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Czechoslovak Mathematical Journal, Vol. 11 (1961), No. 4, 558–569

Persistent URL: http://dml.cz/dmlcz/100484

Чехословацкий математический журнал, т. 11 (86) 1961, Прага

SPECTRAL THEORY OF SEMI-GROUPS CONNECTED WITH DIFFUSION PROCESSES AND ITS APPLICATION

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(Received December 3, 1959)

In this paper a connection between the semi-group defined by transition probabilities of a diffusion process bounded on one side and a semi-group in a Hilbert space is established. This connection is then used to study the limit behaviour of the probability distribution of the process.

The present paper is devoted to the application of the theory of singular boundary problems for second order differential equations to diffusion processes. Processes bounded on one side by a reflecting, absorbing or elastic barrier are studied. The main subject of the paper is the study of the following problem posed by J. HÁJEK for the case of absorbing and elastic barriers:

To find the conditions for the convergence to a limit distribution of the probability distribution of the position of the particle at time t under the condition that the particle was not absorbed before the time t. (Compare also [4].)

We shall study diffusion processes restricted to the interval $(0, \infty)$ which are usually described by the aid of the diffusion equation

$$\frac{\partial}{\partial t} u = \frac{\partial}{\partial x} \left\{ \frac{\partial}{\partial x} u - b(x) u \right\}$$

with boundary condition (1). We shall characterize them by means of semi-groups in the following way:

In the space $\boldsymbol{L}$ of functions integrable on $(0, \infty)$, let us consider the operator $\Omega u = \mathrm{d}/\mathrm{d}x \{\mathrm{d}/\mathrm{d}x\, u - b(x) u\}$. To the domain of definition of Ω there belong those $u \in \boldsymbol{L}$ for which $\mathrm{d}/\mathrm{d}x \{\mathrm{d}/\mathrm{d}x\, u - b(x) u\} \in \boldsymbol{L}$ and which satisfy the boundary condition

$$(1) \qquad (1 - p) u = p(u' - bu)_{x=0}, \quad \text{where} \quad 0 \leqq p \leqq 1,$$

$\mathrm{d}/\mathrm{d}x$ means the derivative of an absolutely continuous function, u' is the derivative with respect to x, p is a constant. We presume that $b(x)$ is a real function which has a continuous derivative $b'(x)$ on $\langle 0, \infty)$. We shall suppose throughout that $b(x)$ satisfies the following condition:

If we set $B(x) = \int_0^x b(s)\,\mathrm{d}s$, then the functions $\int_0^x \exp B(s)\,\mathrm{d}s \exp - B(x)$ and $\int_0^x \exp - B(s)\,\mathrm{d}s \exp B(x)$ are not integrable. (This means that in the terminology of W. FELLER, infinity is the natural boundary of the process.)

It is a classical result of Feller ([2], p. 495) that under this condition the operator Ω is the generator of a semi-group $\{V_t, t \geqq 0\}$ of contraction operators in L. This semi-group defines unique transition densities such that $V_t g = \int_0^\infty f^{(t)}(y, x)\, g(y)\,\mathrm{d}y$.

We shall use the notation

$$v(t, x; g) = \int_0^\infty f^{(t)}(y, x)\, g(y)\,\mathrm{d}y\,.$$

Thus if g is a density of initial probability distribution, then $v(t, x; g)$ is the probability distribution inside $\langle 0, \infty)$ at time t. We remark that the value $p = 0$ in condition (1) corresponds to an absorbing barrier and the value $p = 1$ to a reflecting barrier.

Let us set $\beta(x) = \int_0^x \exp - B(s)\,\mathrm{d}s$ and denote by $\mathscr{L}$ the Hilbert space of functions on $(0, \infty)$, whose squared absolute value is integrable with respect to the measure determined by the distribution function $\beta(x)$. The differential operator

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{\mathrm{d}}{\mathrm{d}x} u - b(x)\, u\right\} = \Omega_0$$

may also be considered as an operator acting on elements u of $\mathscr{L}$. In this space it is formally self-adjoint in the sense that for arbitrary functions φ and ψ with absolutely continuous first derivative and vanishing in some neighborhood of zero and infinity we have

$$\int_0^\infty (\Omega_0\varphi)\,\overline{\psi}\,\mathrm{d}\beta(x) = \int_0^\infty \varphi(\overline{\Omega_0\psi})\,\mathrm{d}\beta(x)\,.$$

All the definitions and theorems of the theory of singular boundary problems (as for instance the notion of the limit point case) (see [1], chap. IX), stated usually for operators formally self-adjoint when the weighing measure is Lebesgue measure, can be carried over to such operators without changes. We can use the substitution mentioned in the proof of Theorem 6 to transform the operator Ω_0 into a self-adjoint one.

