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ON THE EXTENSION OF TRANSI-GROUPS

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The author generalizes the theory of extensions of groups to the transi-groups introduced by the author in the paper [5].

In the theory of transi-groups one can construct the quotient-transi-group of a given transi-group with respect to a normal subtransi-group of the latter. As in the theory of groups one can raise the problem of finding transi-groups which are in a certain sense, extensions of two given transi-groups. The present paper treats and solves a slightly modificated problem.

Coming now to a more precise formulation we remark that, when giving a transi-group $(M, \mathfrak{A}, e)$ and an equivalence relation $\sim$ in M, imprimitive with respect to $\mathfrak{A}$, i.e. a normal subtransi-group $(K, \mathfrak{B}, e, \sim)$, the group $\mathfrak{B}$ is not an (effective) transformation group of K but it is homorphically mapped onto such a group $\mathfrak{B}^*$, obtaining in this manner the transi-group $(K, \mathfrak{B}^*, e)$ associated with $(K, \mathfrak{B}, e, \sim)$. The group $\mathfrak{A}$ is an extension of $\mathfrak{B}$ by $\mathfrak{A}/\mathfrak{B}$ (which acts as a transformation group of $M/\sim$). If the quotient-transi-group $(M/\sim, \mathfrak{A}/\mathfrak{B}, [e])$ (where $[e]$ is the equivalence class of e and coincides as a subset of M with K) is transitive, all the equivalence classes of M with respect to $\sim$ have the same cardinal number, so that M can be identified with the cartesian product $K \times (M/\sim)$.

The present paper deals with the problem of *existence* (and not of finding all solutions) of a transi-group $(M, \mathfrak{A}, e)$ when the group $\mathfrak{A}$ is given as an extension of $\mathfrak{B}$ by $\mathfrak{A}/\mathfrak{B}$. To be more precise, a necessary and sufficient condition is given in order that the given extension $\mathfrak{A}$ act as an (effective) transformation group in the cartesian product $K \times (M/\sim)$.

For the questions of the theory of equivalence relations, the reader may refer to [2] and [1], for those of the theory of groups to [3] and for those of the theory of transi-groups, to the papers [4] and [5] where the definitions of technical terms employed are also given.

1. Let K be a set and let $\mathfrak{B}$ be a group for which there exists a group homomorphism ψ from $\mathfrak{B}$ onto an (effective) transformation group $\mathfrak{B}^*$ of K. Let $(M^*, \mathfrak{A}^*, e^*)$

be a transitive transi-group; let $\mathfrak{A}_0^*$ be the subgroup of all α, $\alpha \in \mathfrak{A}^*$, for which $\alpha(e^*) = e^*$ and let α_λ, $\lambda \in \Lambda$, be a system of representatives of the left residue classes of $\mathfrak{A}^*$ with respect to $\mathfrak{A}_0^*$ (here the representative of $\mathfrak{A}_0^*$ is meant to be ε, the unit element of $\mathfrak{A}^*$).

For any $x^* \in M^*$, there exists a unique λ, $\lambda \in \Lambda$, for which $\alpha_\lambda(e^*) = x^*$.

Let $\mathfrak{A}$ be a given extension of $\mathfrak{B}$ by $\mathfrak{A}^*$; let τ_α, $\alpha \in \mathfrak{A}^*$, be a system of representatives of the extension, where $\tau_\varepsilon = 1$ (here 1 denotes the unit element of $\mathfrak{A}$); let $\mu_{\alpha,\beta}$, α, $\beta \in \mathfrak{A}^*$, be the factor system associated with the τ_α. Let us denote by $\mathfrak{A}_0$ the subgroup of all elements of $\mathfrak{A}$, whose residue classes are in $\mathfrak{A}_0^*$; we have $\mathfrak{A}_0 \supset \mathfrak{B}$. We introduce the notations

$$\sigma^{(\alpha)} = \tau_\alpha \sigma \tau_\alpha^{-1}, \quad \sigma^\alpha = \tau_\alpha^{-1} \sigma \tau_\alpha, \quad \alpha \in \mathfrak{A}^*, \quad \sigma \in \mathfrak{B}.$$

