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Czechoslovak Mathematical Journal, Vol. 14 (1964), No. 3, 431–437

Persistent URL: http://dml.cz/dmlcz/100630

Чехословацкий математический журнал, т. 14 (89) 1964, Прага

A CHARACTERIZATION OF LOCALLY CONNECTED AND LOCALLY ARCWISE CONNECTED TOPOLOGICAL SPACES*)

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(Received May 30, 1963)

1. INTRODUCTION

The purpose of this paper is to obtain a characterization of locally connected and locally arcwise connected topological spaces, using as a tool GAIL S. YOUNG's G-topology ([1])**), and to give sufficient conditions under which the G-topology is the coarsest locally connected or locally arcwise connected topology on a set finer than the original topology on the set.

1.1. Definition. Let G be a collection of subsets of a topological space (X, π). A point p in X is said to be a π_G-limit point of a subset M of X if and only if for every open subset U of (X, π) containing p there exists an element γ in G such that $p \in \gamma \subseteq U$ and $\gamma \cap (M - \{p\}) \neq \emptyset$.

The preceding definition determines a new topology for X, the G-topology, which we shall denote by π_G. To avoid confusion, names of topological properties will be prefixed by π- or π_G- in order to indicate in which topology a subset of X has the given property. If M is any subset of a topological space (X, π), the notation $r - \pi$ will be used to indicate the topology π relativized to M.

2. RELATIONS BETWEEN THE G-TOPOLOGY AND THE ORIGINAL TOPOLOGY ON A SET

Implicit in the definition of the G-topology is the following theorem due to Young. It should be noted that in all theorems due to Young referred to in this paper, the hypothesis that the original space be a T_0-space has been deleted since the results are easily seen to hold without this restriction.

*) This material is a portion of a dissertation submitted to the Graduate School of the University of Michigan.

**) Numbers in square brackets refer to the bibliography.

2.1. Theorem. ([1], theorem 1, p. 481.) *If G is any collection of subsets of a topological space (X, π), then the topology π_G on X is finer than π, i.e., $\pi \subseteq \pi_G$.*

That π and π_G are, in general, distinct topologies on a given set is a consequence of the following example.

2.2. Example. Let $X = \bigcup[X_n : n = 0, 1, 2, \ldots]$ be the subset of the plane where

$$X_0 = [(x, y) : x = 0,\ 0 \leqq y \leqq 1] \quad \text{and} \quad X_n = \left[(x, y) : x = \frac{1}{n},\ 0 \leqq y \leqq 1\right]$$
$$\text{for} \quad n = 1, 2, \ldots ,$$

and let X be endowed with the usual metric topology π. Suppose G is that collection of subsets of (X, π) which consists of all the sub-arcs of each X_n, $n = 0, 1, 2, \ldots$. Then $\pi_G \neq \pi$ since $Y = \bigcup[X_n : n = 1, 2, \ldots]$ is a π_G-closed subset of X which is not π-closed.

The next two theorems provide us with conditions under which π_G and π are identical. Their proofs clearly follow from theorem 2.1 and the fact that $\pi_G \subseteq \pi$ since, in each case, every π-limit point is a π_G-limit point.

2.3. Theorem. *If G is a collection of subsets of a topological space (X, π) and if G contains a base for the family of π-open sets in X, then $\pi_G = \pi$.*

2.4. Theorem. *If (X, π) is a locally arcwise connected topological space and if G is the collection of all arcs in (X, π), then $\pi_G = \pi$.*

3. A CHARACTERIZATION OF LOCALLY CONNECTED TOPOLOGICAL SPACES

3.1. Theorem. ([1], theorem 1, p. 481.) *If G is any collection of connected subsets of a topological space (X, π) then every π-component of a π-open set is π_G-open.*

3.2. Corollary. *If G is any collection of connected subsets of a topological space (X, π) and if $\pi_G = \pi$ then (X, π) is locally connected.*

Proof. If $\pi_G = \pi$ then π-components of π-open sets are π-open and so (X, π) is locally connected.

3.3. Theorem. *If G is the collection of all connected subsets of a locally connected topological space (X, π) then $\pi_G = \pi$.*

Proof. The local connectedness of (X, π) implies the existence of a base B of connected, open sets for the topology π on X. Since G contains B, $\pi_G = \pi$ by theorem 2.3.

3.4. Combining theorem 3.3 and corollary 3.2 we obtain the following characterization of locally connected spaces: if G is the collection of all connected subsets of a topological space (X, π) then (X, π) is locally connected if and only if $\pi_G = \pi$.

