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Zdeněk Frolík

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ON BIANALYTIC SPACES

Zdeněk Frolík, Praha

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Bianalytic spaces are introduced and studied. A metrizable space X is bianalytic if and only if X is a separable Borel subset of some complete metrizable space (and consequently, if $X \subset Y$ and Y is metrizable, then X is a Borel subset of Y). An intrinsic characterization of Borel subsets of complete metrizable separable spaces is given.

1. NOTATION AND TERMINOLOGY

1.1. A *centered family* of sets is a family with the finite intersection property.

1.2. If $\mathcal{M}$ is a family of sets, then $\mathcal{M}_\sigma$ and $\mathcal{M}_\delta$ will be used to denote the family consisting of all countable unions and countable intersections of sets from $\mathcal{M}$. The meaning of $\mathcal{M}_{\sigma\delta}$ is clear.

1.3. Let $\mathcal{M}$ be a family of sets. The symbol $\mathcal{B}(\mathcal{M})$ will be used to denote the smallest family of sets containing $\mathcal{M}$ and closed under countable unions and countable intersections. Let M be the union of $\mathcal{M}$. The complemented part of $\mathcal{M}$, denoted by compl. p. $\mathcal{M}$, is the family $\{P;\ P \in \mathcal{M},\ (M - P) \in \mathcal{M}\}$.

Finally, the symbol $\mathcal{B}^*(\mathcal{M})$ will be used to denote the smallest family $\mathcal{N}$ of sets containing $\mathcal{M}$, closed under countable unions and intersections and such that $P \in \mathcal{N}$ implies $(M - P) \in \mathcal{N}$. Clearly

$$\mathcal{M} \subset \text{compl. p. } \mathcal{B}(\mathcal{M}) \subset \mathcal{B}(\mathcal{M}) \subset \mathcal{B}^*(\mathcal{M})$$

and

$$\text{compl. } p.\ \mathcal{B}^*(\mathcal{M}) = \mathcal{B}^*(\mathcal{M}) .$$

1.4. The letter N always denotes the discrete space of all positive integers. The letter S always denotes the set of all finite sequences of positive integers. The set of all $s \in \mathsf{S}$ of lenght n will be denoted by S_n. The topological product N^N will be denoted by Σ.

If $\sigma = \{\sigma_1, \sigma_2, \ldots\} \in \Sigma$ and $s = \{s_1, \ldots, s_n\} \in \mathsf{S}_n$ then $\sigma \succ s$ means that s is a section of σ, *i.e.* that $s_i = \sigma_i$ for $i \leqq n$.

1.5. A *determining system* in a family of sets $\mathcal{M}$ is a mapping $M = \{M(s)\}$ of S

into $\mathcal{M}$. A determining system is regular if always $M(s_1, \ldots, s_{n+1}) \subset M(s_1, \ldots, s_n)$. The *nucleus* of a determining system M is the set

$$\mathcal{A}(M) = \bigcup_{\sigma \in \Sigma} \bigcap_{s \prec \sigma} M(s) .$$

$\mathcal{A}(\mathcal{M})$ denotes the family of $A(M)$, where M varies over all determining systems in $\mathcal{M}$. The sets from $\mathcal{A}(\mathcal{M})$ are called *$\mathcal{M}$-Souslin*, or *Souslin with respect* to $\mathcal{M}$.

1.6. All topological spaces under consideration are supposed to be *completely regular*. If X is a space and $\mathcal{M}$ is a family of subsets of X, then $\overline{\mathcal{M}}^X$ or merely $\overline{\mathcal{M}}$ denotes the family consisting of closures of all sets from $\mathcal{M}$.

1.7. If X is a space, then

1.7.1. $\mathsf{F}(X)$ and $\mathsf{G}(X)$ denote the family of all closed (all open, respectively) subsets of X.

1.7.2. $\mathsf{Z}(X)$ denotes the family of all zero-sets of X, *i.e.* the family of all $f^{-1}[0]$, where f varies over all continuous functions on X.

1.7.3. $\mathsf{K}(X)$ denotes the family of all compact subspaces of X.

1.8. A mapping of a space X onto a space Y will be called *perfect* if f is both continuous and closed and if the inverse images of points are compact.

1.9. $\beta(X)$ will always denote the Čech-Stone compactification of X.

