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APPLICATIONS OF COMPLETE FAMILIES OF CONTINUOUS FUNCTIONS TO THE THEORY OF Q-SPACES

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In the present paper the concept of a complete family of continuous is introduced and applied to the theory of $N(\mathfrak{m})$-spaces (intersections of $\mathfrak{m}$ N-sets in their Stone-Čech extensions) and, in particular, Q-spaces. $N(\mathfrak{m})$-spaces may be defined as the inverse images under continuous closed compact mappings to the topological product of $\mathfrak{m}$ real lines. The section 3 is devoted to the problem, under what conditions on the mapping is the image of a $N(\mathfrak{m})$-space (in particular, of a Q-space) an $N(\mathfrak{m})$-space (a Q-space, respectively).

In [2] the concept of a complete indexed family of open coverings of a space has been introduced. For convenience, we recall the definition. An indexed family of open coverings

(1) $$\{\mathfrak{W}_a;\ a \in A\}$$

is said to be complete if the following condition is satisfied:

If $\{F\}$ is a centered family of closed subsets of P such that for each a in A there exists a V_a in $\mathfrak{W}_a$ containing some $F_a \in \{F\}$, then $\bigcap\{F\} \neq \emptyset$.

In [2] the following theorem was proved:

A completely regular space P is an intersection of $\mathfrak{m}$ open sets in every compact extension of P if and only if there exists a complete indexed family (1) of open coverings of P such that the potency of A is $\mathfrak{m}$.

In the present paper we investigate spaces possesing *a* complete family of open coverings (1) of a special sort. If f is a continuous real-valued function on P then the open cover consisting of sets

$$\{x;\ |f(x)| < n\}\ ,\quad n = 1, 2, \ldots$$

will be denoted by $\mathfrak{W}(f)$. We shall consider coverings of the form $\mathfrak{W}(f)$ only. We shall prove that a completely regular space P possesses a family of continuous functions $\mathfrak{F}$ such that

$$\{\mathfrak{W}(f);\ f \in \mathfrak{F}\}$$

is complete (such a family $\mathfrak{F}$ is said to be complete) if and only if there exists a indexed family $\{N_f; f \in \mathfrak{F}\}$ of N-sets in βP such that

$$P = \bigcap\{N_f; f \in \mathfrak{F}\}.$$

If the potency of $\mathfrak{F}$ is at most $\mathfrak{m}$, then such spaces will be called $N(\mathfrak{m})$-spaces. A space is a Q-space (for Hewitt's definition of Q-spaces see [3]) if and only if it is a $N(\mathfrak{m})$-space for some cardinal $\mathfrak{m}$.

If f is a continuous function, then f is bounded on a set M if and only if there exists a set in $\mathfrak{W}(f)$ containing M. Thus we obtain a definition of complete families of continuous functions which does not use coverings.

In section 1 we shall study complete families of continuous functions on an arbitrary space. For convenience we shall use a more general definition of a complete family. But for completely regular spaces both definitions are identical.

In section 2 we shall investigate complete families on completely regular spaces, more precisely, we shall study $N(\mathfrak{m})$-spaces (in particular, Q-spaces) using the concept of a complete family of continuous functions.

The section 3 is devoted to the question:

Let Φ be a mapping from a $N(\mathfrak{m})$-space onto a space Q. Under what conditions on Φ may we assert that Q is a $N(\mathfrak{m})$-space.

If $\mathfrak{Z}$ is a family of sets, then the intersection of $\mathfrak{Z}$ will be denoted by $\bigcap\mathfrak{Z}$, that is

$$\bigcap\mathfrak{Z} = \bigcap\{Z; Z \in \mathfrak{Z}\}.$$

For convenience we shall use the following convention: If V is a property of sets, then a indexed family $\{M_a; a \in A\}$ is said to have the property V if the set of all M_a has the property V. If V is a property of indexed families, then a set $\mathfrak{M}$ has the property V if the indexed family $\{M; M \in \mathfrak{M}\}$ has the property V.

A topological space (in the sequel a space, merely) P is said to be an extension of a space R if R is a dense subspace of P. An extension P of R is said to be Hausdorff, regular, completely regular, compact if P is a Hausdorff, regular, completely regular, compact space, respectively. The Čech-Stone extension of a completely regular space P will be denoted by βP. It is well-known that βP is the compact extension of P uniquely determined by the property:

every bounded real-valued continuous function on P has a continuous extension over βP.

It is also well-known that if K is a compact extension of P, then there exists one and only one continuous mapping Φ from βP onto K such that the restriction of Φ to P is the identity mapping. This mapping will be called Čech-Stone mapping.

Function will always mean a real-valued function. A subset M of a space is said to be a Z-set if there exists a continuous function f on P such that

$$M = Z(f) = \{x; f(x) = 0\}.$$

A subset M of a space P is said to be a N-set if $P - M$ is a Z-set. We shall use the notation

$$N(f) = \{x; f(x) \neq 0\} .$$

1. COMPLETE FAMILIES OF FUNCTIONS

1.1. Definition. Let $\mathfrak{F}$ be a family of continuous functions on a space P. $\mathfrak{F}$ is said to be complete if the following conditions is satisfied:

1.1.1. If $\mathfrak{Z}$ is centered family of Z-sets in P and if for each f in $\mathfrak{F}$ there exists a Z_f in $\mathfrak{Z}$ such that f is bounded on Z_f, then $\bigcap \mathfrak{Z} \neq \emptyset$.

Note. We have at once that a family of continuous functions containing a complete family is a complete family.

1.2. Lemma. *Let $\mathfrak{Z}$ be a maximal centered family of Z-sets in a space P such that the intersection of every countable subfamily is non-void. For every continuous function f on P there exists a Z in $\mathfrak{Z}$ on which f is bounded.*

Proof. Let f be a continuous function on P. For every $n = 1, 2, \ldots$ denote by Z_n the set

$$\{x; x \in P, |f(x)| \geqq n\} . \tag{2}$$

If for some n the set Z_n does not belong to $\mathfrak{Z}$, then there exists a Z in $\mathfrak{Z}$ with $Z_n \cap Z = \emptyset$. Then $|f(x)| \leqq n$ for x in Z and hence f is bounded on Z. In the other case we have $Z_n \in \mathfrak{Z}$ for every $n = 1, 2, \ldots$ By our assumption we have

$$Z_0 = \bigcap_{n=1}^{\infty} Z_n \neq \emptyset .$$

According to (2)

$$x \in Z_0 \Rightarrow |f(x)| \geqq n$$

for every n, which is impossible since f is finite-valued.