4. THE COARSEST LOCALLY CONNECTED TOPOLOGY ON A SET FINER THAN THE ORIGINAL TOPOLOGY ON THE SET

4.1. Theorem. ([1], theorem 3, p. 481.) *Let G be a collection of subsets of a topological space (X, π) and suppose that each element of G is π_G-connected. Then the following statements hold.*

(i) (X, π_G) *is locally connected.*

(ii) *If every two points of X lie in an element of G then (X, π_G) is connected.*

4.2. Lemma. *Let (X, π) be a topological space and let π' be a topology on X which is finer than π. If C is any π'-connected subset of X then C is π-connected.*

Proof. The result follows immediately from the definition of a connected subset and the fact that $\pi \subseteq \pi'$.

4.3. Theorem. *Let G be any collection of subsets of a topological space (X, π). If (X, π_G) is connected then (X, π) is connected.*

Proof. The result follows immediately from theorem 2.1 and the preceding lemma.

4.4. Theorem. *Let G be the collection of all π-connected subsets of a topological space (X, π). If every element of G is π_G-connected then π_G is the coarsest locally connected topology for X which is finer than π.*

Proof. By theorems 4.1, (i) and 2.1, π_G is a locally connected topology for X which is finer than π and it remains to be shown that π_G is the coarsest such topology on X. Let π' be any locally connected topology for X which is finer than π. Let M be any subset of X, p a π'-limit point of M, and U any π-open set containing p. Since $\pi \subseteq \pi'$, U is π'-open. Moreover, since π' is a locally connected topology for X, U contains a π'-open, π'-connected set V containing p and, since p is a π'-limit point of M, V must contain a π'-connected set C which intersects $\{p\}$ and M-$\{p\}$. By lemma 4.2, C is π-connected and it follows that p is a π_G-limit point of M. Thus $\pi_G \subseteq \pi'$, i.e., π_G is coarser than π'.

5. A CHARACTERIZATION OF LOCALLY ARCWISE CONNECTED TOPOLOGICAL SPACES

5.1. Theorem. *Let G be the collection of all arcs in a topological space (X, π). If A is an arc in (X, π) then A is an arc in (X, π_G).*

Proof. Consider $(A, r-\pi)$ and $(A, r-\pi_G)$. Let M be a subset of A and let p be a $(r-\pi)$-limit point of M. If U is any π-open set containing p then $U \cap A$ is a $(r-\pi)$-open set containing p and, since p is a $(r-\pi)$-limit point of M and $(A, r-\pi)$ is locally arcwise connected, $U \cap A$ contains a point q such that $q \in M$, $q \neq p$, and p

and q are the end points of an arc γ in $U \cap A$. Hence p is a π_G-limit point of M in A, i.e., p is a $(r-\pi_G)$-limit point of M. It now follows that $(r-\pi_G) \subseteq (r-\pi)$. Combining this with $(r-\pi) \subseteq (r-\pi_G)$ we have $(r-\pi) = (r-\pi_G)$. Therefore A is an arc in (X, π_G).

5.2. Theorem. *Let G be the collection of all arcs in a topological space (X, π). (X, π) is arcwise connected if and only if (X, π_G) is connected.*

Proof. Suppose (X, π) is arcwise connected. Then every two points of X lie in an element of G. Furthermore, if $\gamma \in G$ then γ is an arc in (X, π_G) by theorem 5.1. Hence (X, π_G) is connected. In fact, (X, π_G) is arcwise connected.

Now suppose (X, π_G) is connected. If (X, π) is not covered by arcs., i.e., if there is a point x in (X, π) which is not contained in an element of G, then x is an isolated point of (X, π_G) and so (X, π_G) is not connected. We may therefore assume that (X, π) is covered by arcs. Define a relation R on X as follows: for all x, y in X such that $x \neq y$, xRy if and only if x and y are contained in an element of G. Now assume that (X, π) is not arcwise connected, so that there exists an x in X such that $R_x \neq X$, where $R_x = [y : y \in X \text{ and } yRx]$. $X = R_x \cup (X - R_x)$ where $R_x \cap (X - R_x) = \emptyset$, $R_x \neq \emptyset$ since (X, π) is covered by arcs, and $(X - R_x) \neq \emptyset$ since $R_x \neq X$. Since no point of R_x can be a π_G-limit point of $X - R_x$, and conversely, then R_x and $X - R_x$ must be π_G-closed. Hence (X, π_G) is not connected.

