{t_i}^{t_{i+1}} |\Psi(x(\sigma),\sigma,\lambda) - \Psi(x(t_i),\sigma,\lambda)|\,d\sigma + \right. \\ \left. + \left|\int_{t_i}^{t_{i+1}} [\Psi(x(t_i),\sigma,\lambda) - H(x(t_i),\sigma)]\,d\sigma\right| + \left|\int_{t_i}^{t_{i+1}} [H(x(t_i),\sigma) - H(x(\sigma),\sigma)]\,d\sigma\right|\right\}$$

holds. The first and the last integral are estimated by the difference $|x(\sigma) - x(t_i)|$ the function Ψ being Lipschitzian in x with respect to our assumptions. According to (17), the first integral can be majorized by the sum

$$\sum_{i=0}^{n-1} MA(t_{i+1} - t_i)^2 .$$

We can majorize similarly even the last integral. In fact, as for all $t_*, t^* \in \langle s, T \rangle$ there is

$$\int_{t_*}^{t^*} [H(x_2, t) - H(x_1, t)] \, \mathrm{d}t = \lim_{\lambda \to 0+} \int_{t_*}^{t^*} [\Psi(x_2, t, \lambda) - \Psi(x_1, t, \lambda)] \, \mathrm{d}t ,$$

$$\left| \int_{t_*}^{t^*} [\Psi(x_2, t, \lambda) - \Psi(x_1, t, \lambda)] \, \mathrm{d}t \right| \leqq M|x_2 - x_1| \, |t^* - t_*| ,$$

there is also

(27) $$\left| \int_{t_*}^{t^*} [H(x_2, t) - H(x_1, t)] \, \mathrm{d}t \right| \leqq M|x_2 - x_1| \, |t^* - t_*| .$$

Denote $\mathscr{H}(x(\tau), t) = \int_s^t H(x(\tau), \sigma) \, \mathrm{d}\sigma$; then $\partial \mathscr{H} / \partial t = H(x(\tau), t)$ and thus, with respect to 0.1,

$$\int_s^t \mathrm{D}_\sigma \mathscr{H}(x(\tau), \sigma) = \int_s^t H(x(\sigma), \sigma) \, \mathrm{d}\sigma .$$

Hence, we can rewrite (27) in the form

$$|\mathscr{H}(x_2, t^*) - \mathscr{H}(x_1, t^*) - \mathscr{H}(x_2, t_*) + \mathscr{H}(x_1, t_*)| \leqq M|x_2 - x_1| \, |t^* - t_*| .$$

Substituting here $x_2 = x(t^*)$, $x_1 = x(t_*)$, we get by means of (17)

$$|\mathscr{H}(x(t^*), t^*) - \mathscr{H}(x(t_*), t^*) - \mathscr{H}(x(t^*), t_*) + \mathscr{H}(x(t_*), t_*)| \leqq MA|t^* - t_*|^2$$

so that (according to 0.3)

$$\left| \int_{t_i}^{t_{i+1}} [H(x(\sigma), \sigma) - H(x(t_i), \sigma)] \, \mathrm{d}\sigma \right| =$$

$$= \left| \int_{t_i}^{t_{i+1}} \mathrm{D}_\sigma \mathscr{H}(x(\tau), \sigma) - \mathscr{H}(x(t_i), t_{i+1}) + \mathscr{H}(x(t_i), t_i) \right| \leqq MA(t_{i+1} - t_i)^2 .$$

The sum of the first and the third right-hand integrals of (26) is in this way majorized by the sum

$$2MA \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2 = 2MA \frac{(T - s)^2}{n}$$

so that (26) may be writen in the following way:

$$\left|\int_s^\tau [\Psi(x(\sigma), \sigma, \lambda) - H(x(\sigma), \sigma)]\, d\sigma\right| \leqq 2MA\,\frac{(T-s)^2}{n} + n\,o(1)\,.$$

From this it follows analogously as in the proof of Lemma 2

$$\left|\int_s^\tau [\Psi(x(\sigma), \sigma, \lambda) - H(x(\sigma), \sigma)]\, d\sigma\right| = o(1)\,.$$

Substituting this result into (25), we get

$$\int_s^t \left|\int_s^\tau [\Psi(x(\sigma, \lambda), \sigma, \lambda) - H(x(\sigma), \sigma)]\, d\sigma\right| d\tau \leqq o(1) + M \int_s^t \int_s^\tau |x(\sigma, \lambda) - x(\sigma)|\, d\sigma\, d\tau\,.$$

Making use of the analogous estimate (24) for the function f, (23) is transformed into the form

$$|x(t, \lambda) - x(t)| \leqq o(1) + 2M \int_s^t \int_s^\tau |x(\sigma, \lambda) - x(\sigma)|\, d\sigma\, d\tau\,.$$

The first part of the assertion of Theorem 2 is now an obvious consequence of Lemma 3.

The second part of the assertion follows quite similarly from (11), for according to Lemma 2 and to the assumptions of Theorem 2 there is

$$|\dot{x}(t, \lambda) - \Phi(x(t, \lambda), t, \lambda) - \dot{x}(t)| \leqq o(1) + \left|\int_s^t [f(x(\tau, \lambda), \tau, \lambda) - f(x(\tau), \tau)]\, d\tau\right| +$$
$$+ \left|\int_s^t [\Psi(x(\tau, \lambda), \tau, \lambda) - H(x(\tau), \tau)]\, d\tau\right|.$$

From this inequality we get

$$|\dot{x}(t, \lambda) - \Phi(x(t, \lambda), t, \lambda) - \dot{x}(t)| \leqq o(1) + 2M \int_s^t |x(\tau, \lambda) - x(\tau)|\, d\tau =$$
$$= o(1) + 2M \int_s^t \int_s^\tau |\dot{x}(\sigma, \lambda) - \Phi(x(\sigma, \lambda), \sigma, \lambda) - \dot{x}(\sigma)|\, d\sigma\, d\tau$$

by means of estimates analogous to (24) and (27) and by Lemma 2, as

$$\int_s^t |x(\tau, \lambda) - x(\tau)|\, d\tau = \int_s^t \int_s^\tau |\dot{x}(\sigma, \lambda) - \Phi(x(\sigma, \lambda), \sigma, \lambda) - \dot{x}(\sigma)|\, d\sigma\, d\tau + o(1)\,.$$

From Lemma 3, the second part of (21) follows immediately.

2. In this section we deduce some estimates for the convergence of the solution of equation (1) when $\lambda \to 0_+$. For the sake of simplicity we assume that neither the initial conditions nor the function f depend on λ.

Theorem 3. *In equation* ($\mathfrak{E}$) *let be* $f(x, t, \lambda) = f(x, t)$. *The function* ω *from the assumption* iv) *let be linear, i.e.* $\omega(\eta) = K_3\eta$. *Further let hold*

$$|P(x, t, \lambda)| \leqq B\lambda\,, \quad |P_x(x, t, \lambda)| \leqq B\lambda\,, \tag{28}$$

$$\left|\int_s^t [\Psi(x, \tau, \lambda) - H(x, \tau)]\,\mathrm{d}\tau\right| \leqq B\lambda$$

for all $x \in G$, $t \in \langle s, T\rangle$ *and* $\lambda \in (0, \lambda_0\rangle$.

Let $\tilde{x}_1 \in G$, $\tilde{x}_2(\lambda) - \Phi(\tilde{x}_1, s, \lambda) = \tilde{x}_2$. *Then, if the assumptions of Theorem 2 are fulfilled, an estimate*

$$|x(t, \lambda) - x(t)| \leqq$$

$$\leqq \left\{\frac{2\sqrt{(AB)}}{M\sqrt{3}}\left[K_2\sqrt{(2K_3)} + \sqrt{(2M)} + \frac{3K_1(1 + A)\sqrt{(K_3)}}{\sqrt{(2M)}}\right]\lambda^{\frac{1}{2}} + O(1)\,\lambda\right\} e^{(\sqrt{2M})(t-s)}$$

holds for all $\lambda > 0$ *sufficiently small and all* $t \in \langle s, T\rangle$, *where* $x(t, \lambda)$ *and* $x(t)$ *are solutions of* (1), (20), *respectively, with the initial conditions* $\tilde{x}_1, \tilde{x}_2(\lambda)$; $\tilde{x}_1, \tilde{x}_2$, *respectively.* ($O(1)$ *means a function of* (x, t, λ) *which is bounded independently of* λ *for* $x \in G$ *and* $t \in \langle s, T\rangle$.)

Proof. If all assumptions of Theorem 2 are fulfilled, then it follows from (12)

$$|x(t, \lambda) - x(t)| \leqq \left|\int_s^t \Phi(x(\tau, \lambda), \tau, \lambda)\,\mathrm{d}\tau\right| + \left|\tilde{x}_2 \int_s^t\int_s^\tau \Phi_x(x(\sigma, \lambda), \sigma, \lambda)\,\mathrm{d}\sigma\,\mathrm{d}\tau\right| + \tag{29}$$

$$+ \left|\int_s^t\int_s^\tau \Phi_x(x(\sigma, \lambda), \sigma, \lambda)\int_s^\sigma [f(x(\xi, \lambda), \xi) -\right.$$

$$\left. - \dot{x}(\xi, \lambda)\,\Phi_x(x(\xi, \lambda), \xi, \lambda)]\,\mathrm{d}\xi\,\mathrm{d}\sigma\,\mathrm{d}\tau\right| +$$

$$+ \left|\int_s^t\int_s^\tau [f(x(\sigma, \lambda), \sigma) - f(x(\sigma), \sigma)]\,\mathrm{d}\sigma\,\mathrm{d}\tau\right| +$$

$$+ \left|\int_s^t\int_s^\tau [\Psi(x(\sigma, \lambda), \sigma, \lambda) - H(x(\sigma), \sigma)]\,\mathrm{d}\sigma\,\mathrm{d}\tau\right|$$

where $\Psi = \Phi_x\Phi$ again.

The first right-hand integral may be writen as a generalized integral (cf. proof of Lemma 2) $\int_s^t \mathrm{D}_\sigma\, P(x(\tau, \lambda), \sigma, \lambda)$. As continuous partial derivative $\partial P/\partial x = P_x$

exists, we may write (see 0.2)

$$(30)\quad \left|\int_s^t \mathrm{D}_\sigma P(x(\tau,\lambda),\sigma,\lambda)\right| = \left|P(x(t,\lambda),t,\lambda) - P(\tilde{x}_1,s,\lambda) - \int_s^t \mathrm{D}_\tau P(x(\tau,\lambda),\sigma,\lambda)\right| \leqq$$

$$\leqq |P(x(t,\lambda),t,\lambda)| + |P(\tilde{x}_1,s,\lambda)| + \left|\int_s^t P_x(x(\tau,\lambda),\tau,\lambda)\,\dot{x}(\tau,\lambda)\,\mathrm{d}\tau\right| .$$

Further,

$$\left|\int_s^t \Phi(x(\tau,\lambda),\tau,\lambda)\,\mathrm{d}\tau\right| \leqq 2B\lambda + AB\lambda(t-s) = O(1)\,\lambda$$

according to (16) and (28). If we write Φ_x, P_x instead of Φ, P in (15), we get an estimate for the second integral

$$\left|\tilde{x}_2(\lambda)\int_s^t\int_s^\tau \Phi_x(x(\sigma,\lambda),\sigma,\lambda)\,\mathrm{d}\sigma\,\mathrm{d}\tau\right| \leqq K_2\left[2nB\lambda(t-s) + K_3A\,\frac{(t-s)^3}{3n}\right]$$

and similarly, after changing the order of integration, we get for the third integral as in the proof of Theorem 1

$$\left|\int_s^t\int_s^\tau \Phi_x \int_s^\sigma [f - \dot{x}\Phi_x]\,\mathrm{d}\xi\,\mathrm{d}\sigma\,\mathrm{d}\tau\right| \leqq K_1(1+A)\left[nB\lambda(t-s)^2 + K_3A\,\frac{(t-s)^4}{12n}\right];$$

n may be chosen arbitrarily in each of this estimates.