The elements of $\mathfrak{A}$ may be uniquely represented in the form $\sigma\tau_\alpha$, $\sigma \in \mathfrak{B}$, $\alpha \in \mathfrak{A}^*$ and the product is made according to

$$(\sigma_1 \tau_\alpha)(\sigma_2 \tau_\beta) = \sigma_1 \sigma_2^{(\alpha)} \mu_{\alpha,\beta} \tau_{\alpha\beta}, \quad \alpha, \beta \in \mathfrak{A}^*, \quad \sigma_1, \sigma_2 \in \mathfrak{B}.$$

Let us denote the cartesian product $K \times M^*$ by M.

Finally we make the following hypothesis:

Hypothesis ($\mathscr{S}$): *If* σ, $\sigma \in \mathfrak{B}$, *has the property* $\psi(\sigma^{\alpha_\lambda}) = \psi(1)$ *for all* $\lambda \in \Lambda$, *then* $\sigma = 1$.

The introduction of this hypothesis will be justified at point 2.

2. To justify the introduction of hypothesis ($\mathscr{S}$), we make the following considerations:

Suppose that $(M, \mathfrak{A}, e)$ is a transi-group and that $(K, \mathfrak{B}, e, \sim)$ is a normal subtransi-group, the quotient-transi-group with respect to whom, $(M^*, \mathfrak{A}^*, e^*)$ say, is transitive; it is well-known[1] that in this case, all classes in the decomposition of M by $\sim$ are cardinally equivalent; consequently, M can be identified with the cartesian product $K \times M^*$. The identification is made in the following manner (we employ the notations of point 1, though we have to do here with another situation):

If $x \in M$, let x^* be his equivalence class with respect to $\sim$; suppose that α_λ, $\lambda \in \Lambda$, is such that $\alpha_\lambda(e^*) = x^*$; then

$$\tau_{\alpha_\lambda}^{-1}(x) = k \in K \tag{1}$$

as is easily seen from the definition of quotient-transi-groups; *we then identify* x, $x \in M$, *with the couple* $(k, x^*) \in K \times M^*$.

In particular we identify $k \in K$ with (k, e^*) and if $\alpha_\lambda(e^*) = x^*$ we obtain from (1), $\tau_{\alpha_\lambda}(k, e^*) = (k, x^*)$.

[1]) See, for instance, [4].

With every $\sigma \in \mathfrak{B}$ there is associated, in a natural way a transformation $\psi(\sigma)$ of K, namely that which satisfies $((\psi\sigma)(k), e^*) = \sigma(k, e^*)$, $k \in K$ [2]). Consequently, with the same notations as before, we have

$$\sigma(k, x^*) = \sigma\tau_{\alpha_\lambda}(k, e^*) = \tau_{\alpha_\lambda}((\psi\sigma^{\alpha_\lambda})(k), e^*) = ((\psi\sigma^{\alpha_\lambda})(k), x^*) .$$

Hence, it is sen that if $\psi\sigma^{\alpha_\lambda}$ is the identical transformation of K for all $\lambda \in \Lambda$, then $\sigma = 1$.

Thus, hypothesis $(\mathscr{S})$ is a necessary condition for the construction of the transi-group $(M, \mathfrak{A}, e)$.

It must be also observed that if $\mathfrak{B}$ coincides with $\mathfrak{B}^*$ and ψ is the identity mapping, hypothesis $(\mathscr{S})$ is automatically satisfied since among the σ^{α_λ} may be found σ too.