Lemma 1. *Under our condition that ∞ is a natural boundary of the process, the case of limit point takes place for the operator Ω_0.*

Proof. It suffices to show ([1], chap. IX, th. 2.1), that there exists one solution of the equation $\Omega_0 u = 0$ which does not belong to $\mathscr{L}$. One such solution is $u(x) = \exp B(x) \int_0^x \exp - B(s)\,\mathrm{d}s$. We have

$$\int_0^\infty u^2(x)\,\mathrm{d}\beta(x) = \int_0^\infty \mathrm{e}^{B(x)} \left(\int_0^x \mathrm{e}^{-B(s)}\,\mathrm{d}s\right)^2 \mathrm{d}x\,.$$

It is immediately seen that this integral is infinite when $\int_0^\infty \exp B(x)\,\mathrm{d}x = \infty$. When $\int_0^\infty \exp B(x)\,\mathrm{d}x < \infty$ then $\int_0^\infty \exp - B(x)\,\mathrm{d}x = \infty$. So for x large enough,

$$e^{B(x)} \left(\int_0^x e^{-B(s)}\,\mathrm{d}s \right)^2 \geqq e^{B(x)} \int_0^x e^{-B(s)}\,\mathrm{d}s$$

and the expression on the right side is not integrable by the hypothesis that ∞ represents a natural boundary of the process.

We shall restate in a concise manner the results of the spectral theory of singular boundary problems needed in the sequel. (See [1], chap. IX.) Let $\psi(x, \lambda)$ be the solution of

$$\frac{\mathrm{d}}{\mathrm{d}x} \left\{ \frac{\mathrm{d}}{\mathrm{d}x} \psi - b\psi \right\} + \lambda\psi = 0$$

satisfying $\psi(0, \lambda) = p$, $\psi'(0, \lambda) = 1 - p(1 - b(0))$ and let $\varrho(\lambda)$ be the spectral distribution function corresponding to the boundary problem. We suppose that $\varrho(\lambda)$ is left-continuous and equal to zero for $\lambda \leqq 0$. We denote by $\mathscr{L}_\varrho$ the Hilbert space of functions whose squared absolute value is integrable with respect to the measure determined by $\varrho(\lambda)$. There exists a unitary transformation Ψ of $\mathscr{L}$ onto $\mathscr{L}_\varrho$ which is defined for $f \in \mathscr{L}$ by the relation

$$\Psi f = \lim_{A \to \infty} (\mathscr{L}_\varrho) \int_0^A f(x)\, \psi(x, \lambda)\, \mathrm{d}\beta(x) \,.$$

For $g \in \mathscr{L}_\varrho$ we have

$$\Psi^{-1} g = \lim_{A \to \infty} (\mathscr{L}) \int_{0-}^A \psi(x, \lambda)\, g(\lambda)\, \mathrm{d}\varrho(\lambda) \,.$$

We denote by $\hat{\Omega}$ the operator $\mathrm{d}/\mathrm{d}x\,\{\mathrm{d}/\mathrm{d}x\, u - b(x)\, u\}$, considered in the space $\mathscr{L}$, whose domain of definition is restricted to elements u satysfying (1). The operator $\hat{\Omega}$ is the generator of a semi-group of operators in $\mathscr{L}\{\hat{V}_t, t \geqq 0\}$ such that $\Psi \hat{V}_t f = e^{-\lambda t} \Psi f$. (This is a known fact which may be established for instance by means of exercise 12 to chapter IX in [1] and by the Hille-Yosida theorem. The operators of multiplication by $e^{-\lambda t}$ in $\mathscr{L}_\varrho$ form a semi-group in $\mathscr{L}_\varrho$. We show that the resolvent operator of this semi-group corresponds (in the equivalence of $\mathscr{L}_\varrho$ and $\mathscr{L}$ defined by the transformation Ψ) to $(\lambda E - \hat{\Omega})^{-1}$ and this implies that $\hat{\Omega}$ is a generator of the semi-group in $\mathscr{L}$, which corresponds to multiplication by $e^{-\lambda t}$ in $\mathscr{L}_\varrho$). We denote $\hat{v}(t, x; f) = \hat{V}_t f$.

The norms in $\boldsymbol{L}$ and $\mathscr{L}$ will be denoted by $\|h\|_L$ and $\|h\|_{\mathscr{L}}$. Next we define an auxiliary Banach space $\boldsymbol{l}$, which is the set $\boldsymbol{L} \cap \mathscr{L}$ with the norm $\|h\|_l = \|h\|_L + \|h\|_{\mathscr{L}}$. The differential operator $\mathrm{d}/\mathrm{d}x\,\{\mathrm{d}/\mathrm{d}x\, u - bu\}$ restricted to $\boldsymbol{l}$ and limited by the boundary condition (1) will be denoted by $\tilde{\Omega}$.