As an immediate consequence of 1.2 we have

1.3. Theorem. *If there exists a complete family of continuous functions on a space P, then the following condition is satisfied:*

1.3.1. *If $\mathfrak{Z}$ is a maximal centered family of Z-sets in P such that the intersection of every its countable subfamily is non-void, then $\bigcap \mathfrak{Z} \neq \emptyset$.*

1.4. Lemma. *Let $\mathfrak{Z}$ be a maximal centered family of Z-sets in P. If the intersection of some countable subfamily of $\mathfrak{Z}$ is empty, then there exists a continuous function f on P which is bounded on no Z in $\mathfrak{Z}$.*

Proof. Let $\{Z_n\}$ be a sequence in $\mathfrak{Z}$ such that

$$\bigcap_{n=1}^{\infty} Z_n = \emptyset . \tag{3}$$

Choose continuous functions f_n on P such that $Z_n = Z(f_n)$ and $0 \leqq f_n \leqq 1$ Consider the continuous function $g = 1/f$ where $f = \sum_{u=1}^{\infty} \frac{1}{2^n} f_n$. Clearly:

$$x \in \bigcap_{i=1}^{n} Z_i \Rightarrow f(x) \leqq \frac{1}{2^n},$$

and hence, $g(x) \geqq 2^n$ for each x in $\bigcap_{i=1}^{n} Z_i$. It follows immediately that f is bounded on no Z in $\mathfrak{Z}$.

As a corollary of 1.4 we have:

1.5. Theorem. *If a space P satisfies the condition* 1.3.1, *then the family of all continuous function is complete.*

1.6. Definition. A space is said to be quasi-compact if the intersection of every centered family of Z-sets is non-void. A subspace R of P is said to be relatively quasi-compact in P if the following condition is satisfied:

1.6.1. If $\mathfrak{Z}$ is a family of Z-sets in P and if $\mathfrak{Z} \cap R$ is a centered family, then $\bigcap \mathfrak{Z} \cap R \neq \emptyset$.

Note. Evidently every Z-set of a quasicompact space P is relatively quasi-compact in P. Moreover, every intersection of Z-sets of a quasi-compact space P is relatively quasi-compact in P. For further information see [1], 200—202.

1.7. Theorem. *Let $\mathfrak{F}$ be a family of continuous functions on a space P. $\mathfrak{F}$ is complete if and only if the following two conditions* 1.7.1 *and* 1.7.2 *are satisfied:*

1.7.1. *If F is intersection of Z-sets in P and if every $f \in \mathfrak{F}$ is bounded on F, then F is relatively quasi-compact in P.*

1.7.2. *If $\{Z_f; f \in \mathfrak{F}\}$ is a centered indexed family of Z-sets and if f is bounded on Z_f, then*

$$\bigcap \{Z_f; f \in \mathfrak{F}\} \neq \emptyset .$$

Proof. The necessity of conditions 1.7.1 and 1.7.2 is quite obvious. To prove the sufficiency, suppose that $\mathfrak{Z}$ is a centered family of Z-sets in P and that for each f in $\mathfrak{F}$ there exists a Z_f in $\mathfrak{Z}$ on which f is bounded. By 1.7.2 the set

$$F = \bigcap \{Z_f; f \in \mathfrak{F}\}$$

is non-void. By 1.7.1 the set F is relatively quasi-compact in P. Consequently, to prove $\bigcap \mathfrak{Z} \neq \emptyset$ it is sufficient to show that $\mathfrak{Z} \cap F$ is a centered family. But if both Z_1 and Z_2 belong to $\mathfrak{Z}$, there again by 1.7.2 the set

$$\bigcap \{Z_1 \cap Z_2 \cap Z_f; f \in \mathfrak{F}\} = F \cap Z_1 \cap Z_2$$

is non-void. The proof is complete.

We shall need the following

1.8. Lemma. *If $\mathfrak{Z}$ is a maximal centered family of Z-sets in P and if the Z-sets $Z_1, \ldots, Z_k$ cover some $Z \in \mathfrak{Z}$, then some Z_i belongs to $\mathfrak{Z}$.*

Proof. Suppose on the contrary that no Z_i belongs to $\mathfrak{Z}$. According to the maximality of $\mathfrak{Z}$ there exist $Z'_i \in \mathfrak{Z}$, $i = 1, 2, \ldots, k$, such that $Z_i \cap Z'_i = \emptyset$. Then $Z \cap \bigcap_{i=1}^{k} Z'_i$ belongs to $\mathfrak{Z}$, which is impossible since

$$Z \cap \bigcap_{i=1}^{k} Z'_i \subset Z - \bigcup_{i=1}^{k} Z_i = \emptyset .$$

This contradiction completes the proof.

Now we proceed to characterize complete families in terms of mappings of a special sort.

1.9. Definition. A mapping from P to Q is said to be quasi-compact if the inverse image of every point of Q is relatively quasicompact in P. A mapping from P to Q is said to be a Z-mapping if the image of every Z-set of P is closed in Q.

1.10. Theorem. *Let $\mathfrak{F}$ be a family of continuous functions on a space P. Consider the space*

$$E^{\mathfrak{F}} = \mathsf{X}\{E_f;\, f \in \mathfrak{F}\} ,$$

where the E_f are real lines; and also the continuous mapping $\Phi : P \to E^{\mathfrak{F}}$ defined as follows:

$$\Phi(x) = \{f(x) \; ; \quad f \in \mathfrak{F}\} .$$

The family $\mathfrak{F}$ is complete if and only if Φ is a quasi-compact Z-mapping of P to $E^{\mathfrak{F}}$.

Proof. First let us suppose that $\mathfrak{F}$ is a complete family. To prove quasi-compactness of Φ we shall show that

1.10.1. The inverse image of every compact subspace K of $E^{\mathfrak{F}}$ is relatively quasi-compact in P.

It is easy to see that every function f from $\mathfrak{F}$ is bounded on $\Phi^{-1}[K]$. Indeed, we have we have $f(x) = \pi_f(\Phi(x))$ and $f[\Phi^{-1}[K] = \pi_f[K]$ where π_f, denotes the projetions of $E^{\mathfrak{F}}$ onto E_f. Since π_f is a continuous function and K is a compact space, $\pi_f[K]$ is a compact subspace of E_f, and consequently, $\pi_f[K]$ is a bounded subspace of E_f (in the usual metric). K is a compact subspace of the completely regular space $E^{\mathfrak{F}}$ and therefore K is an intersection of Z-sets in $E^{\mathfrak{F}}$. Since Φ is a continuous mapping, it follows at once that $f^{-1}[K]$ is an intersection of Z-sets in P. By Theorem 1.7 the subspace $f^{-1}[K]$ of P is relatively quasi-compact in P. Thus 1.10.1 holds and Φ is a quasi-compact mapping. It remains to prove that Φ is a Z-mapping. Let Z_0 be a Z-set in P. Suppose on the contrary that $\Phi[Z_0] =$