1.10. A class D of spaces will be called A-closed if D is closed under continuous mappings. D is A^{-1}-closed if inverse images under continuous mappings of spaces from D belong to D.

A class C of spaces is an *A-base* of D if each space from D is a continuous image of a space from C. Using perfect mappings instead of continuous we obtain the definitions of a P-closed class, a P^{-1}-closed class, and a *P^{-1}-base*, respectively.

2. PRELIMINARIES

If X is a metrizable space then

$$\mathcal{B}(F(X)) = \mathcal{B}^*(F(X)) = \mathcal{B}(\mathsf{G}(X)) \tag{1}$$

because every open set is an F_σ. In this case the elements of (1) are called Borel sets of X. The theory of Borel sets was developped in the case of complete metrizable separable spaces. In this case $M \subset X$ is a Borel set in X if and only if both M and $X - M$ are analytic in the classical sense (that means, both M and $X - M$ are continuous images of the space Σ of all irrational numbers of the unit interval $\langle 0, 1 \rangle$ of real numbers). The proofs of the majority of deeper results concerning Borel sets essentially depend on the theory of analytic spaces.

Each of the following families could be considered as a generalization of Borel subsets of metrizable spaces:

$$\mathcal{B}(\mathsf{F}(X)) , \quad \mathcal{B}(\mathsf{G}(X)) , \quad \mathcal{B}^*(\mathsf{F}(X)) = \mathcal{B}^*(\mathsf{G}(X)) ,$$
$$\mathcal{B}(\mathsf{Z}(X)) = \mathcal{B}^*(\mathsf{Z}(X)) ,$$

compl. p. $\mathscr{B}(\mathsf{F}(X))$. All these families are indentical if X is metrizable. In general all these families are different. The study of each of the above listed families is of certain importance. With the exception of the compl. p. $\mathscr{B}(\mathsf{F}(X))$, all of these families has been studied by several authors, usually in connection with measure theory in topological spaces. V. ŠNEIDER introduced the family $\mathscr{B}(\mathsf{K}(X))$ as a generalization of Borel subsets of complete metrizable separable spaces. Continuous images of spaces belonging to $\mathscr{B}(\mathsf{K}(X))$ for some X, the so-called analytic spaces, were studied by G. CHOQUET, M. SION and the author.

In the present note we shall study the above listed families for bianalytic X. A space will be called bianalytic if both X and $K - X$ are analytic for some compact space K containing X, or equivalently, if X is a Baire set[1]) of some compact spaces.

In section 3 an interval definition of analytic spaces is given and some older results of G. Choquet, M. Sion and the author are reproved. Moreover certain new theorems are proved.

Section 4 is devoted to a generalization of the first Luzin separation theorem. It is proved that if $\{X_n\}$ is a disjoint sequence of analytic subspaces of a space Y, then there exists a disjoint sequence $\{B_n\}$ of Baire sets of Y (*i.e.* $B_n \in \mathscr{B}(\mathsf{Z}(Y))$ with $B_n \supset X_n$.

From this fact two theorems concerning the equality of $\mathscr{B}(\mathsf{F}(X))$, $\mathscr{B}(\mathsf{Z}(X))$ and the complemented part of $\mathscr{B}(\mathsf{F}(X))$ are deduced.

In section 5 bianalytic spaces are introduced and studied.

3. ANALYTIC SPACES

By definition, a space X is an E-space if X is an $\mathsf{F}_{\sigma\delta}$ in the Čech-Stone compactification $\beta(X)$ of X. If X is a $\mathsf{K}_{\sigma\delta}(Y)$ for some $Y \supset X$, then X is an E-space. The continuous images of E-spaces are said to be analytic. By [4], a space X is analytic if and only if there exists an analytic structure in X. For convenience, let us recall that an analytic structure in a space X is a complete regular determining system M in X such that $\mathscr{A}(M) = X$, and a complete determining system in a space X is a determining system M, where $M(s) \subset X$, such that the following condition is fulfilled: If $\mathscr{M}$ is a centered family of subsets of X and if there exists a $\sigma \in \Sigma$ with

$$s \in \mathsf{S}\,, \quad s \prec \sigma \Rightarrow M(s) \supset \mathsf{L}(s) \in \mathscr{M} \quad {}^{2})$$

then the intersection of $\mathscr{M}$ is non-void.