Theorem 1. *The operator $\tilde{\Omega}$ is the generator of a contraction semi-group $\hat{v}(t, x; g)$ in $\boldsymbol{l}$. We have for $g \in \boldsymbol{l}$*

$$v(t, x; g) = v(t, x; g) = \hat{v}(t, x; g) \,.$$

Proof. According to the Hille-Yosida theorem we have to show that for $f \in \boldsymbol{l}$ and $\lambda > 0$ there exists exactly one solution of the equation

$$\lambda u - \tilde{\Omega} u = f \tag{2}$$

and that it satisfies $\lambda \|u\|_l \leqq \|f\|_l$. This is the same as to show that the (unique) functions $u_1 \in \boldsymbol{L}$ and $u_2 \in \mathscr{L}$ satisfying

$$\lambda u_1 - \Omega u_1 = f \tag{3}$$

and

$$\lambda u_2 - \hat{\Omega} u_2 = f \tag{4}$$

are identical.

We shall establish the equality of u_1 and u_2 when f is nonnegative. The general case may be treated by decompositing f into its positive and negative parts. It is known (see [2], [3]) that there exists a nonnegative solution v of the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{\mathrm{d}}{\mathrm{d}x} v - bv\right\} - \lambda v = 0$$

satisfying (1). From the unicity of (3) and (4) it follows that v belongs neither to $\boldsymbol{L}$ nor to $\mathscr{L}$. If $f \geqq 0$, then also $u_1 \geqq 0$ and $u_2 \geqq 0$. We must have $u_2 - u_1 = cv$. If $c > 0$, we would have $u_2 \geqq cv \geqq 0$ and so $u_2 \notin \mathscr{L}$. If $c < 0$, we would have $u_1 \geqq -cv \geqq 0$ and so $u_1 \notin \boldsymbol{L}$. Hence $u_1 = u_2$, and $\tilde{\Omega}$ is the generator of a semi-group in $\boldsymbol{l}$ which we denote by $\tilde{v}(t, x; g)$. From the equality of resolvent operators it follows that for $g \in \boldsymbol{l}$

$$v(t, x; g) = \tilde{v}(t, x; g) = \hat{v}(t, x; g).$$

Theorem 1 enables us to transfer to $v(t, x; g)$ the results obtained for $\hat{v}(t, x; f)$ when its spectral representation is used.

In the important case, when $\int_0^\infty e^{B(x)}\,\mathrm{d}x < \infty$, we have by Schwartz's inequality that for $f \in \mathscr{L}$

$$\int_0^\infty |f(x)|\,\mathrm{d}x = \int_0^\infty |f(x)|\, e^{-\frac{1}{2}B(x)} e^{\frac{1}{2}B(x)}\,\mathrm{d}x \leqq \left(\int_0^\infty e^{B(x)}\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_0^\infty |f(x)|^2\,\mathrm{d}\beta(x)\right)^{\frac{1}{2}}.$$

Thus $f \in \boldsymbol{L}$ and $\|f\|_L \leqq c\|f\|_{\mathscr{L}}$. Hence $\boldsymbol{l}$ may be identified with $\mathscr{L}$. In this case we have also that $e^{B(x)} \in \mathscr{L}$ and $\int_0^\infty f(x)\,\mathrm{d}x$ is a linear functional in $\mathscr{L}$.

Let A be a measurable subset of $(0, \infty)$ and let $h \in \mathscr{L}$. We denote

$$\tau_A(\lambda) = \Psi \chi_A e^B = \int_A \psi(x, \lambda)\,\mathrm{d}x$$

and $g(\lambda) = \Psi h$. From the unitarity of the transform Ψ and from the spectral representation of the semi-group $\hat{v}(t, x; .)$ we obtain that

$$\int_A \hat{v}(t, x; h)\,\mathrm{d}x = \int_0^\infty e^{-\lambda t} g(\lambda)\,\tau_A(\lambda)\,\mathrm{d}\varrho(\lambda).$$

We see that for asymptotic behaviour of the integral over A it is decisive, how the functions $\tau_A(\lambda)$ and $\varrho(\lambda)$ behave in the neighborhood of the least point of increase of the function $\varrho(\lambda)$. This point is characterized by the following lemma:

Lemma 2. *Let λ_ϱ be the least point of increase of the function $\varrho(\lambda)$; then λ_ϱ is the greatest of those numbers λ, for which $\psi(x, \lambda)$ does not change sign.*

Proof. The spectral distribution function $\varrho(\lambda)$ is the limit for $s \to \infty$ of spectral distribution functions corresponding to the boundary problem on the interval $\langle 0, s \rangle$, with boundary condition (1) for $x = 0$ and condition $u = 0$ for $x = s$. So we see that there cannot exist a $\lambda > \lambda_\varrho$ for which $\psi(x, \lambda)$ does not change sign. Let $\psi(x, \lambda_\varrho)$ change sign. Then there exist bounded intervals I and J such that for $x \in I$ we have $\psi(x, \lambda_\varrho) > 0$ and for $x \in J$ $\psi(x, \lambda_\varrho) < 0$. Let us choose a nonnegative $f \in \mathscr{L}$ not equivalent to zero, such that $f(x) = 0$ for $x \notin I$, and denote $\Psi f = g(\lambda)$. We see that

$$\lim_{\lambda \to \lambda_\rho +} \tau_J(\lambda)\, g(\lambda) < 0\,.$$

Hence we must have

$$\int_J \hat{v}(t, x; f)\, dx = \int_{\lambda_\rho -}^{\infty} e^{-\lambda t}\, \tau_J(\lambda)\, g(\lambda)\, d\varrho(\lambda) < 0$$

for sufficiently large t, in contradiction with the nonnegativity of f.