5.3. Corollary. *Let G be the collection of all arcs in a topological space (X, π). (X, π) is arcwise connected if and only if (X, π_G) is arcwise connected.*

5.4. Corollary. *Let G be the collection of all arcs in a topological space (X, π). (X, π_G) is connected if and only if (X, π_G) is arcwise connected.*

5.5. Theorem. *Let G be the collection of all arcs in a topological space (X, π). If V is a subset of X which is π-arcwise connected then V is π_G-connected (in fact, π_G-arcwise connected).*

Proof. Since V is π-arcwise connected and since every π-arc is a π_G-arc by theorem 5.1, it follows that V is π_G-connected (in fact, π_G-arcwise connected).

5.6. Example. Let (X, π) be the plane with its usual metric topology and let V be the closure of the graph of $y = \sin 1/x$, $0 < x \leqq 1$. If G is the collection of all arcs in (X, π) then $\pi = \pi_G$ by theorem 2.4 since (X, π) is locally arcwise connected. Thus V is π_G-connected since it is π-connected, but V is not π-arcwise connected. We therefore see that the converse of the preceding theorem does not hold.

5.7. Theorem. *Let (X, π) be a topological space and let V be an open subset of (X, π). If G is the collection of all arcs in (X, π) and if G' is the collection of all arcs in $(V, r-\pi)$ then the topologies $r-\pi_G$ and $(r-\pi)_{G'}$ on V are identical.*

Proof. Let M be any subset of V and let p be any $(r-\pi)_{G'}$-limit point of M. If U is any π-open set containing p then $U \cap V$ is $(r-\pi)$-open and, since p is a $(r-\pi)_{G'}$-limit point of M, there exists an element γ in G' and therefore also in G such that $p \in \gamma \subseteq U \cap V \subseteq U$ and $\gamma \cap (M - \{p\}) \neq \emptyset$. Hence p is a π_G-limit point of M in V, i.e., p is a $(r-\pi_G)$-limit point of M, and it follows that $(r-\pi_G) \subseteq (r-\pi)_{G'}$.

Now let p be any $(r-\pi_G)$-limit point of M, i.e., a π_G-limit point of M in V. If U is any $(r-\pi)$-open set containing p then, since V is π-open, U is π-open. Since p is a π_G-limit point of M in V, U contains an element γ of G such that $p \in \gamma \subseteq U$ and $\gamma \cap (M-\{p\}) \neq \emptyset$. But $\gamma \in G'$ since $\gamma \subseteq U \subseteq V$ and it follows that p is a $(r-\pi)_{G'}$-limit point of M. Hence $(r-\pi)_{G'} \subseteq (r-\pi_G)$.

5.8. Theorem. *Let G be the collection of all arcs in a topological space (X, π). If V is a π-open subset of X and if V is π_G-connected then V is π-arcwise connected.*

Proof. Let V be π_G-connected, i.e., let $(V, r-\pi_G)$ be connected. If G' is the collection of all arcs in $(V, r-\pi)$ then $(V, (r-\pi)_{G'})$ is connected by the preceding theorem and, by theorem 5.2, $(V, r-\pi)$ is arcwise connected, i.e., V is π-arcwise connected.

5.9. Theorem. *Let G be the collection of all arcs in a topological space (X, π). If $\pi_G = \pi$ then (X, π) is locally arcwise connected.*

Proof. Let x be any element of X, U any π-open set containing x, and C the π-component of U containing x. By theorem 3.1 and the fact that $\pi_G = \pi$, C is π-open and also π_G-connected. Thus C is π-arcwise connected, by theorem 5.8, and it follows that (X, π) is locally arcwise connected.

5.10. Synthesizing theorems 2.4 and 5.9 we obtain the following characterization of locally arcwise connected spaces: if G is the collection of all arcs in a topological space (X, π), then (X, π) is locally arcwise connected if and only if $\pi_G = \pi$.

6. THE COARSEST LOCALLY ARCWISE CONNECTED TOPOLOGY ON A SET FINER THAN THE ORIGINAL TOPOLOGY ON THE SET

6.1. Definition. Let (X, π) be an arbitrary topological space, let $x \in X$, and let M be any subset of X which contains x. The arc component of x in M is the set consisting of the point x together with all points in M which can be joined to x by an arc in M.