Proposition 1. *Let M be a regular determining system in a space X and* put

$$M(\sigma) = \bigcap_{s \prec \sigma} \overline{M(s)}\,. \tag{2}$$

If M is complete, then all $M(\sigma)$ are compact and the following condition is satisfied:

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[1]) A Baire set of X is an element of $\mathscr{B}(\mathsf{Z}(X))$.

[2]) Such a family $\mathscr{M}$ will be called an M-Cauchy family.

(*) *If U is an open set containing an $M(\sigma)$, then there exists a neighborhood V of σ in Σ, such that*

$$\tau \in V \Rightarrow M(\tau) \subset U \,. \tag{3}$$

Conversely, if M is a mapping of Σ to $\mathsf{K}(X)$ such that the condition (*) is fulfilled and

$$\bigcup_{\sigma \in \Sigma} M(\sigma) = X \,, \tag{4}$$

then $M = \{M(s)\}$, where

$$M(s) = \bigcup_{\sigma \succ s} M(\sigma) \,, \tag{5}$$

is a complete determining system in X with $\mathscr{A}(M) = X$.

Proof. The first part of the proposition was proved in [5]. Let $M = \{M(s)\}$ satisfy the condition of the second part of Proposition 1. Let $\mathscr{M}$ be a maximal M-Cauchy family. There exists a $\sigma \in \Sigma$ such that

$$s \in \sigma \Rightarrow M(s) \in \mathscr{M} \,. \tag{6}$$

To prove $\bigcap \overline{\mathscr{M}} \neq \emptyset$, it is sufficient to show that $\overline{\mathscr{M}} \cap M(\sigma)$ is a centered family of sets. From condition (*) it follows immediately that if a closed set F meets each $M(s)$, then F meets $M(\sigma)$. Thus $\overline{\mathscr{M}} \cap M(\sigma)$ is centered and the proof is complete.

As a corollary of the preceding Proposition 1 we have:

Theorem 1. *A space X is analytic if and only if there exists a mapping M of Σ to $\mathsf{K}(X)$ such that the union of all $M(\sigma)$, $\sigma \in \Sigma$, is X, and the condition (*) is fulfilled.*

Let us recall (for proofs see [5]), that the class of all analytic spaces is A-closed, P^{-1}-closed, countably productive[3]) and F-hereditary. Every analytic space is a Lindelöf space, and consequently, a normal space. A metrizable space X is analytic if and only if X is analytic in the classical sense, which means that X is the image under a continuous mapping of the space Σ of irrational numbers of the unit interval of real numbers. Finally, the family of all analytical subspaces of a given space is closed under the operation ($\mathscr{A}$), and if X is an analytic subspace of a space Y, then $X \in \mathscr{A}(\mathsf{F}(Y))$.

The following result will not be used in the sequel:

Theorem 2. *A space X is the inverse image under a perfect mapping of Σ (i.e. $X \in P^{-1}(\Sigma)$) if and only if there exists an analytical structure U in X such that the following two conditions are fulfilled:*

(a) *$\{U(s);\ s \in \mathsf{S}_n\}$ is disjoint for every n.*

(b) *all $U(s)$ are open (and hence closed) and non-void.*

Proof. For every $s \in \mathsf{S}$ put

$$\Sigma(s) = \{\sigma;\ \sigma \in \Sigma,\ \sigma \succ s\} \,. \tag{7}$$

Clearly $\Sigma = \{\Sigma(s)\}$ is an analytical structure in Σ satisfying (a) and (b) reading Σ instead of U). Let f be a perfect mapping of a space X onto the space Σ. For every $s \in S$ put

$$U(s) = f^{-1}[\Sigma(s)] .$$

Clearly the conditions (a) and (b) are fulfilled. By proposition 1, U is an analytic structure in X.