We shall use the symbol λ_ϱ in the sense already introduced.

In some of the following theorems we shall restrict ourselves to a set $\mathscr{L}_0 \subset \mathscr{L}$ defined as the set of all nonnegative functions from $\mathscr{L}$, not equivalent to zero, whose transformation Ψf is bounded from below in some right neighborhood of λ_ϱ. The set $\mathscr{L}_0$ contains functions which tend sufficiently quickly to zero when x tends to infinity.

Theorem 2. *For every $f \in \mathscr{L}_0$ and bounded measurable nonzero subsets A and B of $(0, \infty)$ we have*

$$\lim_{t \to \infty} \left(\int_A \hat{v}(t, x; f)\, dx \right) \left(\int_B \hat{v}(t, x; f)\, dx \right)^{-1} = \bar{\tau}_A \bar{\tau}_B\,,$$

where

$$\bar{\tau}_A = \int_A \psi(x, \lambda_\varrho)\, dx\,, \quad \bar{\tau}_B = \int_B \psi(x, \lambda_\varrho)\, dx\,.$$

If $\varrho(\lambda_\varrho +) > 0$, then the assertion is true for every nonnegative nonzero $f \in \mathscr{L}$.

We shall use the notation $f = O(g)$ to express that for two functions $f(t)$ and $g(t)$ we have $0 < c_1 < |f(t)/g(t)| < c_2$ for t large enough. The notation $f = \bar{O}(g)$ will be used to express that we have $|f(t)/g(t)| < c$ for large values of t. We introduce the following notation: For $K > \lambda_\varrho$ we denote

$$I(K, h) = \int_{\lambda_\rho -}^{K} e^{-\lambda t}\, h(\lambda)\, d\varrho(\lambda)$$

and

$$A(K, h) = \int_{\lambda_\rho -}^{K} e^{-\lambda t} \tau_A(\lambda)\, h(\lambda)\, d\varrho(\lambda)\,.$$

1 will be used also to denote the function identically equal to one, and we put $g(\lambda) = \Psi f$. The symbols h^+ and h^- denote the positive and the negative parts of h.

1. When we use the fact that $g(\lambda)$ is bounded from below in some neighborhood of λ_ϱ we obtain $A(\infty, g^-) = \bar{O}(I(K, 1))$.

2. Let $f_0 \in \mathscr{L}$ be a function equal to zero in some neighborhood of infinity, equal to f elsewhere, and let $\Psi f_0 = g_0$. The limit $\lim\limits_{\lambda \to \lambda_\rho +} g_0(\lambda) = \bar{g}_0$ exists and is positive because $\psi(x, \lambda_\varrho)$ is a positive function of x. From this we obtain that

$$\lim_{t \to \infty} I(K, 1)^{-1} A(\infty, g_0) = \bar{\tau}_A \bar{g}_0 > 0\,.$$

3. The existence of the positive limit $\lim\limits_{\lambda \to \lambda_\rho +} \tau_A(\lambda)$ implies that $A(\infty, g^+) = O(I(K, g^+))$ and $A(\infty, g^-) = O(I(K, g^-))$.

4. $I(K, g) = O(I(K, g^+))$ and $I(K, g)$ is positive for t large enough. Let suppose that on the contrary there exists a sequence $\{t_n\}$, $t_n \to \infty$ such that

(5) $$\lim_{t_n} \frac{I(K, g)}{I(K, g^+)} \leqq 0\,.$$

Then we must have

(6) $$\limsup_{t_n} \frac{I(K, g^+)}{I(K, 1)} < C$$

because from 1 and 3 we can deduce that

$$\limsup_{t_n} \frac{I(K, g^+)}{I(K, 1)} = \infty$$

implies

$$\limsup_{t_n} \frac{I(K, g)}{I(K, g^+)} = 1\,.$$

The constant C is independent of K.

We have the following relation

$$\liminf_{t_n} I(S, 1)^{-1} A(\infty, g) = \liminf_{t_n} I(S, 1)^{-1} A(K, g) \leqq$$
$$\leqq \bar{\tau}_A \liminf_{t_n} I(S, 1)^{-1} I(K, g) + \varepsilon \limsup_{t_n} I(S, 1)^{-1} \left(I(K, g^+) + I(K, g^-)\right).$$

Here K is so chosen that for $\lambda_\varrho \leqq \lambda < K$ it is $|\bar{\tau}_A - \tau_A(\lambda)| < \varepsilon$. It is also supposed that $\bar{\tau}_A > \varepsilon > 0$. Hence using (6) we see that the relation (5) implies

$$\liminf_{t_n} I(S, 1)^{-1} A(\infty, g) \leqq 0\,,$$

which contradicts 2, because $A(\infty, g) \geqq A(\infty, g_0)$.

