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INVARIANCE OF G_δ-SPACES UNDER MAPPINGS

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It is proved that the images and inverse images respectively, of G_δ spaces are also G_δ, under certain conditions on the mapping; some related questions are also considered.

The present paper is devoted to the following two questions: Let f be a continuous mapping of a space P onto a space Q. Under what conditions on f may we assert that:

(1) if P is a G_δ-space, then Q is a G_δ-space.

(2) if Q is a G_δ-space, then P is a G_δ-space.

In connection with (2) some characterisations of completely regular G_δ-spaces are given. For example, a completely regular space P is a G_δ-space if and only if P is homeomorphic with some closed subspace of the topological product of a countable family of locally compact completely regular spaces.

All spaces are assumed to be completely regular. The terminology and notation of J. KELLEY, General Topology, is used throughout. $\beta(P)$ always denotes the Čech – Stone compactification of a space P. Let us recall that a space P is said to be a G_δ-space (or topologically complete in the sense of E. ČECH, vide [1]) if P is a G_δ-set in $\beta(P)$. The following facts are well-known (vide [1] or [2]) if a space P is a G_δ in some of its compactifications, then P is a G_δ-space; every G_δ-space is G_δ in each of its extensions (a space R is an extension of a space P if P is a dense subspace of R).

If is well-known that a continuous image of a G_δ-space may fail to be a G_δ-space. Moreover, the image under a linear continuous mapping of a complete normed linear space may fail to be a G_δ-space. Indeed, by well-known theorem, a linear continuous mapping f of a complete normed linear space P is open if and only if $f[P]$ is of the second category in itself.

1. Theorem. *Let f be an open continuous mapping of a space P onto a space Q. If P is a G_δ-space, then Q is a G_δ-space.*

Proof. According to the Čech-Stone theorem, there exists a continuous mapping F of $\beta(P)$ onto $\beta(Q)$ such that f is the restriction of F to P. From the fact that f is open we may conclude at once that, if U is an open subset of $\beta(P)$ containing P, then the interior of $F[U]$ (in $\beta(Q)$) contains the set Q. Now let P be a G_δ-space. Then there exists a sequence $\{U_n\}$ of open subsets of $\beta(P)$ such that $\bigcap_{n=1}^{\infty} U_n = P$. Denoting by V_n the interior of $F[U_n]$, we conclude as above that $V_n \supset Q$. Evidently $Q = \bigcap_{n=1}^{\infty} V_n$. Thus Q is G_δ in $\beta(Q)$, and consequently, Q is a G_δ-space.

As an immediate consequence of the precending theorem and of the fact that a metrizable space P is a G_δ-space if and only if there exists a metric φ for P such that (P, φ) is a complete metric space, we have at once:

2. Theorem. *Let f be an open continuous mapping of a complete metric space P onto a metrizable space Q. Then there exists a metric ψ for Q such that (Q, ψ) is a complete metric space.*

It may be noticed that a continuous mapping of a G_δ-space onto a G_δ-space may fail to be open.

The remainder of this paper is devoted to investigations of inverse images of G_δ-spaces under mappings of a special sort. A mapping f of a space P into a space Q will be called closed if the images of closed sets are closed. We shall need the following

3. Lemma. *Let F be a continuous mapping of a space R onto a space S. Let P be a dense subspace of R. Suppose that the restriction $f = F/P$ of F to P is a closed mapping onto $Q = f[P]$. Finally, let the inverses of points (i. e. sets of the form $f^{-1}[y]$, $y \in Q$) be closed in R. Then $F^{-1}[Q] = P$, or equivalently, $F[R - P] \subset \subset S - Q$.*

Proof. Suppose that there exists a point x in $R - P$ such that the point $y = F(x)$ belongs to Q. Put $K = f^{-1}[y]$. R being a regular space, there exists a neighborhood U of x closed in R and disjoint with K. P being dense in R, we have $x \in \overline{U \cap P}$, and by continuity of F,

$$y = F(x) \in \overline{F[U \cap P]}^S .$$

Since $y \in Q$ and $F[U \cap P] = f[U \cap P] \subset Q$ we obtain at once $y \in \overline{f[U \cap P]}^P$; f being a closed mapping and $U \cap P$ being a closed subset of $P, f[U \cap P]$ is a closed subset of Q. Thus the point y belongs to $f[U \cap P]$. But this is impossible, since the sets $K = f^{-1}[y]$ and U are disjoint. This contradiction establishes the lemma.

