Takesi Isiwata

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SOME CLASSES OF COUNTABLY COMPACT SPACES

Takesi Isiwata, Tokyo

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The present paper investigates the relations between some classes of countably compact spaces introduced by Z. Frolík [1], [2].

The spaces considered here are always completely regular T_1-spaces; **N** denotes the set of positive integers. Recently, Z. Frolík introduced the classes $\check{\mathfrak{P}}$, $\check{\mathfrak{P}}_F$ and $\check{\mathfrak{C}}$ which are characterized by the following: A space X belongs to $\check{\mathfrak{P}}$ (or $\check{\mathfrak{C}}$) if and only if for every pseudocompact (or countably compact) space Y the topological product $X \times Y$ is pseudocompact (or countably compact); a space X belongs to $\check{\mathfrak{P}}_F$ if and only if every closed subspace of X belongs to $\check{\mathfrak{P}}$. Moreover, he gave necessary and sufficient conditions for a space to belong to one of these classes (see § 1 below). Let $\check{\mathfrak{P}}_c$ be the subclass of $\check{\mathfrak{P}}$ consisting of countably compact spaces. It is easy to see that $\check{\mathfrak{P}} - \check{\mathfrak{P}}_c$ is not empty: for instance, $X = [1, \omega] \times [1, \Omega] - \{(\omega, \Omega)\}$ belongs to $\check{\mathfrak{P}} - \check{\mathfrak{P}}_c$, where ω and Ω are the least ordinal numbers of the second and the third classes respectively.

In this paper, we shall give new characterizations of $\check{\mathfrak{P}}_F$ in § 2, and consider, in § 3, the relations between the classes $\check{\mathfrak{P}}_c$, $\check{\mathfrak{P}}_F$ and $\check{\mathfrak{C}}$, and show that, the three classes $\check{\mathfrak{C}} - \check{\mathfrak{P}}_c$, $\check{\mathfrak{P}}_c - \check{\mathfrak{C}}$ and $\check{\mathfrak{P}}_c \cap (\check{\mathfrak{C}} - \check{\mathfrak{P}}_F)$ are not empty. Equivalently, i) there is a countably compact space X such that $X \times Y$ is countably compact for every countably compact space Y, but $X \times Z$ is not pseudocompact for some pseudocompact space Z, ii) there is a countably compact space X such that $X \times Y$ is pseudocompact for every pseudocompact space Y but $X \times Z$ is not countably compact for some countably compact space Z, and iii) there is a countably compact space X such that $X \times Y$ is countably compact (or pseudocompact) for every countably compact (or pseudocompact) space Y but X contains some closed subspace A having the property that $A \times B$ is not pseudocompact for some pseudocompact space B.

1. Preliminary. In this section, for convenience, we shall state Frolík's theorems, and transfer the form 1.3 to the forms 1.5 and 1.6.

1.1 [1,3.6]. $\check{\mathfrak{P}} \ni X$ if and only if X satisfies the following condition: If $\check{\mathfrak{U}}$ is an infinite disjoint family of non-void open subsets of X, then there exists a disjoint sequence

$\{U_n\}$ in $\mathfrak{A}$ such that for every filter $\check{\mathfrak{N}}$ of infinite subsets of $\mathbf{N}$ we have

$$\bigcap_{N_1 \in \check{\mathfrak{N}}} \left(\overline{\bigcup_{n \in N_1} U_n} \right) \neq \emptyset \quad (\emptyset \text{ denotes the empty set}).\ ^{1})$$

1.2 [1,4.2]. $\mathfrak{P}_F \ni X$ if and only if every subset of X contains an infinite subset with a compact closure in X.

1.3 [2,3.3]. $\check{\mathfrak{C}} \not\ni X$ if and only if X satisfies the following condition: there exists an infinite discrete subset N of X such that for every compactification K of X there exists a subset S of $K - X$ such that the subspace $N \cup S$ of K is countably compact.