Still it remains to estimate the last two integrals. By the assumptions of Theorem 2 there is

$$\left|\int_s^t\int_s^\tau [f(x(\sigma,\lambda),\sigma) - f(x(\sigma),\sigma)]\,\mathrm{d}\sigma\,\mathrm{d}\tau\right| \leqq M\int_s^t\int_s^\tau |x(\sigma,\lambda) - x(\sigma)|\,\mathrm{d}\sigma\,\mathrm{d}\tau .$$

For the last integral we get from (25) and (26)

$$\left|\int_s^t\int_s^\tau [\Psi(x(\sigma,\lambda),\sigma,\lambda) - H(x(\sigma),\sigma)]\,\mathrm{d}\sigma\,\mathrm{d}\tau\right| \leqq$$

$$\leqq nB\lambda(t-s) + 2AM\,\frac{(t-s)^3}{3n} + M\int_s^t\int_s^\tau |x(\sigma,\lambda) - x(\sigma)|\,\mathrm{d}\sigma\,\mathrm{d}\tau .$$

Three of the deduced estimates contain terms of the form $c\lambda n + dn^{-1}$. We can assure easily that such a term reaches its minimum for $n = \sqrt{(d/c\lambda)}$; this minimum is $2\sqrt{(cd\lambda)}$. Of course, here n is not necessarily an integer. Neverthless, if we choose such an n that $\sqrt{(d/c\lambda)} < n \leqq \sqrt{(d/c\lambda)} + 1$, there is obviously $c\lambda n + dn^{-1} =$

$= 2\sqrt{(cd\lambda)} + z$ where $z = O(1)\,\lambda$. If we choose n in our estimates in this way, we get

$$|x(t, \lambda) - x(t)| \leqq \sqrt{\left(\frac{AB}{3}\right)}\, \{2[K_2\sqrt{(2K_3)} + \sqrt{(2M)}]\,(t - s)^2 +$$

$$+ K_1\sqrt{(K_3)}\,(1 + A)\,(t - s)^3\}\,\lambda^{\frac{1}{2}} + O(1)\,\lambda + 2M\int_s^t\int_s^\tau |x(\sigma, \lambda) - x(\sigma)|\,\mathrm{d}\sigma\,\mathrm{d}\tau\,.$$

The assertion of Theorem 3 is a consequence of Lemma 3.

Note. The exponent $\frac{1}{2}$ in Theorem 3 cannot be improved in general. As an example we may take the equation

$$\ddot{x} = \lambda^{\frac{1}{2}} \cos\left[\lambda^{-\frac{1}{2}}(t - x)\right].$$

It can be shown easily that for the solution of this equation with the initial conditions $\tilde{x}_1(\lambda) = \tilde{x}_1 = 0$, $\tilde{x}_2(\lambda) = \tilde{x}_2 = 1$ an inequality

$$|x(t, \lambda) - x(t)| \leqq K\lambda^\gamma$$

holds with $\gamma = \frac{1}{2}$ but does not hold for any $\gamma > \frac{1}{2}$. $x(t)$ is the solution of the limit equation $\ddot{x} = 0$.)

Nevertheless, the assumptions of Theorem 3 need not be strengthened much in order to get

Theorem 4. *For equation* $(\mathfrak{E})$ *let the assumptions of Theorem 3 hold. Further, let exist functions* $P_{xx} = \partial^2 P/\partial x^2$, Ψ_x, H_x *continuous in* (x, t) *for which*

$$|P_{xx}(x, t, \lambda)| \leqq B\lambda\,; \quad \left|\int_s^t [\Psi_x(x, \tau, \lambda) - H(x, \tau)]\,\mathrm{d}\tau\right| \leqq B\lambda$$

for all $x \in G$, $t \in \langle s, T\rangle$ *and* $0 < \lambda \leqq \lambda_0$. *Then*

(31)
$$|x(t, \lambda) - x(t)| \leqq$$
$$\leqq \left[2 + \frac{2 + A + 2K_2}{\sqrt{(2M)}} + \frac{A(1 + K_2) + 2K_1(1 + A)}{2M} + \frac{AK_1(1 + A)}{2M\sqrt{(2M)}}\right] B\lambda e^{(\sqrt{2M})(t-s)}.$$

Proof. We make again use of (12) and (29), respectively. If we estimate all the right-hand integrals analogously as $\int_s^t \Phi(x(\tau, \lambda), \tau, \lambda)\,\mathrm{d}\tau$ in (30), we get

$$|x(t, \lambda) - x(t)| \leqq [2 + A(t - s)]\,B\lambda + (1 + K_2)\,[2(t - s) + \tfrac{1}{2}A(t - s)^2]\,B\lambda +$$

$$+ K_1(1 + A)\,[(t - s)^2 + \tfrac{1}{6}A(t - s)^3]\,B\lambda + 2M\int_s^t\int_s^\tau |x(\sigma, \lambda) - x(\sigma)|\,\mathrm{d}\sigma\,\mathrm{d}\tau$$

and (31) follows from Lemma 3.

3. Let us turn now to the case when the function f in (1) depends also on the first derivative of the solution, i.e. on $\dot{x}$. The difference of this case from the case just investigated follows – roughly speaking – from the fact that the function Φ does not in general converge to zero when $\lambda \to 0$ and, consequently, $\dot{x}(t, \lambda) \to \dot{x}(t)$ is not valid.

The differential equation

$$\ddot{x} = f(x, \dot{x}, t, \lambda) + \varphi(x, t, \lambda) \tag{1'}$$

will be denoted by $(\mathfrak{E}')$ if its right-hand side fulfils the following conditions:

The function $f(x, y, t, \lambda)$ is defined for $x \in G$, $y \in G'$, $t \in \langle s, T \rangle$, $\lambda \in (0, \lambda_0)$, G and G' being open subsets of E_n. Further, f is continuous in (x, y, t) on $G \times G' \times \langle s, T \rangle$ for all $\lambda \in (0, \lambda_0\rangle$ and bounded in all its definition domain:

$$|f(x, y, t, \lambda)| \leqq K_1 .$$

The function $\varphi(x, t, \lambda)$ is defined on $G \times \langle s, T \rangle \times (0, \lambda_0\rangle$ continuous in (x, t) on $G \times \langle s, T \rangle$ and fulfils conditions i)–iv) from sec. 1 (p. 126–127).

The proof of the following theorem is quite analogous to that of Theorem 1:

Theorem 5. *Let $x(t, \lambda)$ be a solution of $(\mathfrak{E}')$ in the interval $\langle s, T \rangle$ with initial conditions $x(s, \lambda) = \tilde{x}_1(\lambda)$, $\dot{x}(s, \lambda) = \tilde{x}_2(\lambda)$. Let (7) hold for $0 < \lambda \leqq \lambda_0$. Then*

$$x(t, \lambda) = \tilde{x}_1(\lambda) + [\tilde{x}_2(\lambda) - \Phi(\tilde{x}_1(\lambda), s, \lambda)] (t - s) + \\ + \int_s^t \int_s^\tau f(x(\sigma, \lambda), \dot{x}(\sigma, \lambda), \sigma, \lambda) \, d\sigma \, d\tau - \\ - \int_s^t \int_s^\tau \Phi_x(x(\sigma, \lambda), \sigma, \lambda) \, \Phi(x(\sigma, \lambda), \sigma, \lambda) \, d\sigma \, d\tau + o(1) .$$

Theorem 6. *Let $\tilde{x}_1 \in G$, $\tilde{x}_2 \in G'$, $\tilde{x}_1(\lambda) \to \tilde{x}_1$, $\tilde{x}_2(\lambda) - \Phi(\tilde{x}_1(\lambda), s, \lambda) \to \tilde{x}_2$ when $\lambda \to 0_+$, $G' \supset E[y; |y| \leqq |\tilde{x}_2| + K_1]$.*

Let there exist functions $H(x, t)$, $Q(x, y, t)$ defined and continuous on $G \times \langle s, T \rangle$, $G \times G' \times \langle s, T \rangle$, respectively and such that

$$\lim_{\lambda \to 0+} \int_{t_1}^{t_2} \Phi_x(x, \tau, \lambda) \, \Phi(x, \tau, \lambda) \, d\tau = \int_{t_1}^{t_2} H(x, \tau) \, d\tau ,$$

$$\lim_{\lambda \to 0+} \int_{t_1}^{t_2} f(x, y + \Phi(x, \tau, \lambda), t, \lambda) \, d\tau = (t_2 - t_1) \, Q(x, y, t)$$

uniformly in the whole definition domain.

The functions f and Φ_x let fulfil the Lipschitz condition with respect to x and f with respect to y and t too, with a constant M independent of λ.

Let the equation

$$\ddot{x} = Q(x, \dot{x}, t) - H(x, t) \tag{32}$$

have a unique solution with the initial conditions $x(s) = \tilde{x}_1$, $\dot{x}(s) = \tilde{x}_2$, defined on the whole interval $\langle s, T \rangle$.

Then for all $\lambda > 0$ sufficiently small the solution $x(t, \lambda)$ of $(\mathfrak{E}')$ with the initial conditions $x(s, \lambda) = \tilde{x}_1(\lambda)$, $\dot{x}(s, \lambda) = \tilde{x}_2(\lambda)$ exists. Further,

(33)
$$\lim_{\lambda \to 0_+} x(t, \lambda) = x(t)\,,$$
$$\lim_{\lambda \to 0_+} [\dot{x}(t, \lambda) - \Phi(x(t, \lambda), t, \lambda)] = \dot{x}(t)$$

holds uniformly on $\langle s, T \rangle$.

Proof. Assume first of all that the following assertion holds:

Let $\langle s, T_1 \rangle \subset \langle s, T \rangle$ be an interval on which the solution of $(\mathfrak{E}')$ with the required initial conditions exists. Then Theorem 6 holds if we write $\langle s, T_1 \rangle$ instead of $\langle s, T \rangle$.

It is obvious then that (33) guarantees the existence of the solution $x(t, \lambda)$ on the whole interval $\langle s, T \rangle$. Thus, Theorem 6 will be proved (similarly as Theorem 2) if we prove (33) under assumption that for all $\lambda > 0$ sufficiently small the solution $x(t, \lambda)$ fulfillng our conditions exists on the whole interval $\langle s, T \rangle$.

Denote $\dot{X}(t, \lambda) = \dot{x}(t, \lambda) - \Phi(x(t, \lambda), t, \lambda)$. First we prove the second formula (33), i.e.

$$\lim_{\lambda \to 0_+} |\dot{X}(t, \lambda) - \dot{x}(t)| = 0$$

uniformly on $\langle s, T \rangle$.

Quite similarly as in Theorem 2 we prove a relation analogous to (11). As $\dot{x}(t, \lambda)$ belongs to G', there is $|f| \leqq K_1$ and (17) holds*). Lemma 2 is therefore valid unchanged for equation $(\mathfrak{E}')$. From it follows

(34)
$$|\dot{X}(t, \lambda) - \dot{x}(t)| \leqq o(1) + \left| \int_s^t [f(x(\tau, \lambda), \dot{X}(\tau, \lambda) + \Phi(x(\tau, \lambda), \tau, \lambda), \tau, \lambda) - Q(x(\tau), \dot{x}(\tau), \tau)] \, d\tau \right| + \left| \int_s^t [\Psi(x(\tau, \lambda), \tau, \lambda) - H(x(\tau), \tau)] \, d\tau \right|.$$

Denoting the first right-hand integral by J_1, there holds analogously to (26):

(35)
$$\begin{aligned} |J_1| = \Bigg| & \int_s^t [f(x(\tau, \lambda), \dot{X}(\tau, \lambda) + \Phi(x(\tau, \lambda), \tau, \lambda), \tau, \lambda) - \\ & - f(x(\tau), \dot{x}(\tau) + \Phi(x(\tau), \tau, \lambda), \tau, \lambda)] \, d\tau + \\ & + \sum_{i=0}^{n-1} \left\{ \int_{t_i}^{t_{i+1}} [f(x(\tau), \dot{x}(\tau) + \Phi(x(\tau), \tau, \lambda), \tau, \lambda) - \right. \\ & - f(x(t_i), \dot{x}(t_i) + \Phi(x(t_i), \tau, \lambda), t_i, \lambda)] \, d\tau + \\ & + \int_{t_i}^{t_{i+1}} [f(x(t_i), \dot{x}(t_i) + \Phi(x(t_i), \tau, \lambda), t_i, \lambda) - Q(x(t_i), \dot{x}(t_i), t_i)] \, d\tau + \\ & \left. + \int_{t_i}^{t_{i+1}} [Q(x(t_i), \dot{x}(t_i), t_i) - Q(x(\tau), \dot{x}(\tau), \tau)] \, d\tau \right\} \Bigg| \end{aligned}$$

*) If we use formulae of sec. 1 we suppose that f depends on x, too.

where $s = t_0 < t_1 < \ldots < t_{n-1} < t_n = t$ is an equidistant division of the interval $\langle s, t \rangle$.