A mapping f of a space onto a space Q will be called compact provided that the inverses $f^{-1}[y]$, $y \in Q$, are compact spaces. From the preceding lemma we deduce:

4. Theorem. *Let us suppose that f is a continuous, closed and compact mapping of a space P onto a space Q. If Q is a G_δ-space, then P is a G_δ-space.*

Proof. According to the Čech-Stone theorem, there exists a continuous mapping F of $\beta(P)$ onto $\beta(Q)$ such that f is the restriction of F. It is easy to see that the assumptions of the preceding lemma are fulfilled an hence that

$$(*) \qquad F[\beta(P) - P] \subset \beta(Q) - Q .$$

Let us suppose that Q is a G_δ-space. There exists a sequence $\{U_n\}$ of open subsets of $\beta(Q)$ such that $\bigcap_{n=1}^{\infty} U_n = Q$. According to (*), we have

$$P = \bigcap_{n=1}^{\infty} F^{-1}[U_n] .$$

Thus P is G_δ in $\beta(P)$, and consequently, P is a G_δ-space.

5. **Theorem.** *A space P is a G_δ-space if and only if P is homeomorphic with some closed subspace of the topological product of a countable family of locally compact spaces.*

Proof. Let us suppose that P is a G_δ-space. There exists a sequence $\{U_n\}$ of open subsets of $\beta(P)$ such that $\bigcap_{n=1}^{\infty} U_n = P$. Consider the topological product

$$U = X\{U_n;\ n = 1, 2, \ldots\} .$$

For every x in P denote by $f(x)$ the point $\{x, x, \ldots\}$ of U. The mapping f of P to U is a homeomorphism and $f[P]$ is closed in U. The spaces U_n being locally compact, the necessity is proved. On the other hand, it is well-known (and it may be easily proved) that the topological product of a countable family of G_δ-spaces is a G_δ-space, and that every closed subspace of a G_δ-space is a G_δ-space. The sufficiency follows.

A continuous mapping f of a space P to a space Q will be called non-extensible if there exists no proper extension R of P (that is $P \subsetneq R$ and $\overline{P} = R$) over which f may be continuously extended (in other words: if R is a extension of P and if F is a continuous mapping of R to Q such that f is the restriction of F, then $P = R$).

6. **Theorem.** *A space P is a G_δ-space if and only if there exists a continuous non-extensible mapping of P to a G_δ-space.*

Proof. Let us suppose that f is a continuous non-extensible mapping of P to a G_δ-space Q. According to the Čech-Stone theorem, there exists a continuous mapping F of $\beta(P)$ to $\beta(Q)$. From the non-extensibility of f we obtain that

$$F[\beta(P) - P] \subset \beta(Q) - f[P] .$$

Now by the same argument as in the proof of theorem 4, it may be shown that P is a G_δ-space. Conversely, if P is a G_δ-space then the identity mapping of P (to P) is non-extensible.

It may be noticed that if P is a G_δ-space, then there exists a continuous non-extensible mapping of P to the topological product of a countable family of locally compact spaces. Indeed, the mapping f from the first part of the proof of theorem 5 is non-extensible.

Combining the theorems 4, 5 and 6 we obtain

7. **Theorem.** *The following properties of a space P are equivalent:*

(1) *P is a G_δ-space.*

(2) *There exists a continuous, closed and compact mapping of P to a G_δ-space.*

(3) *P is homeomorphic with some closed subspace of the topological product of a countable family of locally compact spaces.*

(4) *There exists a continuous non-extensible mapping of P to the topological product of a countable family of locally compact spaces.*

(5) *There exists a continuous, closed and compact mapping of P to a topological product of a countable family of locally compact spaces.*

Bibliography

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Резюме

ИНВАРИАНТНОСТЬ G_δ-ПРОСТРАНСТВ ПРИ ОТОБРАЖЕНИЯХ

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Вполне регулярное пространство P называется G_δ-пространством (или топологически полным в смысле Э. Чеха), если оно является G_δ-множеством в своем чеховском бикомпактном расширении $\beta(P)$. В статье доказываются следующие теоремы:

1. *Пусть f — непрерывное открытое отображение вполне регулярного пространства P на вполне регулярное пространство Q. Если P является G_δ-пространством, то Q также является Q_δ-пространством.*

2. *Пусть f — замкнутое непрерывное отображение вполне регулярного пространства P на вполне регулярное пространство Q. Если подпространства $f^{-1}[y]$, $y \in Q$ бикомпактны и Q есть G_δ-пространство, то P тоже является G_δ-пространством.*

Далее даются некоторые эквивалентные определения вполне регулярных G_δ-пространств. Хорошо известно, что метризуемое пространство P является G_δ-пространством тогда и только тогда, когда для некоторой метрики φ метрическое пространство (P, φ) полно. Из 1 в частности следует:

Если существует открытое непрерывное отображение f некоторого полного метрического пространства на метризуемое пространство Q, то для некоторой метрики φ метрическое пространство (Q, φ) полно.