$= F$ is not closed in $E^{\mathfrak{F}}$. Then we may choose $y = \{y_f; f \in \mathfrak{F}\}$ in $\overline{F} - F$. Consider the family

$$\mathfrak{Z} = \{Z_{n,f};\ f \in \mathfrak{F},\ n = 1, 2, \ldots\} \cup (Z_0)$$

of Z-sets in P, where

$$Z_{n,f} = \left\{x;\ x \in P,\ |f(x) - y_f| \leqq \frac{1}{n}\right\}.$$

The point y being an accumulation point of F, the family $\mathfrak{Z}$ is centered. Moreover, each f in $\mathfrak{F}$ is bounded on $Z_{n,f}$. It follows that $\bigcap \mathfrak{Z} \neq \emptyset$. But this is impossible since

$$\bigcap \mathfrak{Z} = \Phi^{-1}[y] \cap Z_0$$

and by our assumption y does not belong to $F = \Phi[Z_0]$, that is, $\bigcap \mathfrak{Z} = \emptyset$. This contradiction completes the proof of necessity.

To prove sufficiency let us suppose that Φ is a quasi-compact Z-mapping. Let $\mathfrak{Z}$ be a maximal centered family of Z-sets in P and suppose that for each f in $\mathfrak{F}$ there exists a Z_f in $\mathfrak{Z}$ such that f is bounded on Z_f. From quasi-compactness of Φ it follows at once that it is sufficient to prove the existence of a point $y = \{y_f; f \in \mathfrak{F}\}$ in $E^{\mathfrak{F}}$ such that $\mathfrak{Z} \cap \Phi^{-1}[y]$ is a centered family. We proceed to construct such a point y.

Choose f in $\mathfrak{F}$. By our assumption f is bounded on Z_f. Hence, there exists a bounded interval I_f of E_f such that

$$f[Z_f] \subset I_f .$$

Let $K_1, \ldots, K_k$ be a finite cover of I_f by closed intervals of lenght less than $\frac{1}{n}$. Since $f^{-1}[K_i]$ are Z-sets in P and

$$\bigcup_{i=1}^{k} f^{-1}[K_i] \supset Z_f \in \mathfrak{Z} ,$$

it follows at once from lemma 1.8 that for some $i = 1, \ldots, k$, $f^{-1}[K_i]$ belongs to $\mathfrak{Z}$.

Thus, for every $n = 1, 2, \ldots$ and for each f in $\mathfrak{F}$ there exists a closed interval $K_{n,f}$ in E_f of length less than $\frac{1}{n}$ such that

$$Z_{n,f} = f^{-1}[K_{n,f}] \in \mathfrak{Z} .$$

Evidently for every f in $\mathfrak{F}$, $\{K_{n,f};\ n = 1, 2, \ldots\}$ is a centered family of compact sets. It follows that

$$\bigcap_{n=1}^{\infty} K_{n,f} \neq \emptyset .$$

This intersection contains only one point, namely y_f, since the lengths of $K_{n,f}$ converge to zero with $n \to \infty$. The point $\{y_f; f \in \mathfrak{F}\}$ will be denoted by y. Since

$\Phi[P]$ is a closed subspace of $E^{\mathfrak{F}}$, y belongs to $\Phi[P]$. It remains to prove that $\mathfrak{Z} \cap \Phi^{-1}[y]$ is a centered family. It is of course sufficient to show that

$$Z \in \mathfrak{Z} \Rightarrow Z \cap \Phi^{-1}[y] \neq \emptyset .$$

Let us suppose on the contrary that some Z in $\mathfrak{Z}$ does not meet $\Phi^{-1}[y]$. The mapping Φ is a Z-mapping and hence $F = \Phi[Z]$ is a closed subspace of $E^{\mathfrak{F}}$. By our assumption y does not belong to F. In consequence, there exists a neighborhood U of y which does not meet F. Since the lengthts of $K_{n,f}$ converge to zero with $n \to 0$, there exist $K_i = K_{n_i,f_i}$ $(i = 1, \ldots, k)$ such that

$$\bigcap_{i=1}^{k} f_i^{-1}[K_i] \cap Z = \emptyset .$$

But this is a contradiction, since $f_i^{-1}[K_i]$ belong to $\mathfrak{Z}$. The proof is complete.

As a corollary of 1.10 and 1.10.1 we have

1.11. Theorem. *If Φ is a quasi-compact Z-mapping from P to the topological product R of a family of real lines, then the inverse image of every compact subspace of R is a relatively quasi-compact subspace of P.*

2. Q-SPACES AND $N(\mathfrak{m})$-SPACES

In this section we shall study complete families of continuous functions on a completely regular space.

2.1. Definition. Let $\mathfrak{m}$ be a cardinal number. A space P is said to be an $N(\mathfrak{m})$-space provided that P is completely regular and there exists a complete family $\mathfrak{F}$ of continuous functions on P such that the potency of $\mathfrak{F}$ is $\leq \mathfrak{m}$. A space is said to be an exact $N(\mathfrak{m})$-space provided that it is an $N(\mathfrak{m})$-space but not an $N(\mathfrak{n})$-space for any cardinal $\mathfrak{n} < \mathfrak{m}$. A space is a Q-space if it is an $N(\mathfrak{m})$-space for some cardinal $\mathfrak{m}$.

Thus a completely regular space is a Q-space if and only if the set of all continuous functions is complete.

2.2. Definition. A mapping Φ of P to Q is said to be compact if the inverse images of points of Q are compact spaces. Φ is closed if the image of every closed subset of P is closed in Q.

2.3. Lemma. *A relatively quasi-compact subspace R of a completely regular space P is a compact space. A quasi-compact mapping from a completely regular space to a space is a compact mapping. A quasi-compact Z-mapping from a completely regular space to a space is a compact closed mapping.*

Proof. Let R be relatively quasi-compact in a completely regular space P. Let $\{F\}$ be a centered family of closed subsets of R. Let $\mathfrak{Z}$ be the family of all Z-sets in P such that for some F in $\{F\}$ the inclusion $\overline{F}^P \subset Z$ holds. Since P is a completely regular space, we have $\bigcap \mathfrak{Z} = \bigcap \{F\}$. By quasicompactness of R we have

$$R \cap \bigcap \mathfrak{Z} \neq \emptyset .$$

Combining the above two relations we obtain $\bigcap \{F\} \neq \emptyset$.

The second statement of the lemma is an immediate consequence of the first.