3. With the data and notations of point 1, let us suppose in addition that *there exists a homomorphism φ, $\varphi : \mathfrak{A}_0 \to \mathfrak{T}(K)$ whose restriction to $\mathfrak{B}$ coincides with ψ.* [3])

1) We define a mapping η, $\eta : \mathfrak{A} \to \mathfrak{T}(M)$ in the following manner: Let $\sigma\tau_\alpha \in \mathfrak{A}$, $\sigma \in \mathfrak{B}$, $\alpha \in \mathfrak{A}^*$ and let $(k, x^*) \in M$, where $k \in K$, $x^* \in M^*$. If the chosen representative of the left residue class with respect to $\mathfrak{A}_0^*$, which contains α, is α_ϱ, $\varrho \in \Lambda$, then $\alpha_\varrho(e^*) = \alpha(e^*)$ hence $\tau_{\alpha_\varrho}^{-1}\tau_\alpha \in \mathfrak{A}_0$; put

$$\delta_\alpha = \varphi(\tau_{\alpha_\varrho}^{-1}\tau_\alpha) ;$$

the δ_α, $\alpha \in \mathfrak{A}^*$, define a function which associates with every $\alpha \in \mathfrak{A}^*$ a $\delta_\alpha \in \mathfrak{T}(K)$. Suppose that $x^* = \alpha_\lambda(e^*)$ and that $\alpha_\nu \equiv \alpha\alpha_\lambda (\text{mod } \mathfrak{A}_{0s}^*)$, [4]) $\lambda, \nu \in \Lambda$.

Then, by definition,[5])

$$(2) \qquad (\eta(\sigma\tau_\alpha))(k, x^*) = ((\sigma\mu_{\alpha,\alpha_\lambda})^{\alpha_\lambda}\, \delta_{\alpha\alpha_\lambda}(k), \alpha(x^*)) .$$

We assert that $\eta(\sigma\tau_\alpha)$, $\sigma \in \mathfrak{B}$, $\alpha \in \mathfrak{A}^*$, is a transformation of M: indeed, if $(k', y^*) \in M$, $k' \in K$, $y^* \in M^*$, then, since $\mathfrak{A}^*$ acts transitively on M, there exist $\alpha, \alpha_\lambda, \alpha_\nu \in \mathfrak{A}^*$ where $\lambda, \nu \in \Lambda$ such that

$$\alpha(x^*) = y^* = \alpha_\nu(e^*) \quad \text{and} \quad \alpha_\lambda(e^*) = x^* ;$$

let k, $k \in K$, be such that

$$(\sigma\mu_{\alpha,\alpha_\lambda})^{\alpha_\nu}\, \delta_{\alpha\alpha_\lambda}(k) = k' ;$$

such a k exists since $\delta_{\alpha\alpha_\lambda}$ and $\psi((\sigma\mu_{\alpha,\alpha_\lambda})^{\alpha_\nu})$ are transformations of K; we have $(\eta(\sigma\tau_\alpha))(k, x^*) = (k', y^*)$.

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[2]) Instead of $\sigma\tau_e$, we have written σ which is the same thing.

[3]) By $\mathfrak{T}(E)$ we denote, generally, the group of all transformations of the set E.

[4]) That is to say, left congruent with respect to $\mathfrak{A}_0^*$.

[5]) Here, instead of $(\psi(\sigma_1))(k_1)$ we have written, in order to simplify notations, $\sigma_1(k_1)$, $\sigma_1 \in \mathfrak{B}$, $k_1 \in K$.

If $(\eta(\sigma\tau_\alpha))(k, x^*) = (\eta(\sigma\tau_\alpha))(k_1, x_1^*)$, it is an immediate fact that $x^* = x_1^*$; further it may be easily checked that $k = k_1$.

2) In order to prove that η is a group homomorphism, let $\sigma_1\tau_\alpha, \sigma_2\tau_\beta \in \mathfrak{A}$ and $(k, x^*) \in M$ where the meaning of notations is obvious; if

$$(3) \qquad x^* = \alpha_\lambda(e^*), \quad \alpha(x^*) = \alpha_\nu(e^*), \quad \beta\alpha(x^*) = \alpha_\pi(e^*),$$