6.2. Theorem. *If G is the collection of all arcs in a topological space (X, π) then (X, π_G) is locally arcwise connected.*

Proof. Let x be any point of X, U any π_G-open set containing x, and V the π_G-arc component of x in U. Suppose V is not π_G-open. Then $X - V$ is not π_G-closed and so there must exist a point y in V such that y is a π_G-limit point of $X - V$. Since each π-arc is a π_G-arc by theorem 5.1, and since V is the π_G-arc component of x in U, no element of G that contains y and intersects $X - V$ lies entirely in U so that each such element intersects $F_G(U)$, the π_G-boundary of U. But then every π_G- open set containing y contains an element of G that intersects $\{y\}$ and $F_G(U)$. Thus y is a π_G-limit point of the π_G-closed set $F_G(U)$ and it follows that $y \in F_G(U)$. This is impossible since $y \in U$ and U is π_G-open. Hence V must be π_G-open and it follows that (X, π_G) is locally arcwise connected.

6.3. The following **example** shows that, in general, it is not true that if G is the collection of all arcs in a topological space (X, π) then π_G is the coarsest locally arcwise connected topology for X which is finer than π.

Let (X, π) be the real line with the minimal T_1-topology and let G be the collection of all arcs in (X, π). Since G is empty, π_G is the discrete topology and π_G is locally arcwise connected. If π' is the usual metric topology on X, π' is locally arcwise connected and $\pi \subseteq \pi' \subseteq \pi_G$.

6.4. Lemma. *Let (X, π) be a topological space in which every compact set is closed, and let π' be a topology for X which is finer than π. If A is an arc in (X, π') then A is an arc in (X, π).*

Proof. Let i be the identity mapping of the arc A in (X, π') onto the same point set in (X, π). The mapping i is one-to-one, onto, and continuous, and since A is π'-compact and π is a topology in which every compact set is closed then i is a homeomorphism. Therefore A is an arc in (X, π).

6.5. Theorem. *Let (X, π) be a topological space in which every compact set is closed. If G is the collection of all arcs in (X, π) then π_G is the coarsest locally arcwise connected topology for X which is finer than π.*

Proof. By theorems 2.1 and 6.2, π_G is a locally arcwise connected topology for X which is finer than π. It remains to be shown that π_G is the coarsest such topology on X. Suppose π' is a locally arcwise connected topology for X which is finer than π. Let M be any subset of X and let p be a π'-limit point of M. If U is any π-open set containing p then U is π'-open since $\pi \subseteq \pi'$. Hence U contains a π'-open set V containing p which is π'-arcwise connected and therefore π-arcwise connected by the preceding lemma. Since p is a π'-limit point of M, V contains a point q in M with $q \neq p$, and since V is π-arcwise connected there is a π-arc in V which has p and q as end points. Hence p is a π_G-limit point of M and it now follows that $\pi_G \subseteq \pi'$.
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Резюме

ХАРАКТЕРИЗАЦИЯ ЛОКАЛЬНО СВЯЗНЫХ И ЛОКАЛЬНО ПО ДУГАМ СВЯЗНЫХ ТОПОЛОГИЧЕСКИХ ПРОСТРАНСТВ

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Пусть G — система подмножеств топологического пространства (X, π). В [1] на множестве X определяется при помощи системы G новая топология π_G, а именно: $p \in X$ является предельной точкой множества $M \subset X$ в топологии π_G, если и только если для каждого открытого подмножества U пространства (X, π), содержащего p, существует $\gamma \in G$ так, что

$$p \in \gamma \subset U\,, \quad \gamma \cap (M - \{p\}) \neq \emptyset\,.$$

В этой статье дается характеризация локально связных и локально по дугам связных пространств при помощи этой топологии π_G и приводится достаточное условие, при котором π_G является самой грубой локально связной (соотв. локально по дугам связной) топологией, более тонкой, чем π. Доказываются следующие теоремы:

Если G — система всех связных подмножеств пространства (X, π), то (X, π) будет локально связным, если и только если $\pi_G = \pi$.

Пусть G — система всех дуг пространства (X, π). Тогда следующие утверждения эквивалентны:

(X, π) по дугам связно; (X, π_G) связно; (X, π_G) по дугам связно.

Если G — система всех дуг пространства (X, π), то (X, π) будет локально по дугам связным, если и только если $\pi = \pi_G$.

Если G — система всех дуг пространства (X, π) и если в пространстве (X, π) каждое компактное подмножество замкнуто, то π_G — самая грубая локально по дугам связная топология, более тонкая, чем π.