Let K be any compactification of X: Then there is a continuous mapping φ_K of βX onto K that leaves X pointwise fixed. We notice that $\varphi_K(X) = X$ and $\varphi_K(\beta X - X) = K - X$. Under this notation, we have

1.4 *Let K and M be any compactifications of X and let N be a discrete subset of X. If there is a subset S of $K - X$ such that $N \cup S$ is countably compact, then $\varphi_K^{-1}(S)$ (or $\varphi_M \varphi_K^{-1}(S)$) is a subset of $\beta X - X$ (of $M - X$) such that the subspace $N \cup \varphi_K^{-1}(S)$ (or $N \cup \varphi_M \varphi_K^{-1}(S)$) of βX (of M, respectively) is countably compact.*

Proof. It is known that if f is a closed mapping from a space P to a countably compact space Q, then the countable compactness of $f^{-1}(y)$ for each point y in Q implies the countable compactness of P (c.f., e.g., [2,1.1]). Consider the two sets $P = \varphi_K^{-1}(N \cap S)$ and $Q = N \cup S$. Since $f = \varphi_K \mid P$ is a closed compact mapping of P onto Q, we have that P is countably compact and $\varphi_K^{-1}(S) = P - N$ is a subset of $\beta X - X$.

The other statement is obvious from the continuity of φ_M.

From 1.3 and 1.4 we have

1.5. Theorem. $\check{\mathfrak{C}} \not\ni X$ *if and only if X satisfies the following condition: there is an infinite discrete subset N of X such that the subspace $N \cup S$ of some compactification K of X is countably compact for some subset S of $K - X$.*

From 1.3 and 1.5 we have

1.6. Theorem. *The following conditions are equivalent:*

i) $\check{\mathfrak{C}} \ni X$,

ii) *for infinite discrete subset N of X, there is a compactification K such that, for every subset S of $K - X$, the subspace $N \cup S$ of K is not countably compact,*

iii) *for every infinite discrete subset N of X, the space $N \cup S$ is not countably compact, where K is any compactification of X and S is any subset of $K - X$.*

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[1]) In the following, the left term of this relation will be denoted by $(\check{\mathfrak{N}}, N_1, U_n, X)$ and if U_n has a form $\{a_n;\ n \in N_1\}$, then by $(\check{\mathfrak{N}}, N_1, \{a_n\}, X)$. The symbol "$X$" denotes the space on which the closure operation is defined.

2. Characterizations of $\check{\mathfrak{P}}_F$. We shall show that the class $\check{\mathfrak{P}}_F$ is contained in $\check{\mathfrak{P}}_c \cap \check{\mathfrak{E}}$. If $X \in \check{\mathfrak{P}}_F$, then every closed subspace A of X belongs to $\check{\mathfrak{P}}$, and hence A must be pseudocompact. Therefore X is countably compact, that is, $\check{\mathfrak{P}}_F \subset \check{\mathfrak{P}}_c$. Let N be any infinite discrete subset of X. By 1.2 there is a compact subset F of X such that $N \cap F$ is infinite. Then, for every subset S of $\beta X - X$, the set $N \cap F$ has no accumulation points in $N \cup S$. Thus $N \cup S$ is not countably compact and hence, by 1.6 (ii), X belongs to $\check{\mathfrak{E}}$. Thus we have $\check{\mathfrak{P}}_F \subset \check{\mathfrak{P}}_c \cap \check{\mathfrak{E}}$.

2.1. Theorem. *The following conditions are equivalent for any space X:*

1) $\check{\mathfrak{P}}_F \ni X$,

2) *for every infinite discrete subset N of X and for every subset S of $K - X$ where K is some compactification of X, the set $N \cup S$ is not pseudocompact,*

3) *for every infinite discrete sequence $\{a_n\}$ of X, there is a subsequence $\{a_{n_i}\}$ such that for every filter $\mathfrak{N}$ of infinite subsets of* $\mathbf{N}$ *we have* $(\check{\mathfrak{N}}, N_1, \{a_{n_i}\}, X) \neq \emptyset$.