where $\lambda, \nu, \pi \in \Lambda$, then according to definition (2), we have

$$(4) \qquad \begin{aligned} \eta(\sigma_2\tau_\beta)\,\eta(\sigma_1\tau_\alpha)(k, x^*) &= \eta(\sigma_2\tau_\beta)([\sigma_1\mu_{\alpha,\alpha_\lambda}]^{\alpha_\nu}\,\delta_{\alpha\alpha_\lambda}(k), \alpha(x^*)) = \\ &= ([\sigma_2\mu_{\beta,\alpha_\nu}]^{\alpha_\pi}\,\delta_{\beta\alpha_\nu}[(\sigma_1\mu_{\alpha,\alpha_\lambda})^{\alpha_\nu}\,\delta_{\alpha\alpha_\lambda}(k)], \beta\alpha(x^*)) \end{aligned}$$

According to the same definition we have

$$(5) \qquad \eta(\sigma_2\tau_\beta\,.\,\sigma_1\tau_\alpha)(k, x^*) = ([\sigma_2\sigma_1^{(\beta)}\mu_{\beta,\alpha}\mu_{\beta\alpha,\alpha_\lambda}]^{\alpha_\pi}\;\delta_{\beta\alpha\alpha_\lambda}(k), \beta\alpha(x^*)) .$$

From relations (3) we obtain

$$(6) \qquad \delta_{\beta\alpha_\nu} = \varphi(\tau_{\alpha_\pi}^{-1}\tau_{\beta\alpha_\nu}), \quad \delta_{\alpha\alpha_\lambda} = \varphi(\tau_{\alpha_\nu}^{-1}\tau_{\alpha\alpha_\lambda}), \quad \delta_{\beta\alpha\alpha_\lambda} = \varphi(\tau_{\alpha_\pi}^{-1}\tau_{\beta\alpha\alpha_\lambda}) .$$

Based on the fact that ψ is a rectriction of φ and on relations (6), the first component of the second member of (4) respectively (5), is the result of the action on k of the transformation

$$\begin{aligned} \varphi(\tau_{\alpha_\pi}^{-1}\sigma_2\mu_{\beta,\alpha_\nu}\tau_{\alpha_\pi})\,\varphi(\tau_{\alpha_\pi}^{-1}\tau_{\beta\alpha_\nu})\,\varphi(\tau_{\alpha_\nu}^{-1}\sigma_1\mu_{\alpha,\alpha_\lambda}\tau_{\alpha_\nu})\,\varphi(\tau_{\alpha_\nu}^{-1}\tau_{\alpha\alpha_\lambda}) &= \\ = \varphi(\tau_{\alpha_\pi}^{-1}\sigma_2\tau_\beta\sigma_1\tau_\alpha\tau_{\alpha_\lambda})& \end{aligned}$$

respectively of the transformation

$$\varphi(\tau_{\alpha_\pi}^{-1}\sigma_2\tau_\beta\sigma_1\tau_\beta^{-1}\mu_{\beta,\alpha}\mu_{\beta\alpha,\alpha_\lambda}\tau_{\alpha_\pi})\,\varphi(\tau_{\alpha_\pi}^{-1}\tau_{\beta\alpha\alpha_\lambda}) = \varphi(\tau_{\alpha_\pi}^{-1}\sigma_2\tau_\beta\sigma_1\tau_\alpha\tau_{\alpha_\lambda}) .$$

Here we used the definition of $\mu_{\xi,\omega}$ namely

$$\tau_\xi\tau_\omega = \mu_{\xi,\omega}\tau_{\xi,\omega} .$$

So it was proved that η is a group homomorphism.

3) Suppose that $\sigma\tau_\alpha$, $\sigma\tau_\alpha \in \mathfrak{A}$, has the property that $\eta(\sigma\tau_\alpha)$ is the identical transformation of M; consequently

$$\eta(\sigma\tau_\alpha)(k, x^*) = (k, x^*)$$

for any $(k, x^*) \in M$.

According to definition (2), this implies, firstly, $\alpha = \varepsilon$. Let $x^* = \alpha_\lambda(e^*)$; since δ_{α_λ} is the identical transformation of K, we obtain from the same definition (2) that $\psi(\sigma^{\alpha_\lambda})$ is the identical transformation of K; since x^* is arbitrarily chosen in M^*, λ may be considered as an arbitrary element of Λ, so that we may apply hypothesis $(\mathscr{S})$ and obtain $\sigma = 1$.