In what follows we make use of the fact that functions f and Q fulfil the Lipschitz condition in x, y, t.

In fact, f is Lipschitzian by the assumption of Theorem 6. Further,

$$Q(x, y, t) = \frac{1}{t_2 - t_1} \lim_{\lambda \to 0+} \int_{t_1}^{t_2} f(x, y + \Phi(x, \tau, \lambda), t, \lambda) \, d\tau .$$

Hence e.g.

$$|Q(x_2, y, t) - Q(x_1, y, t)| =$$

$$= \frac{1}{|t_2 - t_1|} \left| \lim_{\lambda \to 0+} \int_{t_1}^{t_2} [f(x_2, y + \Phi(x_2, \tau, \lambda), t, \lambda) - f(x_1, y + \Phi(x_1, \tau, \lambda), t, \lambda)] \, d\tau \right| \leqq$$

$$\leqq \frac{1}{|t_2 - t_1|} \lim_{\lambda \to 0+} \left\{ \int_{t_1}^{t_2} [f(x_2, y + \Phi(x_2, \tau, \lambda), t, \lambda) - f(x_1, y + \Phi(x_2, \tau, \lambda), t, \lambda)] \, d\tau + \right.$$

$$\left. + \int_{t_1}^{t_2} |f(x_1, y + \Phi(x_2, \tau, \lambda), t, \lambda) - f(x_1, y + \Phi(x_1, \tau, \lambda), t, \lambda)| \, d\tau \right\} \leqq$$

$$\leqq \frac{1}{|t_2 - t_1|} \left\{ M|(t_2 - t_1)(x_2 - x_1)| + M \int_{t_1}^{t_2} |\Phi(x_2, \tau, \lambda) - \Phi(x_1, \tau, \lambda)| \, d\tau \right\} \leqq$$

$$\leqq M(1 + K_1) |x_2 - x_1|$$

because the function Φ fulfils the Lipschitz condition in x with the constant K_1 which bounds the partial derivative of Φ with respect to x. Analogously the validity of Lipschitz condition in the other variables may be proved (with the constant M).

Now we can continue the proof of the Theorem. There is

$$|J_1| \leqq M \int_s^t \{|x(\tau, \lambda) - x(\tau)| + |\dot{X}(\tau, \lambda) - \dot{x}(\tau)| +$$

$$+ |\Phi(x(\tau, \lambda), \tau, \lambda) - \Phi(x(\tau), \tau, \lambda)|\} \, d\tau +$$

$$+ \sum_{i=0}^{n-1} \left\{ o(1) + 2M \int_{t_i}^{t_{i+1}} [|x(\tau) - x(t_i)| + |\dot{x}(\tau) - \dot{x}(t_i)| + \right.$$

$$\left. + |\tau - t_i| + |\Phi(x(\tau), \tau, \lambda) - \Phi(x(t_i), \tau, \lambda)|] \, d\tau \right\} .$$

The function Φ is Lipschitzian in x (Φ_x is bounded). From (17) and from

$$\dot{x}(t) = \tilde{x}_2 + \int_s^t Q(x(\tau), \dot{x}(\tau), \tau) \, d\tau - \int_s^t H(x(\tau), \tau) \, d\tau$$

it follows that $\dot{x}(t)$ and $x(t)$ fulfil the Lipschitz condition in t. Further,

$$(36) \quad \int_s^t |x(\tau, \lambda) - x(\tau)| \, d\tau \leqq \int_s^t |\tilde{x}_1(\lambda) - \tilde{x}_1| \, d\tau + \int_s^t \left| \int_s^\tau [\dot{X}(\sigma, \lambda) - \dot{x}(\sigma)] \, d\sigma \right| d\tau +$$
$$+ \int_s^t \left| \int_s^\tau \Phi(x(\sigma, \lambda), \sigma, \lambda) \, d\sigma \right| d\tau \leqq o(1) + (T - s) \int_s^t |\dot{X}(\tau, \lambda) - \dot{x}(\tau)| \, d\tau$$

and hence

$$|J_1| \leqq c_1 \int_s^t |\dot{X}(\tau, \lambda) - \dot{x}(\tau)| \, d\tau + c_2 \sum_{i=0}^{n-1} \left[\int_{t_i}^{t_{i+1}} (\tau - t_i) \, d\tau + o(1) \right].$$

By an appropriate choice of a positive integer n we prove (cf. the proof of Lemma 2) the following estimate for J_1:

$$|J_1| \leqq o(1) + c^* \int_s^t |\dot{X}(\tau, \lambda) - \dot{x}(\tau)| \, d\tau \, .$$

The other integral on the right-hand side of (34) can be estimated as in the proof of Theorem 2 (cf. (26) and below). Making use of (36) we see immediately that an analogous estimate as for J_1 holds so that we get altogether

$$|\dot{X}(t, \lambda) - \dot{x}(t)| \leqq o(1) + c^{**} \int_s^t |\dot{X}(\tau, \lambda) - \dot{x}(\tau)| \, d\tau \, .$$

From Lemma 1 we get immediately (33). The validity of the first formula is now evident, for

$$|x(t, \lambda) - x(t)| = \left| \tilde{x}_1(\lambda) + \int_s^t \dot{x}(\tau, \lambda) \, d\tau - \tilde{x}_1 - \int_s^t \dot{x}(\tau) \, d\tau \right| \leqq$$
$$\leqq |\tilde{x}_1(\lambda) - \tilde{x}_1| + \int_s^t |\dot{X}(\tau, \lambda) - \dot{x}(\tau)| \, d\tau + \left| \int_s^t \Phi(x(\tau, \lambda), \tau, \lambda) \, d\tau \right| = o(1) \, .$$

This completes the proof of Theorem 6.

Note. Let e.g. $f(x, y, t, \lambda) = ay^2$, $\varphi(x, t, \lambda) = \lambda^{-1} x \cos \lambda^{-1} t$. Then we obtain by direct computation

$$\Phi(x, t, \lambda) = x \sin \lambda^{-1} t \, ,$$

$$\lim_{\lambda \to 0+} \int_{t_1}^{t_2} f(x, y + \Phi(x, \tau, \lambda), t, \lambda) \, d\tau = \lim_{\lambda \to 0+} a \int_{t_1}^{t_2} (y + x \sin \lambda^{-1} \tau)^2 \, d\tau =$$
$$= (t_2 - t_1) \, a(y^2 - \tfrac{1}{2} x^2) \, .$$

The limit equation has the form

$$\ddot{x} = a\dot{x}^2 - \frac{1}{2} a x^2 - \frac{x}{2} \, .$$

On the other hand, if f depends on $\dot{x}$ linearly, no resonant term appears in the limit equation. In fact, if $f(x, y, t, \lambda) = a(x, t, \lambda)\, y + b(x, t, \lambda)$ then according to (5)

$$\lim_{\lambda \to 0+} \int_{t_2}^{t_1} f(x, y + \Phi(x, \tau, \lambda), t, \lambda)\, d\tau = \lim_{\lambda \to 0+} \int_{t_1}^{t_2} f(x, y, t, \lambda)\, d\tau +$$

$$+ \lim_{\lambda \to 0+} a(x, t, \lambda) \int_{t_1}^{t_2} \Phi(x, \tau, \lambda)\, d\tau = (t_2 - t_1) \lim_{\lambda \to 0+} f(x, y, t, \lambda)\,.$$

4. In this section we deduce important inequalities (38) which enable us to prove a theorem on the existence and stability of a periodic solution of equation ($\mathfrak{E}'$) with small λ. However, we have to introduce some further assumptions concerned to the convergence of the right-hand side of ($\mathfrak{E}'$) and to its smoothness with respect to x, $\dot{x}$.

Notation. In this section, the symbol $h(\lambda)$ will denote an arbitrary function for which $\lim_{\lambda \to 0+} h(\lambda) = 0$.

We shall say that the function $U(x, t, \lambda)$ has the property $\mathscr{P}$ if there exists a constant K and a function $h(\lambda)$ so that

i) $|U(x, t, \lambda)| \leqq h(\lambda)$;

ii) $|U(x, t, \lambda) - U(y, t, \lambda)| \leqq |x - y|\, h(\lambda)$;

iii) $|U(x, t_1, \lambda) - U(x, t_2, \lambda) - U(y, t_1, \lambda) + U(y, t_2, \lambda)| \leqq K|t_2 - t_1|\, |x - y|$;

iv) $|U(x + u, t_1, \lambda) - U(x + u, t_2, \lambda) - U(y + u, t_1, \lambda) + U(y + u, t_2, \lambda) - U(x, t_1, \lambda) + U(x, t_2, \lambda) + U(y, t_1, \lambda) - U(y, t_2, \lambda)| \leqq K|t_2 - t_1|\, |x - y|\, |u|$

for all t, t_1, t_2, x, y, λ from the definition domain of U and for $|t_2 - t_1|$, $|x_2 - x_1|$, $|u|$ small enough.

Theorem 7. *Let us have the equation* ($\mathfrak{E}'$)

$$\ddot{x} = f(x, \dot{x}, t, \lambda) + \varphi(x, t, \lambda)$$

which fulfils all assumptions of Theorem 6.

Let functions $P(x, t, \lambda)$, $P_x(x, t, \lambda)$ *and* $W(x, t, \lambda) = \int_s^t [\Psi(x, \tau, \lambda) - H(x, \tau)]\, d\tau$ *have the property* $\mathscr{P}$; *the function* f *let fulfil the following conditions*:

i$_1$) $$\left| \int_{t_1}^{t_2} f(x, u + \Phi(x, \tau, \lambda), t, \lambda)\, d\tau - (t_2 - t_1)\, Q(x, u, t) \right| \leqq h(\lambda)\,;$$

ii$_1$) $$\left| \int_{t_1}^{t_2} [f(x, u + \Phi(x, \tau, \lambda), t, \lambda) - f(y, u + \Phi(y, \tau, \lambda), t, \lambda)]\, d\tau - (t_2 - t_1)\, [Q(x, u, t) - Q(y, u, t)] \right| \leqq |x - y|\, h(\lambda);$$

ii$_2$) $$\left| \int_{t_1}^{t_2} [f(x, u + \Phi(x, \tau, \lambda), t, \lambda) - f(x, v + \Phi(x, \tau, \lambda), t, \lambda)]\, d\tau - (t_2 - t_1)\, [Q(x, u, t) - Q(x, v, t)] \right| \leqq |u - v|\, h(\lambda);$$

iii$_1$) $|f(x, u, t, \lambda) - f(y, u, t, \lambda)| \leqq K|x - y|$;

iii$_2$) $|f(x, u, t, \lambda) - f(x, v, t, \lambda)| \leqq K|u - v|$;

iv$_1$) $|f(x, u, t, \lambda) - f(y, u, t, \lambda) - f(x + z, u, t, \lambda) + f(y + z, u, t, \lambda)| \leqq K|x - y|\,|z|$;

iv$_2$) $|f(x, u, t, \lambda) - f(x, v, t, \lambda) - f(x, u + w, t, \lambda) + f(x, v + w, t, \lambda)| \leqq K|u - v|\,|w|$;

iv$_3$) $|f(x, u, t, \lambda) - f(x, v, t, \lambda) - f(y, u, t, \lambda) + f(y, v, t, \lambda)| \leqq K|x - y|\,|u - v|$

where K is a constant, $t, t_1, t_2 \in \langle s, T\rangle$, $x, y \in G$, $u, v \in G'$, $|x - y|$, $|u - v|$, $|z|$, $|w|$ *sufficiently small.*

Further, for any two solutions $x(t)$, $y(t)$ *of the limit equation* (32) *with initial conditions* $x(s) = \tilde{x}_1$, $\dot{x}(s) = \tilde{x}_2$; $y(s) = \tilde{y}_1$, $\dot{y}(s) = \tilde{y}_2$, *respectively,*

$$|x(t) - y(t)| + |\dot{x}(t) - \dot{y}(t)| \leqq R\|\tilde{x} - \tilde{y}\| \tag{37}$$

let hold for all $t \in \langle s, T\rangle$ *where R is a constant and* $\|u\|$ *denotes a norm of the vector* (u_1, u_2) (*e.g.* $\|u\| = |u_1| + |u_2|$).