Conversely, let U be an analytic structure in X satisfying (a) and (b). Put

$$U(\sigma) = \bigcap_{s \succ \sigma} U(s) . \tag{8}$$

By our assumptions the sets $U(\sigma)$ are compact non-void and disjoint. For $x \in U(\sigma)$ put $f(x) = \sigma$. It is easy to see that f is a perfect mapping of X onto Σ. The continuity is clear from the facts that the family of all sets $\Sigma(s)$, $s \in S$ is an open base of Σ and the sets $U(s) = f^{-1}[\Sigma(s)]$ are open. The sets $f^{-1}[\sigma] = U(\sigma)$ are compact because U is an analytic structure. It remains to prove f is a closed mapping. Let F be closed in X and let σ be a point of $\Sigma - f[F]$. Since $F \cap U(\sigma) = \emptyset$, we have by Proposition 1 that there exists a $s \prec \sigma$ with $U(s) \cap F = \emptyset$. It follows that $\Sigma(s) \cap f[F] = \emptyset$ which shows that σ is not in the closure of $f[F]$. Thus $f[F]$ is closed. This completes the proof.

4. SEPARATION OF ANALYTIC SPACES

By a classical theorem of Luzin (*cf.* [7], 393), if X and Y are disjoint analytic subsets of a complete metrizable space T, then there exists a Borel set B of T such that $X \subset B \subset T - Y$. This result has the following generalization.

Theorem 3. *Let X_1 and X_2 be two disjoint analytic subspaces of a space X. There exists a set $B \in \mathscr{B}(Z(X))$ such that*

$$X_1 \subset B \subset X - X_2 . \tag{9}$$

Proof. For convenience, two subsets X_1 and X_2 of X will be called B-separated if there exists a set $B \in \mathscr{B}(Z(X))$ such that (9) holds. First we shall prove the following simple result:

Lemma. *If $P = \bigcup_{n=1}^{\infty} P_n$ and $Q = \bigcup_{n=1}^{\infty} Q_n$ are subsets of X, and every P_n and Q_n are B-separated, then P and Q are separated.*

Indeed, if B_{nm} B-separates P_n and Q_n, then the set

$$B = \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} B_{nm}$$

separates P and Q.

[3]) Countable products of analytic spaces are analytic.

Now let X_1 and X_2 be two disjoint analytic subspaces of X. Let $P = \{P(s)\}$ and $Q = \{Q(s)\}$ be analytic structures in X_1 and X_2, respectively. For every $t \in S$ put

$$P_1(t) = \bigcup_{\sigma \in \Sigma(t)} \bigcap_{r \prec \sigma} P(r),$$

$$Q_1(t) = \bigcup_{\sigma \in \Sigma(t)} \bigcap_{r \prec \sigma} Q(r).$$

Clearly

(11) $$P_1(s) \subset P(s), \quad Q_1(s) \subset Q(s)$$

and

$$P_1(\{s_1, \ldots, s_n\}) = \bigcup_{k=1}^{\infty} P_1(\{s_1, \ldots, s_n, k\}),$$

$$Q_1(\{s_1, \ldots, s_n\}) = \bigcup_{k=1}^{\infty} Q_1(\{s_1, \ldots, s_n, k\}).$$

Now suppose that X_1 and X_2 are not B-separated. Using the above Lemma one can construct by induction $\sigma, \tau \in \Sigma$ such that for every $n = 1, 2, \ldots$ the sets $P_1(\{\sigma_1, \ldots, \sigma_n\})$ and $Q_1(\{\tau_1, \ldots, \tau_n\})$ are not separated. Put

$$P(\sigma) = \bigcap_{s \prec \sigma} \overline{P(s)}, \quad Q(\tau) = \bigcap_{t \prec \tau} \overline{Q(s)}.$$

The sets $P(\sigma)$ and $Q(\tau)$ are disjoint (because X_1 and X_2 are disjoint) and compact because P and Q are analytical structures. It follows that there exists a zero-set Z such that

$$P(\sigma) \subset \operatorname{int} Z \subset Z \subset X_2 - Q(\tau).$$

By Proposition 1 there exists an n such that

$$P(\{\sigma_1, \ldots, \sigma_n\}) \subset Z,$$

$$Q(\{\tau_1, \ldots, \tau_n\}) \subset X - Z.$$

By (11) $P_1(\{\sigma_1, \ldots, \sigma_n\})$ and $Q_1(\{\tau_1, \ldots, \tau_n\})$ are also B-separated which contradicts our construction of σ and τ and completes the proof.