We have $I(K, g^-) = I(K, g^+) - I(K, g)$. Hence from 4 we obtain $I(K, g^-) = \bar{O}(I(K, g))$.

5. We have $\lim_{t\to\infty} I(s, g)^{-1} A(\infty, g) = \bar{\tau}_A$. This is established be means of 4 and the relations

$$\bar{\tau}_A - \varepsilon \limsup_{t\to\infty} I(s, g)^{-1} (I(K, g^+) + I(K, g^-)) \leqq \liminf_{t\to\infty} I(s, g)^{-1} A(\infty, g) \leqq$$
$$\leqq \limsup_{t\to\infty} I(s, g)^{-1} A(\infty, g) \leqq \bar{\tau}_A + \varepsilon \limsup_{t\to\infty} I(s, g)^{-1} (I(K, g^+) + I(K, g^-))$$

where K and ε are subject to the same conditions as in 4.

The first assertion of the theorem is an immediate consequence of 5. To establish the second we remark that if $\varrho(\lambda_\varrho +) > 0$ then

$$A(\infty, g) = g(\lambda_\varrho)\, \varrho(\lambda_\varrho+)\, e^{-\lambda_\rho t}\bar{\tau}_A + o(e^{-\lambda_\rho t}).$$

We return now to the problem of the convergence to a limit distribution of the probability distribution at the time t under the condition that the particle was not absorbed before the time t. For nonnegative, nonzero $g \in \boldsymbol{L}$ we denote

$$\bar{v}(t, x; g) = \left(\int_0^\infty v(t, y; g)\, \mathrm{d}y\right)^{-1} v(t, x; g).$$

So when g is a probability density, $\bar{v}(t, x; g)$ is the probability density of the conditional distribution examined.

Theorem 3. *If* $\int_0^\infty \psi(x, \lambda_\varrho)\, \mathrm{d}x = \infty$, *then for every* $f \in \mathscr{L}_0 \cap \boldsymbol{L}$ *and* $A > 0$, $\lim_{t\to\infty} \int_0^A \bar{v}(t, x; f)\, \mathrm{d}x = 0$. *This case arises when zero belongs to the spectrum of the problem and, in the condition* (1), *either* $p \neq 1$ *or* $p = 1$ *and at the same time* $\int_0^\infty e^{B(x)}\, \mathrm{d}x$ *is infinite. When* $\varrho(\lambda_\varrho) > 0$, *the assertion holds for every nonnegative nonzero* $f \in \mathscr{L} \cap \boldsymbol{L}$.

Proof. We denote $I = (0, A)$, $J = (0, B)$. From Theorem 2 we obtain

$$\limsup_{t\to\infty} \int_0^A \bar{v}(t, x; f)\, \mathrm{d}x \leqq \lim_{t\to\infty} \left(\int_0^A v(t, x; f)\, \mathrm{d}x\right)\left(\int_0^B v(t, x; f)\, \mathrm{d}x\right)^{-1} = \bar{\tau}_I \bar{\tau}_J^{-1}$$

and, by hypothesis, $\lim_{B\to\infty} \bar{\tau}_J = \infty$. We have

$$\psi(x, 0) = p e^{B(x)} + (1 - p)\, e^{B(x)} \int_0^x e^{-B(s)}\, \mathrm{d}s$$

and this is not an integrable function under the hypothesis of the second assertion of the theorem.

Note. When the hypothesis of the second assertion of the theorem holds for some boundary condition, then it holds also for the remaining conditions for which $p \neq 1$.

This follows from the fact that the derived set of the spectrum is independent of p. (Compare [1], chap. IX, exerc. 8.)

Theorem 4. *Let be* $\int_0^\infty e^{B(x)}\,dx < \infty$. *Then for the existence of* $\lim_{t\to\infty} (\mathscr{L})\,\bar{v}(t, x; h) = w(x)$ *for some probability density* $h(x)$, $h \in \mathscr{L}$ *it is necessary that* $\psi(x, \lambda_\varrho) \in \mathscr{L}$. *If* $\psi(x, \lambda_\varrho) \in \mathscr{L}$, *then for an arbitrary density* $f \in \mathscr{L}$ *we have* $\lim_{t\to\infty} (\mathscr{L})\,\bar{v}(t, x; f) = (\int_0^\infty \psi(x, \lambda_\varrho)\,dx)^{-1}\,\psi(x, \lambda_\varrho)$ *and* $\int_0^\infty \hat{v}(t, x; f)\,dx = O(e^{-\lambda_\rho t})$.