Let $x(t)$, $y(t)$ *be two solutions of* (32) *on* $\langle s, T\rangle$ *with the initial conditions* $\tilde{x}_1, \tilde{x}_2$; $\tilde{y}_1, \tilde{y}_2$, *respectively,* $x(t, \lambda)$, $y(t, \lambda)$ *two solutions of* (1′) *on* $\langle s, T\rangle$ *with the initial conditions* $x(s, \lambda) = \tilde{x}_1$, $\dot{x}(s, \lambda) = \tilde{x}_2(\lambda) = \tilde{x}_2 + \Phi(\tilde{x}_1, s, \lambda)$; $y(s, \lambda) = \tilde{y}_1$, $\dot{y}(s, \lambda) = \tilde{y}_2 + \Phi(\tilde{y}_1, s, \lambda)$, *respectively.*

Then for all $t \in \langle s, T\rangle$ *there holds*

$$\begin{aligned} &|x(t, \lambda) - y(t, \lambda) - x(t) + y(t)| \leqq \|\tilde{x} - \tilde{y}\|\, h(\lambda)\,, \\ &|\dot{X}(t, \lambda) - \dot{Y}(t, \lambda) - \dot{x}(t) + \dot{y}(t)| \leqq \|\tilde{x} - \tilde{y}\|\, h(\lambda) \end{aligned} \tag{38}$$

where $\dot{X}(t, \lambda) = \dot{x}(t, \lambda) - \Phi(x(t, \lambda), t, \lambda)$, $\dot{Y}(t, \lambda) = \dot{y}(t, \lambda) - \Phi(y(t, \lambda), t, \lambda)$, *respectively.*

Proof. In the same way as in sec. 1 we get estimates (cf. (11) and (12))

$$\begin{aligned} &|\dot{X}(t, \lambda) - \dot{Y}(t, \lambda) - \dot{x}(t) + \dot{y}(t)| \leqq \\ &\leqq \left|\int_s^t [f(x(\tau, \lambda), \dot{x}(\tau, \lambda), \tau, \lambda) - f(y(\tau, \lambda), \dot{y}(\tau, \lambda), \tau, \lambda) - f(x(\tau), \dot{x}(\tau), \tau) + \right. \\ &\left. + f(y(\tau), \dot{y}(\tau), \tau)]\, d\tau\right| + \left|\int_s^t [\tilde{x}_2(\lambda)\, \Phi(x(\tau, \lambda), \tau, \lambda) - \tilde{y}_2(\lambda)\, \Phi(y(\tau, \lambda), \tau, \lambda)]\, d\tau\right| + \\ &+ \left|\int_s^t \left\{\Phi_x(x(\tau, \lambda), \tau, \lambda) \int_s^\tau [f(x(\sigma, \lambda), \dot{x}(\sigma, \lambda), \sigma, \lambda) - \dot{x}(\sigma, \lambda)\, \Phi_x(x(\sigma, \lambda), \sigma, \lambda)]\, d\sigma - \right.\right. \\ &\left.\left. - \Phi_x(y(\tau, \lambda), \tau, \lambda) \int_s^\tau [f(y(\sigma, \lambda)\, \dot{y}(\sigma, \lambda), \sigma, \lambda) - \dot{y}(\sigma, \lambda)\, \Phi_x(y(\sigma, \lambda), \sigma, \lambda)]\, d\sigma\right\} d\tau\right| + \\ &+ \left|\int_s^t [\Psi(x(\tau, \lambda), \tau, \lambda) - \Psi(y(\tau, \lambda), \tau, \lambda) - H(x(\tau), \tau) + H(y(\tau), \tau)]\, d\tau\right|\,, \end{aligned} \tag{39}$$

$$\begin{aligned} &|x(t, \lambda) - y(t, \lambda) - x(t) + y(t)| \leqq \\ &\leqq \left|\int_s^t [\Phi(x(\tau, \lambda), \tau, \lambda) - \Phi(y(\tau, \lambda), \tau, \lambda)]\, d\tau\right| + \int_s^t |\dot{X}(\tau, \lambda) - \dot{Y}(\tau, \lambda) - \dot{x}(\tau) + \dot{y}(\tau)|\, d\tau\,. \end{aligned} \tag{40}$$

We shall prove an important lemma which enables us to estimate in the same way (only with formal changes) the integrals on the right-hand side of (39) and (40).

Lemma 4. *Let the function U defined on $G \times \langle s, T\rangle \times (0, \lambda_0\rangle$ have property $\mathscr{P}$. If we use the notation of Theorem 7, then*

$$\left|\int_s^t \mathrm{D}_\sigma[U(x(\tau,\lambda),\sigma,\lambda) - U(y(\tau,\lambda),\sigma,\lambda)]\right| \leqq \tag{41}$$

$$\leqq \|\tilde{x} - \tilde{y}\|\, h(\lambda) + \gamma \int_s^t |x(\tau,\lambda) - y(\tau,\lambda) - x(\tau) + y(\tau)|\, \mathrm{d}\vartheta(\tau,\lambda)$$

where $\vartheta(\tau,\lambda) = \tau + \Theta(\tau,\lambda)$, $\Theta(\tau,\lambda)$ is a non-decreasing piecewise constant function continuous from the left with the variation $t - s$, $\gamma =$ const.

Proof. Let n be a positive integer,

$$s = \alpha_0 < \alpha_1 < \ldots < \alpha_{n-1} < \alpha_n = t$$

an equidistant division of $\langle s, t\rangle$. As (cf. 0.3)

$$J_i = \int_{\alpha_i}^{\alpha_{i+1}} \mathrm{D}_\sigma[U(x(\tau,\lambda),\sigma,\lambda) - U(y(\tau,\lambda),\sigma,\lambda)]$$

exists ($i = 0, 1, 2, \ldots, n-1$), there is

$$J_i = \lim_{k\to\infty} \sum_{j=0}^{k-1} [U(x(\beta_j^{(k)},\lambda), \beta_{j+1}^{(k)}, \lambda) - U(y(\beta_j^{(k)},\lambda), \beta_{j+1}^{(k)}, \lambda) - U(x(\beta_j^{(k)},\lambda), \beta_j^{(k)}, \lambda) +$$
$$+ U(y(\beta_j^{(k)},\lambda), \beta_j^{(k)}, \lambda)]$$

for any sequence of divisions $\alpha_i = \beta_0^{(k)} \leqq \beta_1^{(k)} \leqq \ldots \leqq \beta_{k-1}^{(k)} \leqq \beta_k^{(k)} = \alpha_{i+1}$ for which

$$\lim_{k\to\infty} \max_j |\beta_{j+1}^{(k)} - \beta_j^{(k)}| = 0\,.$$

We can easily make sure that*)

$$J_i = U(x(\alpha_i,\lambda), \alpha_{i+1}) - U(y(\alpha_i,\lambda), \alpha_{i+1}) - U(x(\alpha_i,\lambda), \alpha_i) + U(y(\alpha_i,\lambda), \alpha_i) + Z_i\,,$$

$$Z_i = \lim_{k\to\infty} \sum_{j=0}^{k-1} [U(x(\beta_j,\lambda), \beta_{j+1}) - U(y(\beta_j,\lambda), \beta_{j+1}) - U(x(\beta_j,\lambda), \beta_j) + \tag{42}$$
$$+ U(y(\beta_j,\lambda), \beta_j) - U(x(\alpha_i,\lambda), \beta_{j+1}) + U(y(\alpha_i,\lambda), \beta_{j+1}) +$$
$$+ U(x(\alpha_i,\lambda), \beta_j) - U(y(\alpha_i,\lambda), \beta_j)]\,.$$

*) In what follows we omit the parameter λ and index k provided no misunderstanding may appear.

First let us estimate the remainder Z_i:

$$|Z_i| \leqq \lim_{k\to\infty} \{\sum_{j=0}^{k-1} |U(x(\beta_j, \lambda), \beta_{j+1}) - U(x(\beta_j, \lambda), \beta_j) - U(x(\beta_j, \lambda) +$$
$$+ y(\alpha_i, \lambda) - x(\alpha_i, \lambda), \beta_{j+1}) + U(x(\beta_j, \lambda) + y(\alpha_i, \lambda) - x(\alpha_i, \lambda), \beta_j) -$$
$$- U(x(\alpha_i, \lambda), \beta_{j+1}) + U(y(\alpha_i, \lambda), \beta_{j+1}) + U(x(\alpha_i, \lambda), \beta_j) - U(y(\alpha_i, \lambda), \beta_j)| +$$
$$+ \sum_{j=0}^{k-1} |U(x(\beta_j, \lambda) + y(\alpha_i, \lambda) - x(\alpha_i, \lambda), \beta_{j+1}) -$$
$$- U(x(\beta_j, \lambda) + y(\alpha_i, \lambda) - x(\alpha_i, \lambda), \beta_j) - U(y(\beta_j, \lambda), \beta_{j+1}) + U(y(\beta_j, \lambda), \beta_j)|\} .$$

The first sum we estimate by iv), the second one by iii) from the definition of property $\mathscr{P}$:

$$(43)\qquad |Z_i| \leqq \lim_{k\to\infty} K\{\sum_{j=0}^{k-1} (\beta_{j+1} - \beta_j) |x(\alpha_i, \lambda) - y(\alpha_i, \lambda)| |x(\beta_j, \lambda) - x(\alpha_i, \lambda)| +$$
$$+ \sum_{j=0}^{k-1} (\beta_{j+1} - \beta_j) |x(\alpha_i, \lambda) - y(\alpha_i, \lambda) - x(\beta_j, \lambda) + y(\beta_j, \lambda)|\} .$$

Further, there is obviously

$$|x(\alpha_i, \lambda) - y(\alpha_i, \lambda)| \leqq |x(\alpha_i, \lambda) - y(\alpha_i, \lambda) - x(\alpha_i) + y(\alpha_i)| + |x(\alpha_i) - y(\alpha_i)| ,$$
$$|x(\beta_j, \lambda) - x(\alpha_i, \lambda)| \leqq A(\beta_j - \alpha_i) \leqq A(\alpha_{i+1} - \alpha_i)$$

because (16) and (17) holds in the same way as in sec. 1

$$|x(\alpha_i, \lambda) - y(\alpha_i, \lambda) - x(\beta_j, \lambda) + y(\beta_j, \lambda)| \leqq |x(\alpha_i, \lambda) - y(\alpha_i, \lambda) - x(\alpha_i) + y(\alpha_i)| +$$
$$+ |x(\beta_j, \lambda) - y(\beta_j, \lambda) - x(\beta_j) + y(\beta_j)| + |x(\alpha_i) - y(\alpha_i) - x(\beta_j) + y(\beta_j)| ;$$

for the last term on the right-hand side there holds (cf. (37))

$$|x(\alpha_i) - y(\alpha_i) - x(\beta_j) + y(\beta_j)| =$$
$$= \left|\int_{\alpha_i}^{\beta_j} (\dot{x}(\tau) - \dot{y}(\tau))\, d\tau\right| \leqq (\beta_j - \alpha_i) \sup_{t\in\langle\alpha_i,\beta_j\rangle} |\dot{x}(t) - \dot{y}(t)| \leqq (\alpha_{i+1} - \alpha_i)\, R\|\tilde{x} - \tilde{y}\| .$$

Transforming (43) with respect to (37) and to the inequalities just mentioned and sumarizing with respect to j if possible, we get

$$|Z_i| \leqq AK(\alpha_{i+1} - \alpha_i)^2 \left[|x(\alpha_i, \lambda) - y(\alpha_i, \lambda) - x(\alpha_i) + y(\alpha_i)| + |x(\alpha_i) - y(\alpha_i)|\right] +$$
$$+ K(\alpha_{i+1} - \alpha_i) |x(\alpha_i, \lambda) - y(\alpha_i, \lambda) - x(\alpha_i) + y(\alpha_i)| + KR(\alpha_{i+1} - \alpha_i)^2 \|\tilde{x} - \tilde{y}\| +$$
$$+ \lim_{k\to\infty} K \sum_{j=0}^{k-1} (\beta_{j+1} - \beta_j) |x(\beta_j, \lambda) - y(\beta_j, \lambda) - x(\beta_j) + y(\beta_j)| \leqq$$
$$\leqq R_1(\alpha_{i+1} - \alpha_i)^2 \|\tilde{x} - \tilde{y}\| + R_2(\alpha_{i+1} - \alpha_i) |x(\alpha_i, \lambda) - y(\alpha_i, \lambda) - x(\alpha_i) + y(\alpha_i)| +$$
$$+ K \int_{\alpha_i}^{\alpha_{i+1}} |x(\tau, \lambda) - y(\tau, \lambda) - x(\tau) + y(\tau)|\, d\tau .$$

Here (and further) the letter R (with indices) denotes a constant.