So η is not merely a homomorphism, but a monomorphism.

While identifying $\sigma\tau_\alpha$ with $\eta(\sigma\tau_\alpha)$ and, consequently, $\mathfrak{A}$ with $\eta(\mathfrak{A})$ we may say that $\mathfrak{A}$ is an (effective) transformation group of M. Moreover, if we choose in K a well-determined but otherwise arbitrary element k_0, we obtain the transi-group $(M, \mathfrak{A}, e)$ where $e = (k_0, e^*)$.

4) We define in M the following equivalence relation which we denote by $\sim$: let $(k_1, x^*), (k_2, y^*) \in M$; then, by definition $(k_1, x^*) \sim (k_2, y^*)$ if and only if $x^* = y^*$.

Considering the subgroup $\mathfrak{B}_{\mathfrak{A}}(\sim)$ associated with $\sim$, if $\sigma\tau_\alpha$, $\sigma\tau_\alpha \in \mathfrak{A}$, belongs to this subgroup, then, according to (2), $\alpha(x^*) = x^*$ for any $x^* \in M^*$ and then $\alpha = \varepsilon$ and $\sigma\tau_\alpha \in \mathfrak{B}$; conversely, every element of $\mathfrak{B}$ may be written in the form $\sigma\tau_\varepsilon$ and, consequently, belongs to $\mathfrak{B}_{\mathfrak{A}}(\sim)$; hence $\mathfrak{B}_{\mathfrak{A}}(\sim) = \mathfrak{B}$.

It is easily seen that the equivalence $\sim$ is imprimitive with respect to $\mathfrak{A}$.

On the other hand, the equivalence class of e is $K \times \{e^*\}$; if we identify the couple (k, e^*) with k, for any $k \in K$, then $K \times \{e^*\}$ is obtained to be identified with K. So we have to do with the normal subtransi-group $(K, \mathfrak{B}, e, \sim)$ of $(M, \mathfrak{A}, e)$.

Let $\sigma\tau_\varepsilon \in \mathfrak{B}$; then, from (2) we have

$$\sigma\tau_\varepsilon(k, e^*) = ((\psi\sigma)(k), e^*) \tag{7}$$

so that σ, considered as an element of $\mathfrak{A}$, induces in K the transformation $\psi(\sigma)$. Consequently, the transi-group associated with $(K, \mathfrak{B}, e, \sim)$ is $(K, \mathfrak{B}^*, e)$ where, namely, to σ, $\sigma \in \mathfrak{B}$, we associate $\psi(\sigma)$.

Now, we define a homomorphism of transi-groups, (h, χ), $(h, \chi) : (M, \mathfrak{A}, e) \to$ $\to (M^*, \mathfrak{A}^*, e^*)$, through the relations:

$$h(k, x^*) = x^*\,, \quad \chi(\sigma\tau_\alpha) = \alpha\,, \quad k \in K\,, \quad x^* \in M^*\,, \quad \sigma \in \mathfrak{B}\,, \quad \alpha \in \mathfrak{A}^*\,.$$

It was no error while anticipating on the homomorphism character of (h, χ), since according to definition (2), we have

$$h(\sigma\tau_\alpha(k, x^*)) = \alpha(x^*) = (\chi(\sigma\tau_\alpha))\,(h(k, x^*))$$

and there is no difficulty to prove the other properties of transi-group homomorphisms. Obviously, the kernel of (h, χ) is $(K, \mathfrak{B}, e, \sim)$.