Proof. 1) $\Rightarrow$ 2). Suppose that $\check{\mathfrak{P}}_F \ni X$, N is any infinite discrete subset of X and S is any subset of $K - X$. By assumption, there is a compact subset F of X such that $F \cap N$ is infinite. Let $\{a_n\} \subset F \cap N$. Then $\{a_n\}$ is a family of open sets of the space $N \cup S$ and $\{a_n\}$ has no accumulation points in $N \cup S$. Therefore $\{a_n\}$ is locally finite in $N \cup S$, and hence $N \cup S$ is not pseudocompact.

2) $\Rightarrow$ 3). Let $N = \{a_n\}$ be any infinite discrete sequence of X. $Y = N \cup (\bar{N}(\text{in } K) - - X)$ is not pseudocompact by assumption. Thus there exists a locally finite family of open sets $\{U_n\}$ of Y. Since every point a_n is open in Y and N is dense in Y, each U_n contains a point a_{i_n} of N. Then for every filter $\check{\mathfrak{N}}$ of infinite subsets of $\mathbf{N}$, we have $B = (\check{\mathfrak{N}}, N_1, \{a_{i_n}\}, X) \neq \emptyset$. For, if $B = \emptyset$, then we have $(\check{\mathfrak{N}}, N_1, \{a_{i_n}\}, Y) = = (\check{\mathfrak{N}}, N_1, \{a_{i_n}\}, K) \neq \emptyset$, and hence $\{U_n\}$ is not locally finite in Y. This is a contradiction.

3) $\Rightarrow$ 1). Let $N = \{a_n\}$ be an infinite discrete sequence of X. By assumption, there is a subsequence $N' = \{a_{n_i}\}$ satisfying the relation in (3). For any point a in $\bar{N}'(\text{in } K)$, we take a base $\{U_\alpha\}$ of neighborhoods (in K) of a and put $N_\alpha = \{a_{n_i};\, a_{n_i} \in U_\alpha \cap N'\}$. Then $\{N_\alpha\}$ is a filter $\check{\mathfrak{N}}$. Thus by assumption we have $(\check{\mathfrak{N}}, N_\alpha, \{a_{n_i}\}, X) = \{a\}$, that is, the closure (in X) of N' is compact and hence X belongs to $\check{\mathfrak{P}}_F$.

From 2.1 we have

2.2. Theorem. *X belongs to $\check{\mathfrak{E}} - \check{\mathfrak{P}}_F$ if and only if for every infinite discrete subset N of X, $N \cup S$ is not countably compact for every subset S of $K - X$ but $T \cup N$ is pseudocompact for some subset T of $K - X$ where K is some compactification of X.*

2.3. Corollary. *If $\check{\mathfrak{P}} \ni X$ and Y is a dense subset of X every point of which is isolated in X, then any infinite subset of Y contains a subset with a compact closure in X.*

2.4. Remark. In 2.1(2) and 2.2, we may replace the word "some" compactification K of X by the word "any".

3. Examples. In this section[2]) we assume the *continuum hypothesis.*

3.1. Example. Let M be the set of all P-points of $\beta\mathbf{N} - \mathbf{N}$ and let $X = \beta\mathbf{N} - M$ We shall prove that X belongs to $\check{\mathfrak{C}} - \check{\mathfrak{P}}_c$.

1) $\beta X = \beta\mathbf{N}(= K)$.[2]) This is obvious.

2) *X does not belong to* $\check{\mathfrak{P}}_c$. Every subset L of $\mathbf{N}$ has no subset with compact closure in X by [3; 9M2], and hence, by 2.3, $\check{\mathfrak{P}}_c \not\ni X$.