Note 1. In [9] the following result is proved: If X_1 and X_2 are disjoint subsets of a space X and if $X_1, X_2 \in \mathscr{A}(\mathsf{K}(X))$, then there exists a $B \in \mathscr{B}(\mathsf{K}(X))$ such that $X_1 \subset B \subset X - X_2$. The proof of this theorem is similar to that of Theorem 3.

Note 2. The proof of Theorem 3 yields the following result (in particular, the result from Note 1): Let P and Q be two determining systems in a space X and let $\mathscr{M}$ be a family of subsets of X which is closed under countable unions and intersections. If for every $\sigma \in \Sigma$ and $\tau \in \Sigma$, there exists a positive integer n and an $M \in \mathscr{M}$ such that

$$P(\{\sigma_1, \ldots, \sigma_n\}) \subset M \subset X - Q(\{\tau_1, \ldots, \tau_n\})$$

then there exists an M in $\mathscr{M}$ with

$$\mathscr{A}(P) \subset M \subset X - \mathscr{A}(Q).$$

Theorem 4. *If $\{X_n\}$ is a disjoint sequence of analytic subspaces of a space X then there exists a disjoint sequence $\{B_n\}$ of sets from $\mathscr{B}(\mathsf{Z}(X))$ such that $Z_n \subset B_n$.*

Proof. By Theorem 3 for every (n, m), $n \neq m$ there exists a $B(n, m)$ such that

$$X_n \subset B(n, m) \subset X - X_m .$$

Put

$$B_n = \bigcap_{\substack{m=1 \\ m \neq n}}^{\infty} B(n, m) .$$

Clearly $\{B_n\}$ has the required properties.

Note 3. The preceding theorem will be used essentially in section 5.

Theorem 5. *If X is an analytic space then*

$$\mathscr{B}(\mathsf{Z}(X)) = \text{compl. p. } \mathscr{B}(\mathsf{F}(X)) . \tag{12}$$

Proof. Since $\mathsf{Z}(X) \subset \mathsf{F}(X)$ and $\mathscr{B}(\mathsf{Z}(X)) = \mathscr{B}^*(\mathsf{Z}(X))$ for every space X, we have the inclusion $\subset$. If X is an analytic space and both $M \subset X$ and $X - M$ belong to $\mathscr{B}(\mathsf{F}(X))$, then both M and $X - M$ are analytic, because closed subspaces of analytic spaces are analytic and the family of all analytic subspaces is closed under the operation $\mathscr{A}$ and clearly $\mathscr{A}(\mathsf{F}(X)) \supset \mathscr{B}(\mathsf{F}(X))$. By Theorem 3 there exists a $\mathsf{Z} \in \mathscr{B}(\mathsf{Z}(X))$ with $M \subset \mathsf{Z} \subset X - M$. It follows that $\mathsf{Z} = M$.

Note 4. I do not know of any reasonable necessary and sufficient condition for (12) to hold.

Theorem 6. *If X is an analytic space and*

$$\mathscr{B}(\mathsf{F}(X)) = \mathscr{B}(\mathsf{Z}(X)) \tag{13}$$

then X is a perfectly normal space.

Proof. If (13) holds, then every open set is an analytic space, and hence, a Lindelöf space. Thus every open set is an F_σ. Since X is analytic, X is normal. Thus X is perfectly normal.

Note 5. Obviously, if X is a perfectly normal space, then $\mathsf{F}(X) = \mathsf{Z}(X)$, and consequently (13) holds. I do not know whether (13), implies that X is perfectly normal. (This is an old problem of M. Katětov [6].)

5. BIANALYTIC SPACES

Definition. A space X will be called bianalytic if both X and $\beta(X) - X$ are analytic.

Theorem 7. *If f is a perfect mapping of X onto Y, then X is a bianalytic space if and only if Y is such.*

Proof. Let g be the Čech-Stone mapping of $\beta(X)$ onto $\beta(Y)$. Since f is perfect, by well known result we have

$$g[\beta(X) - X] = \beta(Y) - Y. \tag{14}$$

Thus if both X and $\beta(X) - X$ are analytic, then also both Y and $\beta(Y) - Y$ are analytic. From (14) it follows at once that the restriction of g to $\beta(X) - X$ is a perfect mapping onto $\beta(Y) - Y$. Since the inverse image under a perfect mapping of an analytic space is analytic if Y is bianalytic then X is analytic.