To prove the third statement let us suppose that Φ is a quasi-compact Z-mapping from a completely regular space P to Q. Then Φ is a compact mapping and it remains to show that Φ is a closed mapping. Let F be a closed subspace of P. Denote by $\mathfrak{Z}$ the family of all Z-sets in P containing F. Since P is completely regular, we have $\bigcap \mathfrak{Z} = F$. Put $F_0 = \Phi[F]$. It is sufficient to prove

$$F_1 \stackrel{\text{def}}{=} \bigcap \{\Phi[Z];\ Z \in \mathfrak{Z}\} = F_0 .$$

The inclusion $F_1 \supset F_0$ is trivial. For the other one, suppose that there exists a point y in $F_1 - F_0$. We see at once that

$$\mathfrak{Z} \cap \Phi^{-1}[y] \tag{4}$$

is a centered family of closed subsets of the compact space $\Phi^{-1}[y]$. Thus we may choose a point x in the intersection of the family (4). But this is impossible since

$$x \in \bigcap \mathfrak{Z} = F\,, \quad \Phi(x) = y \text{ non} \in \Phi[F]\,.$$

From 1.7 and 2.3 we have

2.4. Theorem. *Suppose that $\mathfrak{F}$ is a family of continuous functions on a completely regular space P. $\mathfrak{F}$ is complete if and only if the following two conditions are satisfied:*

2.4.1. *If K is closed in P and if each f from $\mathfrak{F}$ is bounded on K, then K is a compact space.*

2.4.2. *If $\{Z_f;\ f \in \mathfrak{F}\}$ is a centered indexed family of Z-sets in P such that f is bounded on Z_f, then*

$$\bigcap \{Z_f;\ f \in \mathfrak{F}\} \neq \emptyset .$$

Definition. $\mathfrak{m}$ being a cardinal number, denote by $E^{\mathfrak{m}}$ the topological product of $\mathfrak{m}$ real lines.

As an immediate consequence of 1.10 and 2.2 we have

2.5. Theorem. *Let $\mathfrak{F}$ be a family of continuous functions on a completely regular space P. Define $E^{\mathfrak{F}}$ and Φ as in* 1.10.

$\mathfrak{F}$ is complete if and only if Φ is a closed compact mapping.

2.6. Theorem. *A space P is an $N(\mathfrak{m})$-space if and only if P is completely regular and there exists a continuous closed compact mapping Φ from P to $E^{\mathfrak{m}}$.*

Proof. First suppose that P is an $N(\mathfrak{m})$-space. Hence P is completely regular and there exists a complete family $\mathfrak{F}$ of continuous function on P such that the potency of $\mathfrak{F}$ at most $\mathfrak{m}$. Without loss of generality we may assume that the potency of $\mathfrak{F}$ is $\mathfrak{m}$. Define $E^{\mathfrak{F}}$ and Φ as in 1.10. By 2.5 Φ is closed and compact.

Evidently Φ is continuous and $E^{\mathfrak{F}} = E^{\mathfrak{m}}$.

Conversely, let Φ be a continuous closed compact mapping from a completely regular space P to

$$E^{\mathfrak{m}} = X\{E_a; a \in A\}$$

where the potency of the index set A is $\mathfrak{m}$ and the E_a are real lines. For each a in A denote by π_a the projection of $E^{\mathfrak{m}}$ onto E_a. Denote by f_a the function $\pi_a(\Phi)$. Every f_a is continuous as the superposition of two continuous mappings. Evidently for each x in P,

$$\Phi(x) = \{f_a(x); a \in A\} .$$

Applying 2.5 we obtain that the family of all f_a is complete.

If Φ is a closed compact mapping from P to Q and if F is a closed subset of P, then the restriction of Φ to F is a closed compact mapping. From this fact and from 2.5 and 2.6 we have at once

2.7. Theorem. *If $\mathfrak{F}$ is a complete family of continuous functions on a completely regular space P and if F is a closed subspace of P, then the family of the restrictions (to F) of all $f \in \mathfrak{F}$ is a complete family on F. Closed subspaces of $N(\mathfrak{m})$-spaces are $N(\mathfrak{m})$-spaces.*

Now we proceed to characterize $N(\mathfrak{m})$-spaces as intersections of $\mathfrak{m}$ N-sets in their Čech-Stone extensions.

2.8. Proposition. Let $\mathfrak{F}$ be a family of continuous functions on a completely regular space P such that $f \geqq 1$ for each f in $\mathfrak{F}$. For each f in $\mathfrak{F}$ denote by f^* the continuous extension of $1/f$ over βP ($1/f$ is bounded).

Then $\mathfrak{F}$ is complete if and only if

2.8.1. $$\bigcap\{N(f^*); f \in \mathfrak{F}\} = P .$$

Proof. First let us suppose that 2.8.1 holds. Let $\mathfrak{Z}$ be a centered system of Z-sets in P such that for each f in $\mathfrak{F}$ there exists a Z_f in $\mathfrak{Z}$ on which f is bounded. βP being a compact space, the set

$$F_0 = \bigcap\{\overline{Z}^{\beta P}; Z \in \mathfrak{Z}\}$$

is non-void. It is sufficient to show that $F \subset P$. According to 2.8.1 it is sufficient to show that

(5) $$\overline{Z}_f^{\beta P} \subset N(f^*)$$

for each f in $\mathfrak{F}$. f is bounded on Z_f,

$$x \in Z_f \Rightarrow |f(x)| \leqq M$$

say, and hence (f^* is continuous)

$$x \in \overline{Z}_f^{\beta P} \Rightarrow |f^*(x)| \geqq M^{-1}$$

which implies (5).

To prove necessity, let us suppose that there exists a point x in

$$\bigcap\{N(f^*);\ f \in \mathfrak{F}\} - P\ .$$

Let $\mathfrak{Z}$ be the family of all Z-sets in βP containing x in their interior. Evidentely

$$\bigcap\{Z;\ Z \in \mathfrak{Z}\} = (x) \subset \beta P - P\ .$$

Thus $\mathfrak{Z} \cap P$ is a centered family of Z-sets in P with empty intersection. To prove that $\mathfrak{F}$ is not complete, it is sufficient to show that for each f in $\mathfrak{F}$ there exists a Z_f in $\mathfrak{Z}$ such that f is bounded on Z_f. Fix $f \in \mathfrak{F}$ Since $f^*(x) \neq 0$, there exist a Z_f in $\mathfrak{Z}$ and an $\varepsilon > 0$ with

$$y \in Z_f \Rightarrow |f^*(y)| \geqq \varepsilon\ .$$

If follows that $y \in Z \cap P \Rightarrow |f(y)| \leqq 1/\varepsilon$. The proof is complete.