3) *X belongs to* $\check{\mathfrak{C}}$. Suppose that $\check{\mathfrak{C}} \not\ni X$, that is, there is, by 1.4 and 1.5, an infinite discrete subset N of X such that the space $N \cup S$ is countably compact for some subset S of $\beta X - X = M$. Since N is discrete, either $N \cap \mathbf{N}$ or $N \cap (X - \mathbf{N})$ contains copies N_n of $\mathbf{N}$ which are mutually disjoint ($n = 1, 2, \ldots$). Thus S contains an accumulation point y_n of N_n for every n. M being a P-space and S being a subset of M, $\{y_n\}$ has no accumulation points in S. On the other hand, $N \cup S$ is countably compact, and hence $\{y_n\}$ has an accumulation point in N. But this contradicts the fact that N is discrete.

3.2. Example. Let A be a copy of $\mathbf{N}$ contained in $\beta\mathbf{N} - \mathbf{N}$ such that every point of A is a P-point of $\beta\mathbf{N} - \mathbf{N}$ and $\beta A \subset \beta\mathbf{N} - \mathbf{N}$. We shall prove that $X = \beta\mathbf{N} - \mathbf{N} - M$ belongs to $\check{\mathfrak{P}}_c \cap (\check{\mathfrak{C}} - \check{\mathfrak{P}}_F)$ where M is a set of all P-points of $\beta A - A$. We notice that $\beta\mathbf{N} - \mathbf{N}$ $(= K)$[2]) is a compactification of X and no point of M is a P-point of $\beta\mathbf{N} - \mathbf{N}$.

1) *X does not belong to* $\check{\mathfrak{P}}_F$. The copy A of $\mathbf{N}$ has no subsets with compact closure in X. For let B be any infinite subset of A, then by [3,9M2] and by the method of construction of M, B does not have a compact closure in X.

2) *X belong to* $\check{\mathfrak{C}}$. Let N be any infinite discrete subset of X and S any subset of M. To prove 3), it is sufficient, by 1.6 (ii), to show that $N \cup S$ is not countably compact. Since N is a discrete subset of $\beta A \subset \beta\mathbf{N}$ and all accumulation points of N are contained in βA, N has a copy of $\mathbf{N}$ by [3,9.10]. Thus, similarly to 3.1(3), we have $\check{\mathfrak{C}} \ni X$.

3) *X belongs to* $\check{\mathfrak{P}}$. Let $\check{\mathfrak{U}}$ be an infinite family of open sets of X such that for every subfamily $\{U_n\}$ of $\check{\mathfrak{U}}$ there is a filter $\check{\mathfrak{N}}$ of infinite subsets of $\mathbf{N}$ such that $(\check{\mathfrak{N}}, N_1, U_n, X) = \emptyset$. Since M is itself a P-space, M does not contain an infinite countably compact subset. Therefore, M, as a subspace of $\beta\mathbf{N} - \mathbf{N}$, has no inner points. Thus every set $U_n - A$ contains a P-point x_n of $\beta\mathbf{N} - \mathbf{N}$ by [3,9M3]. Every point of $\beta A (\subset X \cup M)$ is contained in a closure (in βA) of a (countable) subset of A. Since A is a copy of $\mathbf{N}$, we have $x_n \notin \beta A$ for every n. Moreover we lose no generality by assuming that $\{x_n\}$ is a copy of $\mathbf{N}$ by [3,9.10]. Thus we have two sets $A(= \{a_n\})$ and $B = \{x_n\}$ of P-points of $\beta\mathbf{N} - \mathbf{N}$, and $A \cup B$ is a discrete subset of $\beta\mathbf{N}$. Since $A \cup B$ is countable, there are open sets V_n and W_n of $\beta\mathbf{N}$ such that $V_n \ni x_n$, $W_n \ni a_n$,

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[2]) In this section, if K is a (fixed) compact space and X is dense, then we use the phrase "*a subset P of X is a copy of* $\mathbf{N}$" if $\bar{P}(\text{in } K) - P = \beta P - P = \beta\mathbf{N} - \mathbf{N} \subset K$.