Theorem 8. *The following conditions on a space X are equivalent:*

(1) *X is a bianalytic space.*

(2) *There exists a compactification K of X such that both X and $K - X$ are analytic spaces.*

(3) *X is analytic and for every compactification K of X the space $K - X$ is analytic.*

(4) $X \in \mathscr{B}(\mathsf{Z}(\beta(X)))$.

(5) *For some compactification K of X we have $X \in \mathscr{B}(\mathsf{Z}(K))$.*

(6) *For every space $Y \supset X$, $\overline{X} = Y$, we have $X \in \mathscr{B}(\mathsf{Z}(Y))$.*

Proof. The equivalence of conditions (1)–(3) follows from Theorem 7. From Theorem 6 it follows at once that (1) implies (4). Clearly (4) implies (5). If K is an analytic space, then every set from $\mathscr{A}(\mathsf{F}(K))$ is an analytic space. Since $\mathscr{A}(\mathsf{F}(K))$ contains $\mathscr{B}(\mathsf{Z}(K))$, we have that (5) implies (2). It remains to prove that (6) is equivalent with (1)–(5). Obviously (6) implies (4). Finally, suppose (3). Let $Y \supset X$, $\overline{X} = Y$. Let K be a compactification of Y. By (2), $X \in \mathscr{B}(\mathsf{Z}(K))$. Obviously $X \in \mathscr{B}(\mathsf{Z}(Y))$. This completes the proof.

Theorem 9. *Closed subspaces of bianalytic spaces are bianalytic.*

Proof. Let X be closed in a bianalytic space Y. Then the space

$$Z = \overline{X}^{\beta(Y)} X$$

is a closed subspace of the analytic space $\beta(Y) - Y$ and consequently Z is an analytic space. By Theorem 8, X is a bianalytic space.

Theorem 10. *The topological product of a countable number of bianalytic spaces is a bianalytic space.*

Proof. Let X_n, $n \in \mathsf{N}$, be analytic and let X be the topological product of all X_n. Let K be the topological product of all $\beta(X_n)$. Since the topological product of analytic spaces is analytic, it is easy to see that $K - X$ is the union of a countable number of analytic spaces. Since $\mathsf{A}(K)$ is closed under Souslin's operation $\mathscr{A}$, in particular under countable unions, $K - X$ is an analytic space. By theorem 8 the space X is bianalytic.

Proposition 2. *If Y is a bianalytic space and both $X \subset Y$ and $Y - X$ are analytic, then X is a bianalytic space.*

Proof. Consider the space

$$Z = \overline{X}^{\beta(Y)} - X . \tag{15}$$

We have

$$Z = (Z, Y) \cup (Z \cap (\beta(Y) - Y)) . \tag{16}$$

The first term of the right side of the above equality is closed in the analytic space $Y - X$ and hence it is analytic. The second term is closed in the analytic space $\beta(Y) - Y$ and hence it is also analytic. Thus Z is analytic, and finally by Theorem 6, the space X is bianalytic.

If X is a bianalytic subspace of a space Y, then $Y - X$ may fail to be an analytic space. Moreover, open subspaces of compact spaces, in general, are not analytic. For example, if M is an uncountable discrete space and K is a compactification of M, then M is open in K, but M is not an analytic space because M is not a Lindelöf space. On the other hand we shall prove the following result.

Proposition 3. *If $X \subset Y$, $Y - X$ is dense in Y and both Y and $Y - X$ are bianalytic, then X is analytic (and by Proposition 2 bianalytic.) In particular, if Y is a bianalytic space and both $X \subset Y$ and $Y - X$ are dense in Y, then X is a bianalytic space if and only if Y is such.*

Proof. Let K be a compactification of Y. Obviously

$$X = [K - (Y - X)] \cap Y .$$

Since Y is bianalytic and $Y - X$ is dense in Y and hence in K, the first member of the right side is an analytic space. Since Y is (by our assumption) analytic, the space X is also analytic.

Theorem 11. *A subspace X of a bianalytic space Y is bianalytic if and only if*

$$X \in \text{compl. p. } \mathscr{B}(\mathsf{F}(\overline{X}^Y)) \; (= \mathscr{B}(\mathsf{Z}(\overline{X}^Y))) . \tag{17}$$

The proof follows at once from propositions 2 and 3. As an immediate consequence of the preceding result we have the following assertion.