5) To summarize, we have constructed a transi-group $(M, \mathfrak{A}, e)$ with the following properties:

i) Denoting by $\sim$ the equivalence defined as at 4), the subtransi-group generated by $\sim$ is $(K, \mathfrak{B}, e, \sim)$ where K appears through identification with $K \times \{e^*\}$, the identification having been done as at 4).

ii) For any $\sigma \in \mathfrak{B}$, $k \in K$, we have (7), hence the transi-group associated with $(K, \mathfrak{B}, e, \sim)$ is $(K, \mathfrak{B}^*, e)$ where to σ, $\sigma \in \mathfrak{B}$, we associate $\psi(\sigma)$.

iii) The couple (h, χ) as defined at 4) is a transi-group homomorphism (h, χ) : $(M, \mathfrak{A}, e) \to (M^*, \mathfrak{A}^*, e^*)$, whose kernel is $(K, \mathfrak{B}, e, \sim)$.

4. With the data and notations of point 1, let us suppose that *there exists a transi-group* $(M, \mathfrak{A}, e)$, *where* $e = (k_0, e^*)$ *such that conditions* i)–iii) *from* 3.5) *are fulfilled.*

Let $\sigma\tau_\alpha = \mathfrak{A}_0$, where $\sigma \in \mathfrak{B}$, $\alpha \in \mathfrak{A}_0^*$; if $(k, e^*) \in K$ then, according to condition iii) from 3.5), we have

$$h(\sigma\tau_\alpha(k, e^*)) = \alpha(e^*) = e^* .$$

Hence $\sigma\tau_\alpha$ maps any element of K in an element of K. Since $\mathfrak{A}_0$ is a group, it is not difficult to prove that $\sigma\tau_\alpha$ induces quite a transformation $\varphi(\sigma\tau_\alpha)$ of K. So, we have constructed a mapping φ, $\varphi : \mathfrak{A}_0 \to \mathfrak{T}(K)$, which is obviously a group homomorphism.

If $\sigma\tau_\alpha \in \mathfrak{B}$, then $\alpha \doteq \varepsilon$ and, by definition, $\varphi(\sigma\tau_\alpha)(k, e^*) = \sigma\tau_\varepsilon(k, e^*)$ (where in the second member we mean the result of the action of $\sigma\tau_\varepsilon$ on (k, e^*) as in $(M, \mathfrak{A}, e)$). But according to condition ii) from 3.5) this implies that

$$\varphi(\sigma\tau_\alpha)(k, e^*) = ([\psi\sigma](k), e^*) . \tag{8}$$

With the convention of identification (see i) from 3.5)) and on account of the fact that $\tau_\varepsilon = 1$, relation (8) becomes

$$(\varphi\sigma)(k) = (\psi\sigma)(k) ;$$

as this is valid for any $k \in K$, it follows that ψ is the restriction of φ to $\mathfrak{B}$.

5. We are now quite ready to formulate our fundamental result (we are given the data and notations of point 1):

Theorem. *In order that a transi-group* $(M, \mathfrak{A}, e)$ (*where* $e = (k_0, e^*)$, $k_0 \in K$) *should exist, so that conditions* i)–iii) *from* 3.5) *be satisfied, it is necessary and sufficient that there should exist a group homomorphism* φ, $\varphi : \mathfrak{A}_0 \to \mathfrak{T}(M)$, *whose restriction to* $\mathfrak{B}$ *be* ψ.

6. As particular cases we may consider the following quite trivial ones. Suppose in addition to the conditions stipulated at point 1, that:

1) $\mathfrak{A}^*$ *is a regular*[6]) *transformation group of* M^*; in this case $\mathfrak{A}_0^*$ is the identity subgroup, hence $\mathfrak{A}_0 = \mathfrak{B}$ and we may take $\varphi = \psi$.

2) $\mathfrak{A}$ *is the direct product* $\mathfrak{B} \times \mathfrak{A}^*$; in this case we may define $\varphi(\sigma\tau_\alpha) = \psi(\sigma)$, $\sigma \in \mathfrak{B}$, $\alpha \in \mathfrak{A}_0^*$.

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[6]) For the definition see, for instance, [3], p. 36.

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Резюме

О РАСШИРЕНИИ ТРАНЗИГРУПП

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Автор обобщает теорию расширения групп на случай транзигрупп, введенных им в работе [5].