Let us now return to the integral

$$J = \int_s^t \mathrm{D}_\sigma[U(x(\tau, \lambda)\ \sigma, \lambda) - U(y(\tau, \lambda), \sigma, \lambda)]$$

from inequality (41). There is

$$|J| \leqq \sum_{i=0}^{n-1} |J_i| \leqq \sum_{i=0}^{n-1} |U(x(\alpha_i, \lambda), \alpha_{i+1}) - U(y(\alpha_i, \lambda), \alpha_{i+1}) -$$
$$- U(x(\alpha_i, \lambda), \alpha_i) + U(y(\alpha_i, \lambda), \alpha_i)| + \sum_{i=0}^{n-1} |Z_i| .$$

Let us evaluate the first term:

$$\sum = \sum_{i=0}^{n-1} |U(x(\alpha_i, \lambda), \alpha_{i+1}) - U(y(\alpha_i, \lambda), \alpha_{i+1}) - U(x(\alpha_i, \lambda), \alpha_i) + U(y(\alpha_i, \lambda), \alpha_i)| \leqq$$
$$\leqq \sum_{i=0}^{n-1} |U(x(\alpha_i, \lambda), \alpha_{i+1}) - U(x(\alpha_i, \lambda), \alpha_i) - U(x(\alpha_i), \alpha_{i+1}) + U(x(\alpha_i), \alpha_i) -$$
$$- U(x(\alpha_i, \lambda) + y(\alpha_i) - x(\alpha_i), \alpha_{i+1}) + U(x(\alpha_i, \lambda) + y(\alpha_i) - x(\alpha_i), \alpha_i) +$$
$$+ U(y(\alpha_i), \alpha_{i+1}) - U(y(\alpha_i), \alpha_i)| +$$
$$+ \sum_{i=0}^{n-1} |U(x(\alpha_i, \lambda) + y(\alpha_i) - x(\alpha_i), \alpha_{i+1}) - U(x(\alpha_i, \lambda) + y(\alpha_i) - x(\alpha_i), \alpha_i) -$$
$$- U(y(\alpha_i, \lambda), \alpha_{i+1}) + U(y(\alpha_i, \lambda), \alpha_i)| +$$
$$+ \sum_{i=0}^{n-1} |U(x(\alpha_i), \alpha_{i+1}) - U(y(\alpha_i), \alpha_{i+1})| + \sum_{i=0}^{n-1} |U(x(\alpha_i), \alpha_i) - U(y(\alpha_i), \alpha_i)| .$$

The first right-hand term we estimate by iv), the second by iii) and the third and fourth by ii), from the definition of the property $\mathscr{P}$:

$$\sum \leqq \sum_{i=0}^{n-1} (\alpha_{i+1} - \alpha_i)\ |x(\alpha_i, \lambda) - x(\alpha_i)|\ |x(\alpha_i) - y(\alpha_i)| +$$
$$+ \sum_{i=0}^{n-1} (\alpha_{i+1} - \alpha_i)\ |x(\alpha_i, \lambda) - y(\alpha_i, \lambda) - x(\alpha_i) + y(\alpha_i)| + 2 \sum_{i=0}^{n-1} |x(\alpha_i) - y(\alpha_i)|\ h(\lambda)$$

so that by (37) and by Theorem 6

$$\sum \leqq 2nR \|\tilde{x} - \tilde{y}\|\ h(\lambda) + R_3 \|\tilde{x} - \tilde{y}\|\ h(\lambda) +$$
$$+ \sum_{i=0}^{n-1} (\alpha_{i+1} - \alpha_i)\ |x(\alpha_i, \lambda) - y(\alpha_i, \lambda) - x(\alpha_i) + y(\alpha_i)| .$$

Altogether, there is

$$|J| \leqq \sum + \sum_{i=0}^{n-1} |Z_i| \leqq [2n\, Rh(\lambda) + R_3\, h(\lambda) + R_4 n^{-1}]\, \|\tilde{x} - \tilde{y}\| +$$

$$+ (R_2 + 1) \sum_{i=0}^{n-1} (\alpha_{i+1} - \alpha_i)\, |x(\alpha_i, \lambda) - y(\alpha_i, \lambda) - x(\alpha_i) + y(\alpha_i)| +$$

$$+ K \int_s^t |x(\tau, \lambda) - y(\tau, \lambda) - x(\tau) + y(\tau)|\, \mathrm{d}\tau\, .$$

Choosing n in dependence on $h(\lambda)$ as in Lemma 1, there is

$$2n\, Rh(\lambda) + R_3\, h(\lambda) + R_4 n^{-1} = h(\lambda)\, .$$

The sum

$$\sum_{i=0}^{n-1} (\alpha_{i+1} - \alpha_i)\, |x(\alpha_i, \lambda) - y(\alpha_i, \lambda) - x(\alpha_i) + y(\alpha_i)|$$

may be replaced by an integral

$$\int_s^t |x(\tau, \lambda) - y(\tau, \lambda) - x(\tau) + y(\tau)|\, \mathrm{d}\Theta(\tau, \lambda)$$

where $\Theta(\tau, \lambda)$ is a non-decreasing piecewise constant function continuous from the left, its variation being

$$\sum_{i=0}^{n-1} (\alpha_{i+1} - \alpha_i) = t - s\, .$$

Thus, the proof of Lemma is completed.

Return now to the proof of Theorem 7. Making use of Lemma 4, let us evaluate the right-hand side integrals in (39). First of all let us mention that the estimate (41) of Lemma 4 is valid if we write P or P_x instead of U. Using the fact that

$$\int_s^t \mathrm{D}_\sigma P(x(\tau, \lambda), \sigma, \lambda) = \int_s^t \Phi(x(\tau, \lambda), \tau, \lambda)\, \mathrm{d}\tau$$

we get

(44) $$\left| \int_s^t [\Phi(x(\tau, \lambda), \tau, \lambda) - \Phi(y(\tau, \lambda), \tau, \lambda)]\, \mathrm{d}\tau \right| \leqq$$

$$\leqq \|\tilde{x} - \tilde{y}\|\, h(\lambda) + \gamma \int_s^t |x(\tau, \lambda) - y(\tau, \lambda) - x(\tau) + y(\tau)|\, \mathrm{d}\vartheta(\tau, \lambda)$$

and a quite analogous formula for Φ_x.

Let us transform the integrals in (39) so that we can make use of these estimates. There is

$$(45)\qquad \left|\int_s^t [\tilde{x}_2(\lambda)\, \Phi(x(\tau, \lambda), \tau, \lambda) - \tilde{y}_2(\lambda)\, \Phi(y(\tau, \lambda), \tau, \lambda)]\, \mathrm{d}\tau\right| \leqq$$

$$\leqq \left|\tilde{x}_2(\lambda) \int_s^t [\Phi(x(\tau, \lambda), \tau, \lambda) - \Phi(y(\tau, \lambda), \tau, \lambda)]\, \mathrm{d}\tau\right| +$$

$$+ |\tilde{x}_2(\lambda) - \tilde{y}_2(\lambda)| \left|\int_s^t \Phi(y(\tau, \lambda), \tau, \lambda)\, \mathrm{d}\tau\right|,$$

$$(46)\qquad \left|\int_s^t \left\{\Phi_x(x(\tau, \lambda), \tau, \lambda) \int_s^\tau [f(x(\sigma, \lambda), \dot{x}(\sigma, \lambda), \sigma, \lambda) - \dot{x}(\sigma, \lambda)\, \Phi_x(x(\sigma, \lambda), \sigma, \lambda)]\, \mathrm{d}\sigma -\right.\right.$$

$$\left.\left. - \Phi_x(y(\tau, \lambda), \tau, \lambda) \int_s^\tau [f(y(\sigma, \lambda), \dot{y}(\sigma, \lambda), \sigma, \lambda) - \dot{y}(\sigma, \lambda)\, \Phi_x(y(\sigma, \lambda), \sigma, \lambda)]\, \mathrm{d}\sigma\right\} \mathrm{d}\tau\right| =$$

$$= \left|\int_s^t \left\{[f(x(\sigma, \lambda), \dot{x}(\sigma, \lambda), \sigma, \lambda) - \dot{x}(\sigma, \lambda)\, \Phi_x(x(\sigma, \lambda), \sigma, \lambda)] \,.\right.\right.$$

$$. \int_\sigma^t \Phi_x(x(\tau, \lambda), \tau, \lambda)\, \mathrm{d}\tau - [f(y(\sigma, \lambda), \dot{y}(\sigma, \lambda), \sigma, \lambda) - \dot{y}(\sigma, \lambda)\, \Phi_x(y(\sigma, \lambda), \sigma, \lambda)] \,.$$

$$\left.\left. . \int_\sigma^t \Phi_x(y(\tau, \lambda), \tau, \lambda)\, \mathrm{d}\tau\right\} \mathrm{d}\sigma\right| \leqq \left|\int_s^t f(x(\sigma, \lambda), \dot{x}(\sigma, \lambda), \sigma, \lambda) \,.\right.$$

$$\left. . \int_\sigma^t [\Phi_x(x(\tau, \lambda), \tau, \lambda) - \Phi_x(y(\tau, \lambda), \tau, \lambda)]\, \mathrm{d}\tau\, \mathrm{d}\sigma\right| +$$

$$+ \left|\int_s^t [f(x(\sigma, \lambda), \dot{x}(\sigma, \lambda), \sigma\ \lambda) - f(y(\sigma, \lambda), \dot{x}(\sigma, \lambda), \sigma, \lambda)] \,.\right.$$

$$\left. . \int_\sigma^t \Phi_x(y(\tau, \lambda), \tau, \lambda)\, \mathrm{d}\tau\, \mathrm{d}\sigma\right| +$$

$$+ \left|\int_s^t [f(y(\sigma, \lambda), \dot{x}(\sigma, \lambda), \sigma, \lambda) - f(y(\sigma, \lambda), \dot{y}(\sigma, \lambda), \sigma, \lambda)] \,.\right.$$

$$\left. . \int_\sigma^t \Phi_x(y(\tau, \lambda), \tau, \lambda)\, \mathrm{d}\tau\, \mathrm{d}\sigma\right| +$$

$$+ \left|\int_s^t \dot{x}(\sigma, \lambda)\, \Phi_x(x(\sigma, \lambda), \sigma, \lambda) \int_\sigma^t [\Phi_x(x(\tau, \lambda), \tau, \lambda) - \Phi_x(y(\tau, \lambda), \tau, \lambda)]\, \mathrm{d}\tau\, \mathrm{d}\sigma\right| +$$

$$+ \left|\int_s^t \dot{x}(\sigma, \lambda)\, [\Phi_x(x(\sigma, \lambda), \sigma, \lambda) - \Phi_x(y(\sigma, \lambda), \sigma, \lambda)] \int_\sigma^t \Phi_x(y(\tau, \lambda), \tau, \lambda)\, \mathrm{d}\tau\, \mathrm{d}\sigma\right| +$$

$$+ \left|\int_s^t [\dot{x}(\sigma, \lambda) - \dot{y}(\sigma, \lambda)]\, \Phi_x(y(\sigma, \lambda), \sigma, \lambda) \int_\sigma^t \Phi_x(y(\tau, \lambda), \tau, \lambda)\, \mathrm{d}\tau\, \mathrm{d}\sigma\right|.$$