Proof. Let $\lim_{t\to\infty} (\mathscr{L})\,\bar{v}(t, x; h) = w(x)$. Then we have

$$\lim_{t\to\infty}\left(\int_0^\infty \hat{v}(t + s, y; h)\,dy\right)\left(\int_0^\infty \hat{v}(t, y; h)\,dy\right)^{-1} =$$

$$= \lim_{t\to\infty}\int_0^\infty \hat{v}(s, y; \bar{v}(t, .; h))\,dy = \int_0^\infty \hat{v}(s, y; w)\,dy = \varphi(s)\,.$$

It is easy to see that $\varphi(s)$ satisfies the relations $\varphi(s_1 + s_2) = \varphi(s_1)\,\varphi(s_2)$, $\varphi(s) \leqq 1$. Hence $\varphi(s) = e^{-\alpha s}$ with $\alpha \geqq 0$. Also

$$\bar{v}(t + s, x; h) = \left(\int_0^\infty \hat{v}(t, y; h)\,dy\right)\left(\int_0^\infty \hat{v}(t + s, y; h)\,dy\right)^{-1}\hat{v}(s, x; \bar{v}(t, .; h))\,.$$

By a passage to the limit for $t \to \infty$ we obtain $e^{-\alpha s}\,w(x) = \hat{v}(s, x; w)$. Using the definition of the infinitesimal operator of a semi-group, we obtain $\hat{\Omega}w + \alpha w = 0$. We see that α belongs to the spectrum of the boundary problem and the corresponding eigenfunction is nonnegative. From Lemma 2 it follows that $\alpha = \lambda_\varrho$. Hence $w(x) = k\,\psi(x, \lambda_\varrho)$, and we must have $k = (\int_0^\infty \psi(x, \lambda_\varrho)\,dx)^{-1}$.

If $\psi(x, \lambda_\varrho) \in \mathscr{L}$, then the function $\varrho(\lambda)$ has a jump at the point λ_ϱ. Under our hypothesis we have $\exp B(x) \in \mathscr{L}$. We denote $\Psi \exp B(x) = \tau(\lambda)$. We take an arbitrary density $f \in \mathscr{L}$ and put $\Psi f = g(\lambda)$. We have

$$\int_0^\infty \hat{v}(t, x; f)\,dx = \int_{\lambda_\rho -}^\infty e^{-\lambda t}\,g(\lambda)\,\tau(\lambda)\,d\varrho(\lambda)\,.$$

From the fact that $f \in \mathscr{L}$ and $\exp B \in \mathscr{L}$ it follows that the integrals

$$g(\lambda_\varrho) = \int_0^\infty f(x)\,\psi(x, \lambda_\varrho)\,d\beta(x)$$

and

$$\tau(\lambda_\varrho) = \int_0^\infty \psi(x, \lambda_\varrho)\,dx$$

exist and are evidently positive. It is easy to see that

$$\lim_{t\to\infty} e^{\lambda_\rho t}\int_0^\infty \hat{v}(t, x; f)\,dx = g(\lambda_\varrho)\,\tau(\lambda_\varrho)\,\varrho(\lambda_\varrho +)$$

and that

$$\lim_{t\to\infty} (\mathscr{L}_\varrho)\, e^{\lambda_\rho t}\, \psi\, \hat{v}(t, x; f) = k(\lambda)\,,$$

where $k(\lambda_\varrho) = g(\lambda_\varrho)$ and $k(\lambda) = 0$ for $\lambda \neq \lambda_\varrho$. The desired result follows by means of the formula for Ψ^{-1} and from the unitarity of Ψ.

Note. The conditions of Theorem 4 are satisfied, when $\int_0^\infty \exp B(x)\, \mathrm{d}x < \infty$ and we have a reflecting barrier at zero. Then $\exp B \in \mathscr{L}$ is the eigenfunction of the problem, corresponding to the eigenvalue $\lambda_\varrho = 0$. In this case for $h \in \mathscr{L}$

$$\int_0^\infty \hat{v}(t, y; h)\, \mathrm{d}y = \int_0^\infty h(y)\, \mathrm{d}y\,.$$

This follows from the fact that in the case $p = 1$ the semi-group $\{V_t, t \geqq 0\}$ preserves norm in $\boldsymbol{L}$. (See [2].) From Theorem 4 it follows that

$$\lim_{t\to\infty} \hat{v}(t, x; h) = \int_0^\infty h(s)\, \mathrm{d}s \left(\int_0^\infty e^{B(s)} \mathrm{d}s\right)^{-1} e^{B(x)}$$

in the norm of $\mathscr{L}$. We now use the fact that $\mathscr{L}$ is a dense set in the space $\boldsymbol{L}$ and that the convergence in $\mathscr{L}$ is stronger than that in $\boldsymbol{L}$. From the contractivity of the semigroup $v(t, x; g)$ we find that the limit relation holds for every $g \in \boldsymbol{L}$, in the norm of $\boldsymbol{L}$. (Compare [4].)