As an immediate consequence of 2.8 we have:

2.9. Theorem. *A completely regular space P is an $N(\mathfrak{m})$-space if and only if there exists a set $\mathfrak{N}$ of N-sets in βP such that the potency of $\mathfrak{N}$ is at most $\mathfrak{m}$ and $\bigcap\{N;\ N \in \mathfrak{N}\} = P$.*

Now we shall proceed to give the usual characterisation of $N(\mathfrak{m})$-spaces. First we prove the following crucial property of continuous closed compact mappings.

2.10. Theorem. *Let Φ be a continuous closed compact mapping from a regular space P to a space Q. There exists no proper regular extension R of P on which Φ may be continuously extended.*

Proof. Let us suppose, on the contrary, that there exists a proper regular extension R of P and a continuous mapping Φ^* from R to Q such that Φ is the restriction of Φ^*. Choose x in $R - P$. Since $\Phi[P]$ is a closed subset of Q (Φ is closed) and since by continuity of Φ^*

$$\Phi^*[R] \subset \overline{\Phi[P]}$$

we have at once that $\Phi^*[R] = \Phi[P]$. Hence, there exists a y in $\Phi[P]$ such that $\Phi^*(x) = y$. Denote by K the inverse image under Φ of y (that is, the set $\Phi[y]$). Φ is a compact mapping, and consequently, K is a compact space. It follows that

$$x \text{ non} \in \overline{K}^R = K\ .$$

Since R is a regular space, we may choose a closed (in R) neighborhood F of x with $F \cap K = \emptyset$. Consider the set $F \cap P$. Φ being a closed mapping, $\Phi[F \cap P]$ is a closed subset of Q. Since $F \cap K = \emptyset$, it follows that

$$y \text{ non} \in \Phi[F \cap P] = \Phi^*[F \cap P]\ .$$

But this is impossible, since Φ^* is continuous, $x \in \overline{F \cap P}$ and $\Phi^*(x) = y$. This contradiction establishes the theorem.

2.11. Definition. Suppose that P and Q are completely regular spaces. A continuons mapping Φ from P to Q is said to be non-extensible if for any proper completely regular extension R of P and any continuous mapping Φ^* from R to Q, the restriction of Φ^* to P is different from Φ.

Combining 2.6 and 2.10 we obtain at once:

2.12. Proposition. *If P is an $N(\mathfrak{m})$-space then there exists a continuous non-extensible mapping from P to $E^{\mathfrak{m}}$.*

In the converse direction we shall prove:

2.13. Proposition. *Let us suppose that there exists a continuous non-extensible mapping Φ from a completely regular space P to $E^{\mathfrak{m}}$. Then P is an $N(\mathfrak{m})$-space.*

Proof. Introduce the same notation as in the proof of 2.6:

$$E^{\mathfrak{m}} = X\{E_a; a \in A\}\,, \quad \Phi(x) = \{f_a(x); a \in A\}\,.$$

It is sufficient to show that $\{f_a; a \in A\}$ is a complete family. Suppose, on the contrary, that $\{f_a; a \in A\}$ is not complete. Thus, there exists a maximal centered family $\mathfrak{Z}$ of Z-sets in P such that

$$\bigcap\{Z; Z \in \mathfrak{Z}\} = \emptyset$$

and for each a in A there is a Z_a in $\mathfrak{Z}$ such that f_a is bounded on Z_a. βP being compact and $\mathfrak{Z}$ being a maximal centred family of Z-sets, the intersection of the family $\{\overline{Z}^{\beta P}; Z \in \mathfrak{Z}\}$ contains exactly one point, namely x. Since Φ is a non-extensible continuous mapping, there exists a f_a which is non-extensible over $P \cup (x)$, and clearly, since every bounded continuous function of P is extensible over βP, there must be

$$\lim_{\substack{z \to x \\ z \in P}} f_a(z) = \pm \infty\,.$$

But, x is contained in the closure of every Z in $\mathfrak{Z}$, and consequently, we have

$$Z \in \beta \Rightarrow \lim_{\substack{z \in Z \\ z \to x}} f_a(z) = \pm \infty\,.$$

Particulary, f_a is not bounded on Z_a. This contradiction establishes the Theorem.

Combining 2.12 a 2.13 we obtain:

2.14. Theorem. *A completely regular space P is an $N(\mathfrak{m})$-space if and only if there exists a continuous non-extensible mapping from P to $E^{\mathfrak{m}}$.*

We shall need the following proposition (see [4] and [3]).

2.15. *A space is a Q-space if and only if it is homeomorphic with some closed subspace of $E^{\mathfrak{m}}$ for some $\mathfrak{m}$.*

Proof. Let $\mathfrak{F}$ be the set of all continuous functions on P. Define $E^{\mathfrak{F}}$ and Φ as in 1.10 It is well known that Φ is a homeomorphic mapping if and only if P is a completely regular space. Now the statement follows from the note preceding 1.2 and 2.5.

Now we are prepared to prove the following theorem.

2.16. Theorem. *Let Φ be a continuous mapping from a completely regular space P to a Q-space Q. The following two conditions on Φ are equivalent:*

2.15.1. *Φ is closed and compact.*

2.15.2. *Φ is non-extensible.*

Proof. By 2.10 the assertion 2.15.1 implies 2.15.2. Conversely, suppose that Φ is non-extensible. By 2.15 there exists a homeomorphic mapping Ψ of Q onto a closed subspace of $E^{\mathfrak{m}}$ for some cardinal $\mathfrak{m}$. We see at once that the superposition $\Psi(\Phi)$ of Ψ and Φ is a non-extensible mapping from P to $E^{\mathfrak{m}}$. By 2.13 $\Psi(\Phi)$ is a closed compact mapping, and consequently, Ψ being homeomorphic, Φ is a closed compact mapping.

2.17. Theorem. *Let P, Q and R be completely regular spaces. If Φ is a continuous closed compact mapping from P to Q and if Ψ is a continuous closed compact mapping from Q to R, then the superposition of Ψ and Φ is a continuous closed compact mapping.*

2.18. *If Φ is a continuous closed compact mapping of a space P onto a compact space Q, then P is a compact space. (It may be noticed that if Φ is a continuous mapping from P onto a compact space Q, then P is compact if and only if Φ is closed and compact.)*

The proof of 2.18 is quite routine and may be left to the reader.