Theorem 12. *A metrizable space X is bianalytic if and only if X is separable and an absolute Borel set, i.e., if Y is a separable metrizable space and $X \subset Y$, then $X \in \mathscr{B}(F(Y))$.*

Note 6. The union of two bianalytic subspaces of a given space may fail to be bianalytic. Indeed, N is a bianalytic space and every one-point set is bianalytic. However, $X = N \cup (x) \subset \beta(\mathsf{N})$, where $x \in \beta(\mathsf{N}) - \mathsf{N}$ is not bianalytic, because $\beta(\mathsf{N}) = \beta(X)$ and $\beta(X) - X$ is not a Lindelöf space.

Note 7. One-to-one continuous images of a bianalytic space may fail to be bianalytic. Indeed, the space X from Note 6 is a one-to-one continous image of N.

Theorem 13. *The intersection of a countable number of bianalytic subspaces of a given space is a bianalytic space.*

Proof. Let X_n, $n \in \mathsf{N}$ be bianalytic subspaces of Y and let X be the intersection of all X_n. Without a loss of generality we may assume that Y is compact. Let K be the closure of X in Y. Clearly $Y_n = K \cap X_n$ are also bianalytic. Since X is dense in K and $X \subset Y_n \subset K$, the space K is a compactification of each Y_n. Thus $K - Y_n$ are analytic, and consequently, the set

$$K - X = \bigcup_{n=1}^{\infty} (K - Y_n)$$

is analytic.

6. INTERNAL CHARACTERIZATION OF METRIZABLE BIANALYTIC SPACES

By a well-known classical theorem the image under a one-to-one continuous mapping of an absolute Borel set is an absolute Borel set.

By Note 6 of Section 5 the image under a one-to-one continuous mapping of a bianalytic space may fail to be a bianalytic space. In this section a class of spaces invariant under one-to-one continuous mappings is defined, such that the metrizable spaces from this class are precisely the absolute Borel separable sets.

Proposition 4. *Let X be a subspace of a space Y. Let there exists an analytic structure M in X such that*

(a) *$\{M(s);\ s \in \mathsf{S}_n\}$ are disjoint,*

(b) *every $M(s)$ is an analytic space.*

Then $X \in \mathscr{B}(\mathsf{F}(X))$. If, in addition, the closures of $M(s)$ in Y are zero-sets, then $X \in \mathscr{B}(\mathsf{Z}(X))$.

Proof. By Theorem 4 there exist sets $\mathsf{Z}(s) \in \mathscr{B}(\mathsf{Z}(X))$ such that $\mathsf{Z}(s) \supset M(s)$ and that the families

$$\{\mathsf{Z}(s);\ s \in \mathsf{S}_n\} \tag{18}$$

are disjoint. We may assume $\mathsf{Z}(i_1, \ldots, i_{n+1}) \subset \mathsf{Z}(i_1, \ldots, i_n)$. Put

$$F(s) = \mathsf{Z}(s) \cap \overline{M(s)}^Y .$$

Since the families (18) and hence also the families $\{F(s),\ s \in \mathsf{S}_n\}$ are disjoint, we have

$$\mathscr{A}(F) = \bigcap_{n=1}^{\infty} \bigcup_{s \in \mathsf{S}_n} F(s) . \tag{19}$$

But

$$X = \mathscr{A}(M) \supset \mathscr{A}(F) \supset X .$$

Thus

$$X = \bigcap_{n=1}^{\infty} \bigcup_{s \in S_n} F(s) .$$

Clearly $M(s) \in \mathscr{B}(\mathsf{F}(Y))$ and hence $X \in \mathscr{B}(\mathsf{F}(Y))$. If in addition the sets $\overline{M(s)}^Y$ are zero-sets in Y, or more generally, if $\overline{M(s)}^Y \in \mathscr{B}(\mathsf{Z}(Y))$, then $F(s) \in \mathscr{B}(\mathsf{Z}(Y))$. This completes the proof.