Theorem 5. *Let* $\int_0^\infty e^{B(x)}\, \mathrm{d}x < \infty$ *and* $\Psi e^B = \tau(\lambda)$. *If*

(7) $$\lim_{\lambda\to\lambda_\rho+} \tau(\lambda) = \int_0^\infty \psi(x, \lambda_\varrho)\, \mathrm{d}x < \infty\,,$$

then for every $h \in \mathscr{L}_0$ *we have*

$$\lim_{t\to\infty} \int_0^z v(t, x; h)\, \mathrm{d}x = \left(\int_0^\infty \psi(s, \lambda_\varrho)\, \mathrm{d}s\right)^{-1} \int_0^z \psi(s, \lambda_\varrho)\, \mathrm{d}s\,.$$

Note. The expression (7) has the meaning that in the class of functions equivalent to $\tau(\lambda)$ there exists one for which this relation holds.

Proof. The proof of Theorem 5 is exactly the same as that of Theorem 2 if we put $\tau_B(\lambda) = \tau(\lambda)$.

If we wish to deduce from the coefficient $b(x)$ the limit behaviour of the probability distribution, the following theorem may be useful:

Theorem 6. *If* $\int_0^\infty e^{B(x)}\, \mathrm{d}x < \infty$ *and* $\lim_{x\to\infty} \frac{1}{2}b^2(x) + b'(x) = \infty$, *then for every density*

$$f \in \mathscr{L}\ \lim_{t\to\infty} (\mathscr{L})\, \bar{v}(t, x; f) = \left(\int_0^\infty \psi(x, \lambda_\varrho)\, \mathrm{d}x\right)^{-1} \psi(x, \lambda_\varrho)\,.$$

If $\int_0^\infty |\frac{1}{2}b^2(x) + b'(x)|\,dx < \infty$ *and* $p \neq 1$, *then for every* $f \in \mathscr{L}_0 \cap L$

$$\lim_{t\to\infty} \int_0^A v(t, x; f)\,dx = 0.$$

Proof. By means of the known substitution $u(x) = e^{\frac{1}{2}B(x)}y(x)$ the equation $\Omega_0 u + \lambda u = 0$ transforms to $y'' - \{\frac{1}{4}b^2 + \frac{1}{2}b'\}\,y + \lambda y = 0$ and the boundary condition (1) to $(1 - p)\,y = p(y' - \frac{1}{2}by)|_{x=0}$. The spectral distribution function of this problem remains $\varrho(\lambda)$. So the condition $\lim_{x\to\infty} \frac{1}{4}b^2(x) + b'(x) = \infty$ is the condition of Weyl for discreteness of the spectrum (compare [1], chap. IX, exerc. 1), and Theorem 4 may be applied. The condition $\int_0^\infty |\frac{1}{2}b^2(x) + b'(x)|\,dx < \infty$ implies (compare [1], chap. IX, exerc. 4) that zero belongs to the spectrum of the problem. Hence the second assertion of Theorem 3 may be used.

Example 1. We may have $\int_0^\infty |\frac{1}{2}b^2(x) + b'(x)|\,dx < \infty$ and $\int_0^\infty e^{B(x)}\,dx < \infty$ simultaneously. This is the case of e. g. $b(x) = -(x + 1)^{-\alpha}$ with $1 > \alpha > \frac{1}{2}$. Thus if there is a reflecting barrier at zero, the distribution of particles tends to a limit distribution, but for every $p \neq 1$, the conditional distribution tends to infinity.

Example 2. Brownian motion in a gravitational field.

$$\Omega_0 u = \frac{d^2}{dx^2}\,u + \beta\,\frac{d}{dx}\,u\,,$$

i. e. $b(x) \equiv -\beta$, $\beta > 0$. The boundary condition at the origin is $(1 - \beta p - p)\,u - pu' = 0|_{x=0}$. $e^{B(x)} = e^{-\beta x}$ so that $\int_0^\infty e^{B(x)}\,dx < \infty$. We denote $\gamma = (\frac{1}{4}\beta^2 - \lambda)^{\frac{1}{2}}$. Then for $0 \leqq \lambda < \frac{1}{4}\beta^2$ we have

$$\psi(x, \lambda) = \frac{1 + p(\gamma - \frac{1}{2}\beta - 1)}{2\gamma}\,e^{-\frac{1}{2}\beta x + \gamma x} - \frac{1 + p(\frac{1}{2}\beta + \gamma + 1)}{2\gamma}\,e^{-\frac{1}{2}\beta x - \gamma x}\,,$$

$$\psi(x, \tfrac{1}{4}\beta^2) = [(1 - \tfrac{1}{2}\beta p - p)\,x + p]\,e^{-\frac{1}{2}\beta x}\,.$$