As an immediate consequence of 2.17 we have:

2.19. Theorem. *A completely regular space P is an $N(\mathfrak{m})$-space if and only if there exists a continuous closed compact mapping from P to an $N(\mathfrak{m})$-space.*

2.20. Theorem. *Let $\{P_a; a \in A\}$ be an indexed family such that P_a is an $N(\mathfrak{m}_a)$-space. Then the topological product $P = X\{P_a; a \in A\}$ is an $N(\mathfrak{m})$-space, where $\mathfrak{m} = \Sigma\{\mathfrak{m}_a; a \in A\}$.*

To prove 2.20 it is sufficient to show that:

2.21. Theorem. *Let $\{P_a; a \in A\}$ and $\{Q_a; a \in A\}$ be indexed families of completely regular spaces. For each a in A let Φ_a be a continuous closed compact mapping from P_a to Q_a.*

Consider the product spaces $P = X\{P_a; a \in A\}$ and $Q = X\{Q_a; a \in A\}$ and the mapping $\Phi = \{\Phi_a; a \in A\}$ defined as follows:

$$\Phi(x) = \{\Phi_a(x_a); a \in A\} .$$

The mapping Φ is continuous, closed and compact.

Proof. The proof of continuity is quite routine and may be left to the reader. Denote by π_a the projection of Q onto Q_a. Let y be an element of Q. Clearly

$$\Phi^{-1}[y] = X\{\Phi_a^{-1}[\pi_a(y)];\ a \in A\} .$$

The spaces $\Phi_a^{-1}[\pi_a(y)]$ being compact, the space $\Phi^{-1}[y]$ is compact by Tychonoff's theorem. Thus Φ is a compact mapping. It remains to prove that Φ is a closed mapping. First, let F be a closed subset of P of the form

$$X\{F_a;\ a \in A\} \tag{6}$$

where F_a is a closed subset of P_a. Clearly

$$\Phi[F] = X\{\Phi_a[F_a];\ a \in A\} .$$

Φ_a being closed, the set $\Phi_a[F_a]$ is closed in Q_a, and consequently, $\Phi[F]$ is closed in Q. Now, let F be an arbitrary closed subset of P. Let $\mathfrak{M}$ be the family of all closed subsets of P of the form (6), and containing F. $\Phi[M]$ being closed in Q, the set

$$F_0 = \bigcap\{\Phi[M];\ M \in \mathfrak{M}\}$$

is closed in Q, and consequently, it is sufficient to show that $F_0 = \Phi[F]$. Clearly $F_0 \supset \Phi[F]$. Suppose that there exists a y in $F_0 - \Phi[F]$. Thus $\Phi^{-1}[y] = K$ is a compact subspace of P disjoint with F. Since

$$\bigcap\{M;\ M \in \mathfrak{M}\} = F ,$$

there exists a M in $\mathfrak{M}$ with $M \cap K = \emptyset$. $\Phi[M]$ being closed, we have at once that y non $\in \Phi[M] \supset F_0$. This contradiction completes the proof of 2.21.

Now we give another proof of 2.20 using 2.9 and Stone-Čech theorem (and also Tychonoff's theorem). By 2.9, for each a in A there exists a family $\mathfrak{N}_a$ of N-sets in βP_a such that the potency of $\mathfrak{N}_a$ is at most $\mathfrak{m}_a$ and

$$\bigcap\{N;\ N \in \mathfrak{N}_a\} = P_a .$$

Consider the space $K = X\{\beta P_a;\ a \in A\}$. Denote by π_a the projection of K onto βP_a. Let $\mathfrak{N}_a^*$ be the family of all $\pi_a^{-1}[N]$ where $N \in \mathfrak{N}_a$. Denote by $\mathfrak{N}^*$ the union of the indexed family $\{\mathfrak{N}_a^*;\ a \in A\}$. Evidently $\mathfrak{N}^*$ is a family of N-sets in K and

$$\bigcap\{N;\ N \in \mathfrak{N}^*\} = P .$$

Since the potency of $\mathfrak{N}^*$ is at most $\mathfrak{m}$, the space P is the intersection of $\mathfrak{m}$ N-sets in K. Let Φ be the Čech-Stone mapping from βP onto K. Evidently

$$\bigcap\{\Phi^{-1}[N];\ N \in \mathfrak{N}^*\} = P$$

and $\Phi^{-1}[N]$ are N-sets in βP. By 2.9, the space P is an $N(\mathfrak{m})$-space. The second proof of 2.20 is complete.

In conclusion we give a summary of definitions of $N(\mathfrak{m})$-spaces:

2.22. Theorem. *The following condition on a completely regular space P are equivalent*:

(1) *There exists a complete family* $\mathfrak{F}$ *of continuous functions on* P *such that the potency of* $\mathfrak{F}$ *is* $\leqq \mathfrak{m}$.

(2) *There exists a continuous closed compact mapping from* P *to* $E^{\mathfrak{m}}$.

(3) *There exists a continuous non-extensible mapping from* P *to* $E^{\mathfrak{m}}$.

(4) *There exists a continuous closed compact mapping from* P *to an* $N(\mathfrak{m})$-*space.*

(5) *There exists a continuous non-extensible mapping from* P *to an* $N(\mathfrak{m})$-*space.*

(6) P *is an intersection of* $\mathfrak{m}$ N-*sets in some compactification of* P.

(7) P *is an intersection of* $\mathfrak{m}$ N-*sets in* βP.

3. IMAGES OF Q-SPACES

All spaces are assumed to be completely regular. Let Φ be a continuous mapping from a Q-space P onto Q. Under what conditions on Φ may we assert that Q is a Q-space?

We recall that a subspace P' of P is said to be relatively pseudocompact in P, if for every sequence $\{Z_n\}$ of Z-sets in P such that $\{Z_n \cap P'\}$ is centered, the intersection $P' \cap \bigcap_{n=1}^{\infty} Z_n$ is non-void. Equivalently, P' is relatively pseudocompact if and only if every continuous function on P is bounded on P'.

3.1. Theorem. *Let* Φ *be a continuous mapping from* P *onto* Q *such that*

3.1.1. *The images of* Z-*sets are* Z-*sets, that is, if* Z *is a* Z-*set in* P, *then* $\Phi[Z]$ *is a* Z-*set in* Q.

3.1.2. *The inverses of points under* Φ *are relatively pseudocompact, that is, for each* y *in* Q *the subspace* $\Phi^{-1}[y]$ *of* P *is relatively pseudocompact in* P.

Then if P *is a* Q-*space,* Q *is also a* Q-*space.*

Proof. Let us suppose that $\mathfrak{Z}$ is a maximal centered family of Z-sets in Q such that the intersection of every countable subfamily of $\mathfrak{Z}$ is non-void. Let $\mathfrak{Z}'$ be the family of all $\Phi^{-1}[Z]$ where $Z \in \mathfrak{Z}$. Evidently, $\mathfrak{Z}'$ is a centered family of Z-sets in P. Let $\mathfrak{Z}''$ be a maximal centered family of Z-sets in P containing $\mathfrak{Z}'$. We shall prove that the intersection of every countable subfamily of $\mathfrak{Z}''$ is non-void. Indeed, let $\{Z_n'\}$ be a sequence of Z-sets in $\mathfrak{Z}''$. By 3.1.1 the sets $Z_n = \Phi[Z_n']$ are Z-sets in Q, and clearly $Z_n \in \mathfrak{Z}$. Choose a point y in $\bigcap_{n=1}^{\infty} Z_n$. By 3.1.2 we have

$$\Phi^{-1}[y] \cap \bigcap_{n=1}^{\infty} Z_n' \neq \emptyset .$$

P being a Q-space, the set $\bigcap\{Z'; Z' \in \mathfrak{Z}''\}$ is non-void. Choose a point x in this intersection. Evidently

$$\Phi(y) \in \bigcap\{Z; Z \in \mathfrak{Z}\} .$$

The theorem is proved.