The last but one integral in (39) we can estimate by (41), too:

$$(47)\quad \int_s^t [\Psi(x(\tau, \lambda), \tau, \lambda) - \Psi(y(\tau, \lambda), \tau, \lambda) - H(x(\tau), \tau) + H(y(\tau), \tau)] \, d\tau =$$

$$= \int_s^t D_\sigma \int_s^\sigma [\Psi(x(\tau, \lambda), \xi, \lambda) - \Psi(y(\tau, \lambda), \xi, \lambda) - H(x(\tau, \lambda), \xi) +$$

$$+ H(y(\tau, \lambda), \xi)] \, d\xi +$$

$$+ \int_s^t [H(x(\tau, \lambda), \tau) - H(y(\tau, \lambda), \tau) - H(x(\tau), \tau) + H(y(\tau), \tau)] \, d\tau .$$

The function $W = \int_s^t [\Psi(x, \tau, \lambda) - H(x, \tau)] \, d\tau$ has the property $\mathscr{P}$ (cf. assumptions of Theorem 7) which gives us the required estimate for the first integral. Further,

$$\left| \int_s^t [\Psi(x(\tau, \lambda), \tau, \lambda) - \Psi(y(\tau, \lambda), \tau, \lambda) - \Psi(x(\tau), \tau, \lambda) + \Psi(y(\tau), \tau, \lambda)] \, d\tau \right| \leqq$$

$$\leqq \left| \int_s^t [\Phi(x(\tau, \lambda), \tau, \lambda) - \Phi(x(\tau), \tau, \lambda)] [\Phi_x(x(\tau, \lambda), \tau, \lambda) - \Phi_x(y(\tau, \lambda), \tau, \lambda)] \, d\tau \right| +$$

$$+ \left| \int_s^t \Phi(x(\tau), \tau, \lambda) [\Phi_x(x(\tau, \lambda), \tau, \lambda) - \Phi_x(x(\tau), \tau, \lambda) - \Phi_x(y(\tau, \lambda), \tau, \lambda) + \right.$$

$$\left. + \Phi_x(y(\tau), \tau, \lambda)] \, d\tau \right| +$$

$$+ \left| \int_s^t [\Phi_x(y(\tau, \lambda), \tau, \lambda) - \Phi_x(y(\tau), \tau, \lambda)] [\Phi(x(\tau, \lambda), \tau, \lambda) - \Phi(y(\tau, \lambda), \tau, \lambda)] \, d\tau \right| +$$

$$+ \left| \int_t^s \Phi_x(y(\tau), \tau, \lambda [\Phi(x(\tau, \lambda), \tau, \lambda) - \Phi(x(\tau), \tau, \lambda) - \Phi(y(\tau, \lambda), \tau, \lambda) + \right.$$

$$\left. + \Phi(y(\tau), \tau, \lambda)] \, d\tau \right| \leqq$$

$$\leqq R_5 \|\tilde{x} - \tilde{y}\| \, h(\lambda) + R_6 \int_s^t |x(\tau, \lambda) - y(\tau, \lambda) - x(\tau) + y(\tau)| \, d\tau .$$

In fact, $\Phi = \partial P / \partial t$, $\Phi_x = \partial P_x / \partial t$ so that from iii), iv) from the definition of $\mathscr{P}$

$$|\Phi(x, t, \lambda) - \Phi(y, t, \lambda)| \leqq K|x - y|$$

$$|\Phi(x, t, \lambda) - \Phi(y, t, \lambda) - \Phi(x + u, t, \lambda) + \Phi(y + u, t, \lambda)| \leqq K|x - y| \, |u|$$

and

$$|\Phi(x(t, \lambda), t, \lambda) - \Phi(y(t, \lambda), t, \lambda) - \Phi(x(t), t, \lambda) + \Phi(y(t), t, \lambda)| \leqq$$

$$\leqq |\Phi(x(t, \lambda), t, \lambda) - \Phi(y(t, \lambda), t, \lambda) - \Phi(x(t), t, \lambda) + \Phi(x(t) + y(t, \lambda) - x(t, \lambda), t, \lambda)| +$$

$$+ |\Phi(y(t), t, \lambda) - \Phi(x(t) + y(t, \lambda) - x(t, \lambda), t, \lambda)| .$$

An analogous result holds for Φ_x. By a limiting process so as in the proof of Theorem 6 we get the required estimate for the second right-hand side integral in (47).

The first integral on the right-hand side of (39) may be evaluated by $i_1)-iv_3)$:*)

$$\left|\int_s^t [f(x_\lambda, \dot{x}_\lambda) - f(y_\lambda, \dot{y}_\lambda) - Q(x, \dot{x}) + Q(y, \dot{y})]\,d\tau\right| =$$

$$= \left|\int_s^t [f(x_\lambda, \dot{X}_\lambda + \Phi(x_\lambda)) - f(y_\lambda, \dot{Y}_\lambda + \Phi(y_\lambda)) - Q(x, \dot{x}) + Q(y, \dot{y})]\,d\tau\right| \leqq$$

$$\leqq \left|\int_s^t [f(x_\lambda, \dot{X}_\lambda + \Phi(x_\lambda)) - f(x, \dot{X}_\lambda + \Phi(x_\lambda)) - f(y + x_\lambda - x, \dot{X}_\lambda + \Phi(x_\lambda)) +\right.$$

$$\left. + f(y, \dot{X}_\lambda + \Phi(x_\lambda))]\,d\tau\right| +$$

$$+ \left|\int_s^t [f(y + x_\lambda - x, \dot{X}_\lambda + \Phi(x_\lambda)) - f(y_\lambda, \dot{X}_\lambda + \Phi(x_\lambda))]\,d\tau\right| +$$

$$+ \left|\int_s^t [f(y_\lambda, \dot{X}_\lambda + \Phi(x_\lambda)) - f(y, \dot{X}_\lambda + \Phi(x_\lambda)) - f(y_\lambda, \dot{Y}_\lambda + \Phi(y_\lambda)) +\right.$$

$$\left. + f(y, \dot{Y}_\lambda + \Phi(y_\lambda))]\,d\tau\right| +$$

$$+ \left|\int_s^t [f(x, \dot{X}_\lambda + \Phi(x_\lambda)) - f(x, \dot{x} + \Phi(x_\lambda)) - f(y, \dot{X}_\lambda + \Phi(x_\lambda)) +\right.$$

$$\left. + f(y, \dot{x} + \Phi(x_\lambda))]\,d\tau\right| +$$

$$+ \left|\int_s^t [f(y, \dot{X}_\lambda + \Phi(x_\lambda)) - f(y, \dot{Y}_\lambda + \Phi(y_\lambda)) - f(y, \dot{x} + \Phi(x_\lambda)) +\right.$$

$$\left. + f(y, \dot{x} + \dot{Y}_\lambda - \dot{X}_\lambda + \Phi(y_\lambda))]\,d\tau\right| +$$

$$+ \left|\int_s^t [f(y, \dot{y} + \Phi(y_\lambda)) - f(y, \dot{x} + \dot{Y}_\lambda - \dot{X}_\lambda + \Phi(y_\lambda))]\,d\tau\right| +$$

$$+ \left|\int_s^t [f(x, \dot{x} + \Phi(x_\lambda)) - f(x, \dot{x} + \Phi(x)) - f(x, \dot{y} + \Phi(y_\lambda)) +\right.$$

$$\left. + f(x, \dot{y} + \Phi(x) + \Phi(y_\lambda) - \Phi(x_\lambda))]\,d\tau\right| +$$

$$+ \left|\int_s^t [f(x, \dot{y} + \Phi(y)) - f(x, \dot{y} + \Phi(x) + \Phi(y_\lambda) - \Phi(x_\lambda))]\,d\tau\right| +$$

*) For the sake of brevity let us omit for this moment the variables τ, λ in the functions f, Φ, Q and denote $x(\tau, \lambda) = x_\lambda$, $y(\tau, \lambda) = y_\lambda$, $x(\tau) = x$, $y(\tau) = y$ and analogously $\dot{x}_\lambda, \dot{y}_\lambda, \dot{x}, \dot{y}, \dot{X}_\lambda, \dot{Y}_\lambda$.

$$+\left|\int_s^t [f(x, \dot{y} + \Phi(y_\lambda)) - f(x, \dot{y} + \Phi(y)) - f(y, \dot{y} + \Phi(y_\lambda)) + \right.$$

$$\left. + f(y, \dot{y} + \Phi(y))] \, d\tau \right| +$$

$$+ \left| \int_s^t [f(x, \dot{x} + \Phi(x)) - f(y, \dot{y} + \Phi(y)) - Q(x, \dot{x}) + Q(y, \dot{y})] \, d\tau \right| \leqq$$

$$\leqq \int_s^t K\{|x_\lambda - x| \, |x - y| + |x_\lambda - y_\lambda - x + y| +$$
$$+ |y_\lambda - y| \, [|\dot{X}_\lambda - \dot{Y}_\lambda| + |\Phi(x_\lambda) - \Phi(y_\lambda)|] + |x - y| \, |\dot{X}_\lambda - \dot{x}| +$$
$$+ |\dot{X}_\lambda - \dot{x}| \, [|\dot{X}_\lambda - \dot{Y}_\lambda| + |\Phi(x_\lambda) - \Phi(y_\lambda)|] + |\dot{X}_\lambda - \dot{Y}_\lambda - \dot{x} + \dot{y}| +$$
$$+ |\Phi(x_\lambda) - \Phi(x)| \, [|\dot{x} - \dot{y}| + |\Phi(x_\lambda) - \Phi(y_\lambda)|] + |\Phi(x_\lambda) - \Phi(y_\lambda) -$$
$$- \Phi(x) + \Phi(y)| + |x - y| \, |\Phi(y_\lambda) - \Phi(y)|\} \, d\tau + J .$$

Now we make use of the fact that Φ and Φ_x fulfil the Lipschitz condition in x (with a constant independent of λ), of the inequalities

$$|x(t, \lambda) - y(t, \lambda)| \leqq |x(t, \lambda) - y(t, \lambda) - x(t) + y(t)| + |x(t) - y(t)| ,$$
$$|\dot{X}(t, \lambda) - \dot{Y}(t, \lambda)| \leqq |\dot{X}(t, \lambda) - \dot{Y}(t, \lambda) - \dot{x}(t) + \dot{y}(t)| + |\dot{x}(t) - \dot{y}(t)| ,$$

of the assumption (37) and, finally, of the inequalities

$$|\Phi(x(t, \lambda), t, \lambda) - \Phi(y(t, \lambda), t, \lambda) - \Phi(x(t), t, \lambda) + \Phi(y(t), t, \lambda)| \leqq$$
$$\leqq K|x(t, \lambda) - y(t, \lambda) - x(t) + y(t)| ,$$

$$|\Phi_x(x(t, \lambda), t, \lambda) - \Phi_x(y(t, \lambda), t, \lambda) - \Phi_x(x(t), t, \lambda) + \Phi_x(y(t), t, \lambda)| \leqq$$
$$\leqq K|x(t, \lambda) - y(t, \lambda) - x(t) + y(t)|$$

which are a consequence of the fact that P, P_x have the property $\mathscr{P}$. In this way, we get the final estimate for the first integral in (39)

$$\left| \int_s^t [f(x(\tau, \lambda), \dot{x}(\tau, \lambda), \tau, \lambda) - f(y(\tau, \lambda), \dot{y}(\tau, \lambda), \tau, \lambda) - Q(x(\tau), \dot{x}(\tau), \tau) + \right.$$

$$\left. + Q(y(\tau), \dot{y}(\tau), \tau)] \, d\tau \right| \leqq \|\tilde{x} - \tilde{y}\| \, h(\lambda) + C_1 \int_s^t |x(\tau, \lambda) - y(\tau, \lambda) - x(\tau) + y(\tau)| \, d\tau +$$

$$+ C_2 \int_s^t |\dot{X}(\tau, \lambda) - \dot{Y}(\tau, \lambda) - \dot{x}(\tau) + \dot{y}(\tau)| \, d\tau ,$$

for the integral denoted by J can be majorized by the same expression if we use an analogous method as in the proof of Theorem 6 (cf. (35) and below).