If we denote $\delta = (\lambda - \frac{1}{4}\beta^2)^{\frac{1}{2}}$, we obtain for $\frac{1}{4}\beta^2 < \lambda$ that

$$\psi(x, \lambda) = \frac{1 - p(1 + \frac{1}{2}\beta)}{\delta}\,e^{-\frac{1}{2}\beta x}\sin\delta x + pe^{-\frac{1}{2}\beta x}\cos\delta x\,.$$

Let us first find the value of the smallest point of increase of $\varrho(\lambda)$, using Lemma 2. It is easy to see that for $\lambda > \frac{1}{4}\beta^2$, $\psi(x, \lambda)$ has infinitely many changes of sign. The function $\psi(x, \frac{1}{4}\beta^2)$ is nonnegative for $0 \leqq p \leqq 2/(2 + \beta)$. So for such p, $\lambda_\varrho = \frac{1}{4}\beta^2$. For $2/(2 + \beta) < p \leqq 1$, the greatest λ with nonnegative $\psi(x, \lambda)$ is such that $1 + p(\gamma - \frac{1}{2}\beta - 1) = 0$, so that

$$\lambda_\varrho = \frac{p(2 + \beta) - p^2(1 + \beta) - 1}{p^2}\,.$$

For $0 \leqq p \leqq 2/(2 + \beta)$ the function $\psi(x, \lambda_\varrho)$ does not belong to $\mathscr{L}$, but it is easily seen that the conditions of Theorem 5 are satisfied. For $2/(2 + \beta) < p \leqq 1$ we have $\psi(x, \lambda_\varrho) \in \mathscr{L}$ and this is the case of Theorem 4. Particularly for the absorption barrier at the origin ($p = 0$), we see that the weak limit of conditional distributions is the distribution given by the density $(\frac{1}{2}\beta)^2 \, xe^{-\frac{1}{2}\beta x}$.

Example 3. The process of Uhlenbeck:

$$\Omega_0 u = \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{\mathrm{d}}{\mathrm{d}x} u + \beta x u\right\},$$

where $\beta > 0$. In this case

$$e^{B(x)} = e^{-\frac{1}{2}\beta x^2}\,, \quad \tfrac{1}{2}b^2(x) + b'(x) = \tfrac{1}{2}\beta^2 x^2 - \beta\,.$$

The boundary condition for the process with an absorbing boundary at zero is $u(0) = 0$. By means of the substitution

$$y(x) = e^{x^2} u\left(x \sqrt{\frac{2}{\beta}}\right)$$

we find that the eigenvalues of the problem are $\lambda_k = (2k + 1)\,\beta$, $k = 0, 1, 2, \ldots$ and the coresponding eigenfunctions are

$$u_k(x) = e^{-\beta x^2/2}\, H_{2k+1}\left(\sqrt{\tfrac{1}{2}\beta}x\right) \quad \text{for} \quad x \geqq 0\,.$$

Here $H_n(y)$ is the Hermite polynomial. When we use Theorem 4 we find that

$$\lim_{t \to \infty} \bar{v}(t, x; g) = \beta x e^{-\beta x^2/2} \quad \text{and} \quad \int_0^\infty \hat{v}(t, x; g)\, \mathrm{d}x = O(e^{-\beta t})\,.$$

The paper [5] is devoted to this process.

Literature

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Резюме

СПЕКТРАЛЬНАЯ ТЕОРИЯ ПОЛУГРУПП СВЯЗАННЫХ С ДИФФУЗИОННЫМИ ПРОЦЕССАМИ И ЕЁ ПРИМЕНЕНИЕ

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В работе исследуются методами теории полугрупп диффузионные процессы на интервале $(0,\infty)$, которые описаны уравнением

$$\frac{\partial}{\partial t} u = \frac{\partial}{\partial x} \left\{ \frac{\partial}{\partial x} u - b(x)\, u \right\}.$$

Главное внимание уделено случаю, когда в нуле помещена поглощающая или эластическая стенка. Тогда изучается предельное поведение распределения вероятностей положения диффундирующей частицы во время t при условии, что до этого времени поглощение частицы на стенке еще не произошло. Используется спектральная теория сингулярных краевых задач для дифференциальных уравнений второго порядка.

Получены некоторые касающиеся спектральной функции распределения и решений уравнения

$$\frac{\mathrm{d}}{\mathrm{d}x} \left\{ \frac{\mathrm{d}}{\mathrm{d}x} \psi - b(x)\, \psi \right\} + \lambda \psi = 0$$

условия для того, чтобы условное распределение стремилось к предельному или к бесконечности. Применением этих условий получается утверждение:

Пусть $\exp \int_0^x b(s)\,\mathrm{d}s$ *интегрируема и* $\lim_{x\to\infty} \left(\frac{1}{2}b(x)^2 + b'(x)\right) = \infty$, *тогда условное распределение стремится по норме к предельному. Если* $\int_0^\infty |\frac{1}{2}b^2(x) + b(x)|\,\mathrm{d}x < \infty$, *то оно стремится к бесконечности.*

К статье добавлены три примера.