We proceed to quotient mappings:

3.2. *The image of a Q-space under an open continuous mapping may fail to be a Q-space.*

Proof. Let us suppose that Q is not a Q-space and let

$$Q = \bigcup\{K_a;\ a \in A\}\ ,$$

where K_a are compact subspaces of Q. Finally, suppose the indexed set A endowe with the discrete topology is a Q-space. Under these assumptions we shall construct a Q-space P and a continuous open mapping Φ from P onto Q. Consider the product space $R = Q \times A$ and the subspace

$$P = \bigcup\{K_a \times a;\ a \in A\}$$

of R. By 2.19 the space P is a Q-space. Indeed, the mapping $x \in K_a \times a \to a$ is a closed compact continuous mapping of P onto A. Denote by Φ the projection map of P onto Q, i. e. Φ is the restriction of the projection of R onto Q. It is easy to show that the mapping Φ is open and continuous. The proofs of existence of Q, K_a and A may be left to the reader.

Modifying the construction in 3.2 (to consider the disjoint union) we obtain at once:

3.3. The image of a Q-space under a compact open continuous mapping may fail to be a Q-space.

Note. If the topological product $P \times Q$ is a Q-space ($N(\mathfrak{m})$-space), then both P and Q are Q-spaces ($N(\mathfrak{m})$-spaces, respectively).

We shall need the following assertion:

3.4. Lemma. *Let Φ be an open, closed and continuous mapping from P onto Q. Let f be a continuous function on P. For each y in Q put*

$$F(y) = \sup\{f(x);\ \Phi(x) = y\}\ .$$

If $F(y)$ is a real number for each y in Q, then F is a continuous function on Q.

Proof. Let y_0 be an element of Q and let ε be a positive real number. For each x in $X = \Phi^{-1}[y_0]$ choose an open neighborhood $U(x)$ of x on which f varies less than ε. Denote by U the union of all $U(x)$, $x \in X$. Put

$$V = \bigcup\{\Phi^{-1}[y];\ \Phi^{-1}[y] \subset U\}\ .$$

Φ being closed, V is an open subset of P. Evidently

$$y \in \Phi[V] \Rightarrow F(y) \leqq F(y_0) + \varepsilon\ .$$

Φ being an open mapping, $\Phi[V]$ is an open neighborhood of y_0. Thus F is an upper semi-continuous function. It remains to prove that F is lower semi-continuous. Choose a point x_0 in X such that $f(x_0) > F(y_0) - \frac{\varepsilon}{2}$. Choose an open neighborhood W of x_0 such that

$$x \in W \Rightarrow f(x) > f(x_0) - \frac{\varepsilon}{2} \,.$$

Φ being open, $W' = \Phi[W]$ is an open neighborhood of y_0. We have at once

$$y \in W' \Rightarrow F(y) > F(y_0) - \varepsilon \,.$$

This establishes lower semi-continuity of F and completes the proof of 3.4.

3.5. Proposition. Let us suppose that Φ is a closed open and continuous mapping from P onto Q. Let $\mathfrak{F}$ be a complete family of continuous non-negative functions on P. Suppose that for each f in $\mathfrak{F}$ the function F defined as in 3.4 is real-valued, that is, F is finite. Denote by $\mathfrak{F}'$ the family of all F where $f \in \mathfrak{F}$.

Then $\mathfrak{F}'$ is a complete family of continuous functions on Q.

Proof. By 3.4 the functions $F \in \mathfrak{F}'$ are continuous. To prove completness of $\mathfrak{F}'$, let $\mathfrak{Z}$ be a centered family of closed subsets of Q such that for each F in $\mathfrak{F}'$ there exists a Z_F in $\mathfrak{Z}$ with F bounded on Z_F. Denote by $\mathfrak{Z}'$ the family of all $\Phi^{-1}[Z]$ where $Z \in \mathfrak{Z}$. Evidently, $\mathfrak{Z}'$ is a centered family of closed subsets of P. Moreover, for each f in $\mathfrak{F}$ there exists a Z_f in $\mathfrak{Z}'$ such that f is bounded on Z_f. Indeed, if F is a function corresponding to f, we may put $Z_f = \Phi^{-1}[Z_F]$. $\mathfrak{F}$ being a complete family, we have

$$F_0 = \bigcap\{Z; Z \in \mathfrak{Z}'\} \neq \emptyset \,.$$

Clearly $\Phi[F_0] \subset \bigcap\{Z; Z \in \mathfrak{Z}\}$. Thus $\mathfrak{F}'$ is complete.

As an immediate consequence of 3.5 we have:

3.6. Theorem. *Let Φ be a closed, open and continuous mapping from P onto Q. Suppose that the tranches of Φ (that is, the sets of the form $\Phi^{-1}[y]$, $y \in Q$) are relatively pseudocompact spaces. If P is an $N(\mathfrak{m})$-space, then Q is an $N(\mathfrak{m})$-space. In particular, if P is a Q-space, then Q is a Q-space.*

4. $N(1)$-SPACES AND $N(\aleph_0)$-SPACES

4.1. Theorem. *Let $\mathfrak{m} \geqq 1$. A discrete space M is an $N(\mathfrak{m})$-space if and only if it is homeomorphic with some closed subspace (discrete, of course) of $E^{\mathfrak{m}}$.*

Proof. The theorem is obvious for finite $\mathfrak{m}$. Suppose $\mathfrak{m} \geqq \aleph_0$. We shall use 2.22, condition 2.22.2. To prove necessity let us suppose that Φ is a continuous, closed and compact mapping from M to E^m. The tranches of Φ being compact and discrete, they are finite. Thus M and $\Phi[M]$ has the same potency. The image under closed mappings of a discrete space is a discrete space. The discrete spaces M and $\Phi[M]$ have the same potency, and consequently, they are homeomorphic. The sufficiency is obvious.