Using the facts that f, Φ, Φ_x are bounded and Lipschitzian in x, the boundedness of the first derivative $\dot{x}(t, \lambda)$ of the solution and finally Lemma 2, we get by substituting our results into (39) and (40) and after elementary transformations

$$|\dot{X}(t, \lambda) - \dot{Y}(t, \lambda) - \dot{x}(t) + \dot{y}(t)| \leqq \|\tilde{x} - \tilde{y}\| \, h(\lambda) + \\ + \gamma_1 \int_s^t |x(\tau, \lambda) - y(\tau, \lambda) - x(\tau) + y(\tau)| \, d\vartheta(\tau, \lambda) + \\ + \gamma_2 \int_s^t |\dot{X}(\tau, \lambda) - \dot{Y}(\tau, \lambda) - \dot{x}(\tau) + \dot{y}(\tau)| \, d\tau ,$$

$$|x(t, \lambda) - y(t, \lambda) - x(t) + y(t)| \leqq \|\tilde{x} - \tilde{y}\| \, h(\lambda) + \\ + \gamma \int_s^t |x(\tau, \lambda) - y(\tau, \lambda) - x(\tau) + y(\tau)| \, d\vartheta(\tau, \lambda) + \\ + \int_s^t |\dot{X}(\tau, \lambda) - \dot{Y}(\tau, \lambda) - \dot{x}(\tau) + \dot{y}(\tau)| \, d\tau .$$

Denote

$$q_1(t, \lambda) = |x(t, \lambda) - y(t, \lambda) - x(t) + y(t)| ,$$
$$q_2(t, \lambda) = |\dot{X}(t, \lambda) - Y(t, \lambda) - \dot{x}(t) + \dot{y}(t)| ,$$
$$q(t, \lambda) = q_1(t, \lambda) + q_2(t, \lambda) ;$$

then

$$q_1(t, \lambda) \leqq \|\tilde{x} - \tilde{y}\| \, h(\lambda) + \gamma \int_s^t q_1(\tau, \lambda) \, d\vartheta(\tau, \lambda) + \int_s^t q_2(\tau, \lambda) \, d\tau ,$$

$$q_2(t, \lambda) \leqq \|\tilde{x} - \tilde{y}\| \, h(\lambda) + \gamma_1 \int_s^t q_1(\tau, \lambda) \, d\vartheta(\tau, \lambda) + \gamma_2 \int_s^t q_2(\tau, \lambda) \, d\tau ,$$

$$q(t, \lambda) \leqq \|\tilde{x} - \tilde{y}\| \, h(\lambda) + (\gamma + \gamma_1) \int_s^t q_1(\tau, \lambda) \, d\vartheta(\tau, \lambda) + (1 + \gamma_2) \int_s^t q_2(\tau, \lambda) \, d\tau .$$

As $q_1 \geqq 0$, $q_2 \geqq 0$, there is

$$q(t, \lambda) \leqq \|\tilde{x} - \tilde{y}\| \, h(\lambda) + \int_s^t q(\tau, \lambda) \, d\vartheta^*(\tau, \lambda)$$

where

$$\vartheta^*(\tau, \lambda) = (\gamma + \gamma_1) \, \vartheta(\tau, \lambda) + (1 + \gamma_2) \, \tau .$$

The assertion of Theorem 7, i.e. the inequalities (38) are a consequence of the following lemma:

Lemma 5. *Let $u(t)$ be a bounded non-negative function in $\langle s, T)$, $c_1 \geqq 0$, $c_2 > 0$. Let*

$$u(t) \leqq c_1 + c_2 \int_s^t u(\tau) \, d\vartheta(\tau)$$

for all $t \in \langle s, T \rangle$, $\vartheta(\tau)$ *being a non-decreasing and continuous from the left. Then*

$$u(t) \leqq c_1 c$$

for all $t \in \langle s, T \rangle$. *The constant* c *dependes only on* c_2 *and on the variation* v *of* ϑ *in the interval* $\langle s, T \rangle$:

$$c = 1 + c_2 v e^v .$$

Proof is quite analogous to that of Lemma 1. The respective estimates follow from Lemma 3.4 [4, p. 371].

From this Lemma we get immediately

$$q(t, \lambda) \leqq \|\tilde{x} - \tilde{y}\| \, h(\lambda) ;$$

as $q_1 \geqq 0$, $q_2 \geqq 0$, analogous inequalities hold for $q_i(t, \lambda)$ $(i = 1, 2)$ too. According to our notation, these are the inequalities (38). The proof of Theorem 7 is completed.

5. In this section we prove a theorem on the existence of a stable periodic solution of equation $(\mathfrak{E}')$ with a periodic right-hand side.

Lemma 6. *Let for the right-hand side of a differential equation*

$$\ddot{x} + a\dot{x} + bx = \psi(x, \dot{x}, t) \tag{48}$$

the following relation hold:

$$\lim_{\|u\| + \|v\| \to 0} \frac{\psi(u_1, u_2, t) - \psi(v_1, v_2, t)}{\|u - v\|} = 0 \tag{49}$$

where again $\|u\| = |u_1| + |u_2|$. *The linear equation*

$$\ddot{x} + a\dot{x} + bx = 0$$

let have characteristic roots with negative real parts only.

Then for any two solutions of (48) $x(t)$, $y(t)$ *with the initial conditions* $x(s) = \tilde{x}_1$, $\dot{x}(s) = \tilde{x}_2$; $y(s) = \tilde{y}_1$, $\dot{y}(s) = \tilde{y}_2$, *respectively, which are sufficiently near to zero,*

$$|x(t) - y(t)| + |\dot{x}(t) - \dot{y}(t)| \leqq K\|\tilde{x} - \tilde{y}\| \, e^{-\eta t}$$

holds, K, η *being positive constants.*

Proof of this lemma can be performed by usual methods of the theory of differential equations (cf. e.g. [2], Chap. XIII, Theorem 1.1).

Theorem 8. *Let an equation* $(\mathfrak{E}')$ *fulfil the assumptions of Theorem 7. Its limit equation* (32) *let fulfil the assumptions of Lemma 6 if we put*

$$Q(x, y, t) - H(x, t) = -ay - bx + \psi(x, y, t)$$

in a convenient way, $\psi(0, 0, t) = 0$.

Let the right-hand side of $(\mathfrak{E}')$ *be* $2\pi\lambda$*-periodic in* t.

Then there exist $\lambda_1 > 0$, $\varepsilon > 0$ *such that for* $\lambda \in (0, \lambda_1)$ $(\mathfrak{E}')$ *has one and only one stable periodic solution with the initial conditions bounded in absolute value by* ε, *the period of which is* $2\pi\lambda$.

Proof. There exists $\Delta > 0$ such that the assertion of Lemma 6 is valid for equation (32) if only $\|\tilde{x}\| < \Delta$, $\|\tilde{y}\| < \Delta$ holds for the initial conditions of $x(t)$, $y(t)$. The equation $(\mathfrak{E}')$ fulfils the assumptions of Theorem 7. Hence

$$|x(t, \lambda) - y(t, \lambda)| \leqq |x(t) - y(t)| + \|\tilde{x} - \tilde{y}\| \, h(\lambda) \leqq$$
$$\leqq \|\tilde{x} - \tilde{y}\| \left[Ke^{-\eta t} + h(\lambda)\right].$$

As $h(\lambda) \to 0$ when $\lambda \to 0_+$, $e^{-\eta t} \to 0$ when $t \to \infty$, a positive integer n and $\lambda_1 > 0$ exist such that for $0 < \lambda < \lambda_1$

$$|x(s + 2n\pi\lambda, \lambda| - y(s + 2n\pi\lambda, \lambda)| \leqq \min\left(\frac{1}{4}, \frac{1}{4M}\right) \|\tilde{x} - \tilde{y}\| .$$

In the same way an analogous inequality for the first derivative of the solution of $(\mathfrak{E}')$ (diminished by Φ) can be proved:

$$|\dot{X}(s + 2n\pi\lambda, \lambda) - \dot{Y}(s + 2n\pi\lambda, \lambda)| \leqq \tfrac{1}{4}\|\tilde{x} - \tilde{y}\| .$$

As $\dot{X}(t, \lambda) = \dot{x}(t, \lambda) - \Phi(x(t, \lambda), t, \lambda)$, $\dot{Y}(t, \lambda) = \dot{y}(t, \lambda) - \Phi(y(t, \lambda), t, \lambda)$, respectively, and Φ fulfils the Lipschitz condition in x,

$$|\dot{x}(s + 2n\pi\lambda, \lambda) - \dot{y}(s + 2n\pi\lambda, \lambda)| \leqq$$
$$\leqq \tfrac{1}{4}\|\tilde{x} - \tilde{y}\| + M|x(s + 2n\pi\lambda, \lambda) - y(s + 2n\pi\lambda, \lambda)| \leqq \tfrac{1}{2}\|\tilde{x} - \tilde{y}\| .$$

Consider a transformation in which to every point $\tilde{x}_1, \tilde{x}_2 + \Phi(\tilde{x}_1, s, \lambda)$, $\|\tilde{x}\| < \Delta$ corresponds the point $x(s + 2n\pi\lambda, \lambda)$, $\dot{x}(s + 2n\pi\lambda, \lambda)$ where $x(t, \lambda)$ is the solution of $(\mathfrak{E}')$ with the initial conditions $x(s, \lambda) = \tilde{x}_1$, $\dot{x}(s, \lambda) = \tilde{x}_2 + \Phi(\tilde{x}_1, s, \lambda)$. According to the fixed-point theorem there exists a solution for which $x(s, \lambda) = x(s + 2n\pi\lambda, \lambda)$, $\dot{x}(s, \lambda) = \dot{x}(s + 2n\pi\lambda, \lambda)$; this solution is unique. According to the assumption of the periodicity of the right-hand side of $(\mathfrak{E}')$, this solution is obviously periodic with the period $2n\pi\lambda$.

We can make sure easily that this period is $2\pi\lambda$, i.e. $n = 1$.

In fact, let us suppose that $n \neq 1$. Let $\mathfrak{B}$ be a transformation which transforms each point $z = (z_1, z_2)$, $\|z\| < \Delta$ to the point $(x_z(s + 2\pi\lambda, \lambda), \dot{x}(s + 2\pi\lambda, \lambda))$ where x_z is the solution of $(\mathfrak{E}')$ with the initial conditions $x_z(s, \lambda) = z_1$, $\dot{x}_z(s, \lambda) = z_2$. The intial conditions of the periodic solution (the existence of which we have just proved) let be $p = (p_1, p_2)$ and assume that $\mathfrak{B}(p) = p^* \neq p$ (which is equivalent to $n \neq 1$). However, then $\mathfrak{B}^n(p^*) \neq p^*$ because of the unicity of the periodic solution.

The period of the right-hand side of $(\mathfrak{E}')$ being $2\pi\lambda$, there is

$$\mathfrak{V}^n(p^*) = \mathfrak{V}^{n+1}(p),$$
$$\mathfrak{V}^{n+1}(p) = \mathfrak{V}(p) = p^*,$$

a contradiction. Consequently, the period must be $2\pi\lambda$.