$V_n \cap W_m = \emptyset$, $V_n \cap V_m = \emptyset$ and $W_n \cap W_m = \emptyset (n \neq m)$. Then $V(A) = \bigcup_n V_n$ and $V(B) = \bigcup_n W_n$ are disjoint open sets of $\beta\mathbf{N}$. Put $N(A) = \mathbf{N} \cap V(A)$ and $N(B) = \mathbf{N} \cap W(B)$. Since $\mathbf{N}$ is dense in $\beta\mathbf{N}$, both $N(A)$ and $N(B)$ are infinite and we have that $\beta\mathbf{N} \supset \beta N(A) \supset A$ and $\beta\mathbf{N} \supset \beta N(B) \supset B$. On the other hand, $\beta N(A) \cap \beta N(B) = \emptyset$ because $N(A)$ and $N(B)$ are subspaces of $\mathbf{N}$. Now suppose that $(\mathfrak{N}, N_1, U_n, X) = \emptyset$. Since $\beta\mathbf{N} - \mathbf{N}$ is compact, we have $C = (\mathfrak{N}, N_1, U_n, \beta\mathbf{N} - \mathbf{N}) \neq \emptyset$. Since C is a compact subset of M and M is itself a P-space, we can assume that C consists of only one point a. Thus we have $\{a\} = \{\mathfrak{N}, N_1, \{x_n\}, \beta\mathbf{N} - \mathbf{N})$ and hence $\beta N(B) \ni a$. On the other hand we have that $\beta N(A) \supset \beta A \supset M \ni a$. This is a contradiction.

3.3. Example. By [2,2.9] there is a countably compact subset R, containing $\mathbf{N}$, of $\beta\mathbf{N}$ whose cardinality is $\leqq \mathfrak{c}$. Let $\{x_n\}$ be a discrete set in $R - \mathbf{N}$. Then we shall show that $X = \beta\mathbf{N} - (R - \{x_n\})$ belongs to $\mathfrak{P}_c - \mathfrak{C}$. It is obvious that $\beta\mathbf{N} - \mathbf{N}(= K)$ [2]) is a compactification of X.

1) *X is countably compact.* If X is not countably compact, there is a countable discrete closed subset having a copy of $\mathbf{N}$ as subset, and hence $M = R - \{x_n\}$ contains a compact subset with cardinality $2^{\mathfrak{c}}$. This is a contradiction.

2) *X does not belong to* $\mathfrak{C}$. This follows from 1.5 and the countable compactness of $(R - \{x_n\}) \cup \{x_n\}$.

3) *X belongs to* $\mathfrak{P}$. Let $\mathfrak{A}$ be an infinite disjoint family of open sets of X such that for every subfamily $\{U_n\}$ of A, there is a filter $\mathfrak{N}$ of infinite subsets of $\mathbf{N}$ such that $(\mathfrak{N}, N_1, U_n, X) = \emptyset$. Since $\beta\mathbf{N} - \mathbf{N}$ is compact, $C = (\mathfrak{N}, N_1, U_n, \beta\mathbf{N} - \mathbf{N})$ is a compact subset of $\beta\mathbf{N} - \mathbf{N}$, and hence C is a compact subset of R. Since the cardinality of R is $\leqq \mathfrak{c}$, C must be a finite set by [3,9.11]. Thus we can assume that C consists of only one point a. Every U_n contains distinct P-points x_n, y_n of $\beta\mathbf{N} - \mathbf{N}$. From this, we obtain a contradiction as in the proof of (3) in 3.2.

References

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Резюме

НЕКОТОРЫЕ КЛАССЫ СЧЕТНО КОМПАКТНЫХ ПРОСТРАНСТВ

ТАКЕСИ ИСИВАТА (Takesi Isiwata), Токио

В настоящей заметке исследуются соотношения между некоторыми классами счетно компактных пространств, введенных З. Фроликом.