As an immediate consequence of Proposition 4 we have the following results:

Theorem 14. *If there exists an analytic structure M in a space X such that the conditions* (a) *and* (b) *from Proposition* 4 *are fulfilled, and if exists a perfectly normal compactification of X (in particular, if X is metrizable), then X is a bianalytic space.*

Note 8. If $M = \{M(s)\}$ is an analytic structure in X such that the families $\mathscr{M}_n = \{M(s);\ s \in S_n\}$ are disjoint, then $\{\mathscr{M}_n\}$ is a complete sequence[4]) of countable disjoint coverings of X such that $\mathscr{M}_{n+1}$ refines $\mathscr{M}_n$. Conversely, if $\{\mathscr{M}_n\}$ is a complete sequence of countable disjoint coverings of X such that $\mathscr{M}_{n+1}$ refines $\mathscr{M}_n$, then there exists an analytical structure M in X such that

$$\mathscr{M}_n = \{M(s);\ s \in S_n\} .$$

Theorem 15. *A metrizable space X is bianalytic (= absolute Borel separable space) if and only if there exists a complete sequence $\{\mathscr{M}_n\}$ of countable disjoint coverings of X such that all sets from $\bigcup_{n=1}^{\infty} \mathscr{M}_n$ are analytic.*

Proof. By Theorem 14 and the preceding Note 8, the condition is sufficient. Conversely, let X be bianalytic. By a well-known classical theorem X is a disjoint union of a countable set X_1 and a set X_2 which is a one-to-one continous image of Σ. Denoting this mapping by f, let $\mathscr{M}_n$ be the covering of X consisting of all one-point sets (x), $x \in X_1$ and all $f[\Sigma(s)]$, $s \in S_n$. Clearly $\{\mathscr{M}_n\}$ is a complete sequence of countable disjoint coverings of X, the sets from $\bigcup_{n=1}^{\infty} \mathscr{M}_n$ are analytic and $\mathscr{M}_{n+1}$ refines $\mathscr{M}_n$. This completes the proof.

We have proved that any bianalytic metrizable space has a complete sequence $\{\mathscr{M}_n\}$ of countable disjoint coverings such that all $M \in \bigcup_{n=1}^{\infty} \mathscr{M}_n$ are analytic. All one-to-one images of inverse images under perfect mappings (see Theorem 2) also have such complete sequences. For the sake of completeness we shall prove the following result:

[4]) For definition see [5], [4] or [3].

Theorem 16. *A space X is the one-to-one continuous image of the inverse image under a perfect mapping of the space Σ (of all irrational numbers) if and only if there exists an analytical structure M in X such that*

(a) *the coverings $\{M(s); s \in \mathsf{S}_n\}$ are disjoint, and*

(b) *the sets $M(\sigma) = \bigcap_{s \prec \sigma} M(s)$ are non-void and disjoint.*

Proof. By Theorem 2 the condition is necessary. Conversely, let M be an analytic structure in X such that the conditions (a) and (b) are fulfilled. Let us define a new topology in X such that $M(\sigma)$ are subspaces and the sets $M(s)$ are open. Denote this space by Y. It is easy to see that M is an analytic structure in Y satisfying the conditions (a) and (b) from Theorem 2. Thus Y is the inverse image under a perfect mapping of Σ. This completes the proof.

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Резюме

О БИАНАЛИТИЧЕСКИХ ПРОСТРАНСТВАХ

ЗДЕНЕК ФРОЛИК (Zdeněk Frolík), Прага

Если X — пространство, то $\mathsf{Z}(X)$ обозначает совокупность всех множеств вида $f^{-1}(0)$, где f – вещественная непрерывная функция на X. Наименьшая система множеств, содержащая данную систему $\mathscr{M}$ и замкнутая по отношению к счетным пересечениям и соединениям, обозначается через $\mathscr{B}(\mathscr{M})$. Следуя

М. Катетову, множества, принадлежащие системе $\mathscr{B}(\mathsf{Z}(X))$, называются множествами Бэра пространства X.

В статье рассматриваются пространства, так наз. бианалитические, которые являются множествами Бэра в некотором компактном пространстве. Оказывается, что вполне регулярное пространство X является бианалитическим, если и только если для одного и, следовательно, для всякого компактного K, содержащего X как плотное множество, пространства X и $K - X$ являются аналитическими пространствами (в смысле Шоке). Доказательства основаны на обобщении первой теоремы Лузина об отделимости аналитических пространств.

В заключение дается внутренняя характеризация борелевских подмножеств полно метризуемых сепарабельных пространств.