As a corollary of 4.1 and of the fact that $E^{\aleph_0}$ is a metrizable and separable space we have

4.2. Theorem. *The following conditions on a discrete space are equivalent:*

4.2.1. *M is a $N(1)$-space.*

4.2.2. *M is a $N(\aleph_0)$-space.*

4.2.3. *The potency of M is at most $\aleph_0$.*

4.3. Theorem. *The following conditions on a space P are equivalent*

4.3.1. *P is a $N(1)$-space.*

4.3.2. *There exists a continuous function f on P such that every closed subspace K of P is compact if and only if the function f is bounded on K.*

4.3.3. *There exists a sequence $\{K_n\}$ of compact subspaces of P such that $K_n \subset \subset \operatorname{int} K_{n+1}$ and $\bigcup_{n=1}^{\infty} K_n = P$.*

4.3.4. *P is locally compact and σ-compact.*

Proof. By 2.4 the conditions 4.3.1 and 4.3.2 are equivalent. If f is the function from 3.1.2 and if we put

$$K_n = \{x;\ |f(x)| \leqq n\}\ ,$$

we obtain a sequence $\{K_n\}$ satisfying 3.1.3. Thus 3.1.2 implies 3.1.3. Suppose 3.1.3. Choose continuous functions f_n, $n = 3, 4, \ldots$, such that

$$f_n(x) = \begin{cases} n & \text{for} \quad x \text{ non} \in K_{n-1}\ , \\ 0 & \text{for} \quad x \in K_{n-2} \end{cases}$$

and $0 \leqq f_n(x) \leqq n$ for every x. Put $f = \sum_{n=3}^{\infty} f_n$. Evidently, f is bounded on a set M if and only if the set M is contained is some finite union $\bigcup_{i=1}^{n} K_i$. It follows at once that f satisfies 3.1.2. Thus 3.1.3 implies 3.1.2. The proof of equivalence of 3.1.3 and 3.1.4 is quite routine and may be left to the reader.

4.4. Theorem. *A metrizable space is an $N(1)$-space if and only if it is separable and locally compact.*

Proof. First suppose that P is a metrizable $N(1)$-space. Evidently P is locally compact. By 4.2 and 2.7 the space P contains no uncountable discrete closed subset. Thus P is separable. Conversely, P being a separable and metrizable space, P has a compact metrizable extension K. P being locally compact, P is open in K. An open subset of a metrizable space is an N-set. Thus P is an N-set of a compact space K. It follows that P is an $N(1)$-space.

Recall that a space P is said to be a G_δ-space if it is a G_δ-set in every extension. It is well-known that a metrizable space is a G_δ-space if and only if there exists a metric φ for P such that (P, φ) is a complete metric space (for further informations see [2]). By [2], theorem 2.8, a completely regular space P is a G_δ-space if and only if it is a G_δ-subset of some compact space. Thus every $N(\aleph_0)$-space is a G_δ-space.

4.5. Theorem. *The following two conditions on a metrizable space P are equivalent*:

4.5.1. *P is an $N(\aleph_0)$-space.*

4.5.2. *P is a separable G_δ-space.*

Proof. As we note above, an $N(\aleph_0)$-space is a G_δ-space. Thus, to prove that 4.5.1 implies 4.5.2, it is sufficient to show that every metrizable $N(\aleph_0)$-space is separable. By 4.2 and 2.7 the space P contains no uncountable discrete closed subspace. It follows that P is separable. Conversely, suppose 4.5.2. P being separable, there exists a compact metrizable extension K of P. P being a G_δ-space, P is a G_δ-subset of K, and consequently, P is an $N(\aleph_0)$-set in K. Thus P is an $N(\aleph_0)$-space.

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Резюме

ПРИЛОЖЕНИЯ ПОЛНЫХ СЕМЕЙСТВ ФУНКЦИЙ В ТЕОРИИ ФУНКЦИОНАЛЬНО ЗАМКНУТЫХ ПРОСТРАНСТВ

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Подмножество N пространства P называют N-множеством, если существует непрерывная функция f на P так, что

$$N = N(f) = \{x; x \in P, f(x) \neq 0\}.$$

Если N является N-множеством, то $P - N$ называют Z-множеством. Пространство P называется функционально замкнутым (или Q-пространством, см. [3]) если выполнено следующие условие: всякая максимальная

счетно центрированная система Z-множеств (т. е. максимальная система Z-множеств такая, что любая счетная подсистема имеет непустое пересечение) имеет непустое пересечение. В статье дается определение Q-пространств при помощи полных семейств непрерывных функций. Семейство $\mathfrak{F}$ непрерывных функций называется полным, если выполнено следующее условие:

Если $\mathfrak{Z}$ — такая центрированная система Z-множеств, что всякая функция из $\mathfrak{F}$ ограничена на некотором множестве из $\mathfrak{Z}$, то пересечение системы $\mathfrak{Z}$ не пусто.

Оказывается, что вполне регулярное пространство является Q-пространством тогда и только тогда, если семейство всех непрерывных функций полно.

В статье определены т. наз. $N(\mathfrak{m})$-пространства ($\mathfrak{m}$ — некоторое кардинальное число). P называется $N(\mathfrak{m})$-пространством если существует полное семейство $\{f_\varrho; \in A\}$ непрерывных функций на P такое, что мощность множества A равна $\mathfrak{m}$. Итак, вполне регулярное пространство является Q-пространством тогда и только тогда, если оно является $N(\mathfrak{m})$-пространством для некоторого m. Пусть Φ — отображение пространства P в пространство Q; Φ называется бикомпактным, если прообразы точек бикомпактны, замкнутым, если образы замкнутых множеств замкнуты; наконец, непрерывное Φ называется нерасширимым, если, какого бы ни было пространство R, $R > P$, $R = \bar{P}$, $R \neq P$, отображение Φ нельзя расширить до непрерывного отображения пространства R в Q. Доказана следующая

Теорема. *Следующие свойства вполне регулярного пространства P эквивалентны ($\mathfrak{m}$ — кардинальное число):*

(1) *P является $N(\mathfrak{m})$-пространством.*

(2) *Существует бикомпактное, замкнутое и непрерывное отображение пространства P в топологическое произведение $\mathfrak{m}$ прямых.*

(3) *Существует непрерывное нерасширимое отображение пространства P в топологическое произведение $\mathfrak{m}$ прямых.*

(4) *P является пересечением $\mathfrak{m}$ N-множеств в некотором своем бикомпактном расширении.*

(5) *P является пересечением $\mathfrak{m}$ N-множеств с своем чеховском бикомпактном расширении.*

В последней части рассматривается вопрос, при каких условиях непрерывный образ $N(\mathfrak{m})$-пространства является $N(\mathfrak{m})$-пространством. Указывается, что достаточно предполагать, что отображение замкнуто, открыто, и полные прообразы точек относительно псевдокомпактны.