As an example we may consider the equation

$$\ddot{\Theta} + a\dot{\Theta} = gL^{-1}\sin\Theta - AL^{-1}\omega^2\sin\omega t\sin(\Theta - \alpha), \tag{50}$$

$a > 0$, when $\omega \to \infty$, $A\omega = \text{const.}$*) The limit equation has the form

$$\ddot{\Theta} + a\dot{\Theta} = gL^{-1}\sin\Theta - \left(\frac{A\omega}{2L}\right)^2\sin 2(\Theta - \alpha). \tag{51}$$

At first, let $\alpha = 0$. If $2gL < A^2\omega^2$, then (51) has four equilibrium points, two of them being asymptotically stable ($\Theta = 0$ and $\Theta = \pi$) and the others unstable ($\cos\Theta = 2gLA^{-2}\omega^{-2}$). The stable equilibrium points are preserved even in the equation (50).

The roots Θ of equation

$$gL^{-1}\sin\Theta - \left(\frac{A\omega}{2L}\right)^2\sin 2(\Theta - \alpha) = 0$$

depending continuously on its coefficients and thus on α, too, equation (51) has for $\alpha \neq 0$ four equilibrium points again which differ little from those of (51) with $\alpha = 0$. Their stability (instability, respectively) is preserved, too. Hence, equation (50) has for large ω two periodic solutions with period $2\pi\omega^{-1}$ in the vicinity of the equilibrium points of (51).

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*) If $\alpha = 0$ then this is the equation of Kapica's pendulum in which friction (of the "slow" motion) is considered.

Резюме

О ЗАВИСИМОСТИ ОТ ПАРАМЕТРА РЕШЕНИЙ ОДНОГО КЛАССА ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ВТОРОГО ПОРЯДКА

ИРЖИ ЯРНИК (Jiří Jarník), Прага

Настоящая работа связана с результатами П. Л. Капицы и С. Лояшевича [7], которые исследовали дифференциальные уравнения, аналогичные уравнению движения математического маятника, точка подвеса которого колеблется с большой частотой и малой амплитудой. Существенно исползуются методы теории обобщенных дифференциальных уравнений, разработанной Я. Курцвейлем [3]–[6].

1. В работе исследуется дифференциальное уравнение второго порядка (1), где f, φ определены для $x \in G$ (G – открытое подмножество E_n), $t \in \langle s, T\rangle$, $0 < \lambda \leqq \lambda_0$ и непрерывны по (x, t) в $G \times \langle s, T\rangle$. Функция f удовлетворяет (2) в своей области определения (K_1 – постоянная).

Функция φ выполняет следующие предположения:

i) Существуют функции Φ, P, Φ_x, P_x от (x, t, λ), определенные в $G \times \langle s, T\rangle \times (0, \lambda_0)$ и непрерывные в $G \times \langle s, T\rangle$ так, что выполнено (3) (индекс x обозначает дифференцирование по x).

ii) Выполняется (4) во всей области определения Φ, Φ_x.

iii) Выполняется (5) равномерно в $G \times \langle s, T\rangle$.

iv) Существует $\eta_0 > 0$ и непрерывная неубывающая функция $\omega(\eta)$ в $\langle 0, \eta_0\rangle$, $\omega(0) = 0$ так, что для $|t_2 - t_1| \leqq \eta_0$, $|x_2 - x_1| \leqq \eta_0$ и для всех $\lambda \in (0, \lambda_0\rangle$ выполняется (6).

Уравнение (1), правая часть которого выполняет приведенные условия, будем обозначать ($\mathscr{E}$).

На основании соотношения (13) доказываются следующие теоремы:

Теорема 1. *Пусть $x(t, \lambda)$ – решение уравнения ($\mathscr{E}$) на интервале $\langle s, T\rangle$ с начальными значениями $x(s, \lambda) = \tilde{x}_1(\lambda)$, $\dot{x}(s, \lambda) = \tilde{x}_2(\lambda)$, для которых выполняется (7) (K_2 – постоянная). Тогда выполняется (8).*

Теорема 2. *Пусть $\tilde{x}_1(\lambda), \tilde{x}_1 \in G$, $\tilde{x}_1(\lambda) \to \tilde{x}_1$, $\tilde{x}_2(\lambda) - \Phi(\tilde{x}_1(\lambda), s, \lambda) \to \tilde{x}_2$ если $\lambda \to 0_+$. Пусть существуют функции $H(x, t), f(x, t)$, определенные и непрерывные в $G \times \langle s, T\rangle$ так, что (9) выполнено равномерно для $x \in G$, $t \in \langle s\, T\rangle$. Пусть функции f, Φ_x выполняют условие Липшица по x с постоянной M (независимой от λ). Пусть уравнение (20) с начальными значениями $x(s) = \tilde{x}_1$, $\dot{x}(s) = \tilde{x}_2$ имеет однозначное решение $x(t)$, определенное в $\langle s, T\rangle$ и принадлежащее области G.*

Тогда, для $\lambda > 0$ достаточно малых, существует решение $x(t, \lambda)$ уравнения $(\mathscr{E})$ в $\langle s, T\rangle$ с начальными условиями $\tilde{x}_1(\lambda)$, $\tilde{x}_2(\lambda)$, и соотношение (21) выполняется равномерно в $\langle s, T\rangle$.

2. Для простоты предположим, что f и начальные условия не зависят от λ т.е. $f(x, t, \lambda) = f(x, t)$, $\tilde{x}_1(\lambda) = \tilde{x}_1$, $\tilde{x}_2(\lambda) = \tilde{x}_2 + \Phi(\tilde{x}_1, s, \lambda)$. Далее, пусть $\omega(\eta) = K_3(\eta)$ и пусть (28) выполняется для всех $x \in G$, $t \in \langle s, T\rangle$ и $\lambda \in (0, \lambda_0\rangle$ (B — постоянная).

Тогда, если выполняются предположения Теоремы 2, имеет место оценка

$$|x(t, \lambda) - x(t)| \leqq \text{const.}\ \lambda^{\frac{1}{2}} \exp\left[\sqrt{(2M)}\,(t - s)\right]$$

для всех $\lambda > 0$ достаточно малых и $t \in \langle s, T\rangle$. Показатель $\frac{1}{2}$ нельзя улучшить, как показывает пример уравнения $\ddot{x} = \lambda^{\frac{1}{2}} \cos\left[\lambda^{-\frac{1}{2}}(t - x)\right]$ с начальными условиями $\tilde{x}_1(\lambda) = \tilde{x}_1 = 0$, $\tilde{x}_2(\lambda) = \tilde{x}_2 = 1$.

Если ввести несколько более строгие предположения, касающиеся гладкости функций P, Ψ, H по x, то можно доказать аналогичную оценку, в которой λ представится линейно.

3. Далее будем исследовать уравнение (1′), правая часть которого подчинена аналогичным условиям, как в статье 1; функция f зависит от первой производной решения. Доказывается теорема, аналогичная Теореме 1 и

Теорема 6. *Пусть $\tilde{x}_1 \in G$, $\tilde{x}_1(\lambda) \to \tilde{x}_1$, $\tilde{x}_2(\lambda) - \Phi(\tilde{x}_1(\lambda), s, \lambda) \to \tilde{x}_2$, если $\lambda \to 0_+$, $f(x, y, t, \lambda)$ определена для $y \in G'$, $G' \supset \mathrm{E}[y;\ |y| \leqq |\tilde{x}_2| + K_1]$. Пусть существуют функции $H(x, t)$, $Q(x, y, t)$, определенные в $G \times \langle s, T\rangle$ или $G \times G' \times \langle s, T\rangle$,*

$$\lim_{\lambda \to 0+} \int_{t_1}^{t_2} \Phi_x(x, \tau, \lambda)\, \Phi(x, \tau, \lambda)\, \mathrm{d}\tau = \int_{t_1}^{t_2} H(x, \tau)\, \mathrm{d}\tau\,,$$

$$\lim_{\lambda \to 0+} \int_{t_1}^{t_2} f(x, y + \Phi(x, \tau, \lambda), t, \lambda)\, \mathrm{d}\tau = (t_2 - t_1)\, Q(x, y, t)$$

равномерно в области определения.

Пусть функции f, Φ_x выполяют условие Липшица по x (f тоже по y, t) с постоянной, независящей от λ.

Пусть уравнение (32) имеет однозначное решение с начальными условиями $\tilde{x}_1$, $\tilde{x}_2$, определенное на интервале $\langle s, T\rangle$.

Тогда для всех достаточно малых $\lambda > 0$ существует решение $x(t, \lambda)$ уравнения $(\mathscr{E}')$ с начальными условиями $\tilde{x}_1(\lambda)$, $\tilde{x}_2(\lambda)$, и (33) имеет место равномерно в $\langle s, T\rangle$.

4. Будем говорить, что функция U обладает свойством $\mathscr{P}$, если существует постоянная K и функция $h(\lambda)$, $\lim\limits_{\lambda \to 0+} h(\lambda) = 0$ так, что неравенства i)—iv) выполняются для всех $t, t_1, t_2, x, y, \lambda$ из области определения U и для $|t_2 - t_1|$ $|x - y|$, $|u|$ достаточно малых.

Теорема 7. *Пусть уравнение* (7) *удовлетворяет предположениям Теоремы* 6. *Пусть* Φ_x *выполяет условие Липшица по* x *с постоянной, не зависящей от* λ.

Пусть функции $P, P_x, W = \int_s^t [\Psi - H] \, d\tau$ *имеют свойство* $\mathscr{P}$; *для функции* $f(x, u, t, \lambda)$ *пусть имеют место неравенства* i)–iv$_3$) (*стр.* 126–127) (K – *постоянная*, $\lim_{\lambda \to 0} h(\lambda) = 0$).

Далее, пусть для каждых двух решений $x(t), y(t)$ *предельного уравнения* (32) *с начальными значениями* $\tilde{x}_1, \tilde{x}_2$; $\tilde{y}_1, \tilde{y}_2$ *имеет место* (37) *для всех* $t \in \langle s, T \rangle$ (R – *постоянная*, $\|u\|$ – *норма вектора* (u_1, u_2), *например*, $\|u\| = |u_1| + |u_2|$).

Пусть $x(t), y(t)$ – *решения* (32) *в* $\langle s, T \rangle$ *с начальными значениями* $\tilde{x}_1, \tilde{x}_2$; $\tilde{y}_1, \tilde{y}_2$. *Пусть* $x(t, \lambda), y(t, \lambda)$ – *решения* (1') *в* $\langle s, T \rangle$, *принадлежащие* G *и с начальными значениями* $\tilde{x}_1, \tilde{x}_2 + \Phi(\tilde{x}_1, s, \lambda)$; $\tilde{y}_1, \tilde{y}_2 + \Phi(\tilde{y}_1, s, \lambda)$.

Тогда для всех $t \in \langle s, T \rangle$ *имеет место неравенство* (38), *где* $\dot{X}(t, \lambda) = \dot{x}(t, \lambda) - \Phi(x(t, \lambda), t, \lambda)$, $\dot{Y}(t, \lambda) = \dot{y}(t, \lambda) - \Phi(y(t, \lambda), t, \lambda)$, $\lim_{\lambda \to 0+} h(\lambda) = 0$.

Доказательство теоремы 7 основано на соотношении (41), где U – функция, обладающая свойством $\mathscr{P}$, и $\vartheta(t, \lambda) = t + \Theta(t, \lambda)$, $\Theta(t, \lambda)$ – неубывающая по частям, постоянная, слева непрерывная функция с вариацией $t - s$; γ – постоянная и $\lim_{\lambda \to 0+} h(\lambda) = 0$.

5. Теорема 8. *Пусть уравнение* ($\mathscr{E}'$) *удовлетворяет предположениям Теоремы* 7. *Для его предельного уравнения* (32) *пусть имеет место соотношение* (49), *если написать его удобным способом в форме* (48); *все характеристические корни линейного уравнения* $\ddot{x} + a\dot{x} + bx = 0$ *пусть имеют отрицательные действительные части.*

Пусть правая часть уравнения (E') $2\pi\lambda$ – *периодическая по* t.

Тогда существует $\varepsilon > 0$ *такое, что* (E') *имеет одно и только одно устойчивое периодическое решение с периодом* $2\pi\lambda$ *и с начальными значениями по абсолютной величине* $< \varepsilon$.