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GENERALIZATIONS OF THE G_δ-PROPERTY OF COMPLETE METRIC SPACES

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The present paper is devoted to generalisations of two important properties of complete metric spaces.

The investigations of the present paper are motivated by the following two well-known theorems:

0.1. If P is a G_δ-subset of a complete metric space, then for some metric φ for P the metric space (P, φ) is complete. Conversely, if a subset P of a metrizable space R is homeomorphic with a complete metric space, then P is a G_δ-subset of R.

0.2. If f is a continuous mapping from a subspace P of a metrizable space R to a complete metric space Q, then there exist a G_δ-subset S of R and a continuous mapping F from S to Q such that f is a restriction of F.

In view of 0.1 we define: A Hausdorff (topological) space is said to be a G_δ-space if it is a G_δ-set in every one of its Hausdorff extensions, that is, if P is a dense subset of a Haurdorff space R, then P is a G_δ-set in R. The concept of a G_δ-space is a generalization of "topological complete space" introduced by E. ČECH [1]. In general an open (or closed, G_δ) subset of a G_δ-space may fail to be a G_δ-space (see 3.3). This disadvantage disappears if we consider completely regular spaces only.

We introduce the concept of a complete sequence of open coverings. A sequence $\{\mathfrak{B}_n\}$ of open coverings is complete if for every open filter $\mathfrak{A}$ meeting every $\mathfrak{B}_n$ the intersection $\bigcap \overline{\mathfrak{A}}$ is non-void. A Hausdorff space possessing a complete sequence of open coverings is a G_δ-space. For completely regular spaces the converse is true. This is the main result of the present paper.

Without additional difficulties $G(\mathfrak{m})$-spaces can be studied, that is, spaces which are the intersection of $\mathfrak{m}$ open subsets in every Hausdorff extension. In connection with 0.2, the concept of a $\mathfrak{m}$-space is introduced.

In section 2 we study $G(\mathfrak{m})$-spaces, in section 3 G_δ-spaces, in section 4 spaces containing a dense $G(\mathfrak{m})$-space and the final section 5 is devoted to extensions of continuous mappings.

1. TERMINOLOGY AND NOTATION

The terminology and notation of J. Kelley will be used throughout. For convenience we shall use a few not quite usual symbols and terms which are listed below.

The potency of a set M will be denoted by card M. The union and the intersection of a family $\mathfrak{A}$ of sets will be denoted by $\bigcup \mathfrak{A}$ and $\bigcap \mathfrak{A}$ respectively. If $\mathfrak{A}$ is a family of sets and M is a set then $\mathfrak{A} \cap M$ is used to denote the family of all $A \cap M$ with A in $\mathfrak{A}$. A *system* is a synonym for indexed family. For systems we shall use a notation such as $\{P_a; a \in A\}$ or merely $\{P_a\}$. A is the index set of the system $\{P_a; a \in A\}$ and its elements are indexes.

1.1. If $\mathfrak{m}$ is a cardinal number, then an $\mathfrak{m}$-system is a system whose index set is of potency $\mathfrak{m}$.

1.2. Centered families. A family $\mathfrak{A}$ of sets has the finite intersection property if the intersection of every finite subfamily is not empty. A centered family is a family of sets having the finite intersection property. We shall use the following lemma without further references:

1.2.1. Lemma. *Let $\mathfrak{A}$ be a centered subfamily of a family $\mathfrak{B}$ of sets. There exists a maximal centered subfamily $\mathfrak{C}$ of $\mathfrak{B}$ containing $\mathfrak{A}$; that is, if $\mathfrak{C}'$ is a centered subfamily of $\mathfrak{B}$ containing $\mathfrak{C}$, then $\mathfrak{C}' = \mathfrak{C}$.*

This lemma is an immediate consequence of Tukey's lemma since the property of being a centred family is of finite character.

1.3. All (topological) spaces will be supposed to be Hausdorff. The closure of a subset M of a space P will be denoted by $\overline{M}^P$ or merely $\overline{M}$. If $\mathfrak{A}$ is a family of subsets of a space P then the family of closures of all sets of $\mathfrak{A}$ will be denoted by $\overline{\mathfrak{A}}^P$ or merely $\overline{\mathfrak{A}}$. An open (closed) family of a space P is a family consisting of open (closed, respectively) subsets of P. Analogous conventions will be used for systems.

1.4. A space P is an *extension* of a space R if R is a dense subspace of P. If moreover P is a regular (compact) space, then P is said to be a *regular* (*compact*, respectively) *extension* of R. "Compact extension" and "compactification" are synonymous. The Stone-Čech compactification of a (completely regular) space P will always be denoted by $\beta(P)$.

2. $G(\mathfrak{m})$-SPACES

2.1. Definition. A subset G of a space P is said to be a $G(\mathfrak{m})$*-subset* of P, if it is the intersection of some open $\mathfrak{m}$-system in P. A space is said to be a $G(\mathfrak{m})$-*-space* if it is a $G(\mathfrak{m})$-subset of its every extension.

2.2. Definition. A system $\{\mathfrak{V}_\iota; \iota \in I\}$ of open coverings of a space P is said to be *complete* if the following condition is satisfied:

2.2.1. If $\mathfrak{A}$ is an open centered family in P such that $\mathfrak{A} \cap \mathfrak{B}_\iota \neq \emptyset$ for each ι in I, then $\bigcap \overline{\mathfrak{A}} \neq \emptyset$.

If may be noticed that a space possesses a complete 0-system (1-system respectively) of open coverings if and only if it is (locally) H-closed. From the following theorem we may conclude that a regular space possesses a complete 0-system (1-system, respectively) of open coverings if and only if it is (locally) compact. If $\mathfrak{m} \geqq 1$ is a finite potency, then P possesses a complete $\mathfrak{m}$-system of open coverings if and only if it possesses a complete 1-system of open coverings.

2.3. Proposition. *Let $\{\mathfrak{B}_\iota; \iota \in I\}$ be a complete system of open coverings of a regular space P. Suppose that $\mathfrak{M}$ is a centered family of subsets of P such that $\mathfrak{M} \cap \mathfrak{B}_\iota \neq \emptyset$ for each ι in I. Then $\bigcap \overline{\mathfrak{M}} \neq \emptyset$.*

Proof. Consider the family $\mathfrak{A}$ of all open subsets A of P such that $A \supset M$ for some M in $\mathfrak{M}$. Evidently this family has the finite intersection property and $\mathfrak{A} \cap \mathfrak{B}_\iota \neq \emptyset$ for each ι in I. Since $\{\mathfrak{B}_\iota\}$ is complete, the set $F = \bigcap \overline{\mathfrak{A}}$ is non-void. In consequence, it is sufficient to show that $F \subset \bigcap \overline{\mathfrak{M}}$. But this is an immediate consequence of regularity. Indeeed, every closed subset of a regular space is the intersection of the family of all its closed neighborhoods. It follows that for each M in $\mathfrak{M}$ we have $\overline{M} \supset \bigcap \overline{\mathfrak{A}} = F$, that is, $\bigcap \overline{\mathfrak{M}} \supset F$, which completes the proof.

2.4. Theorem. *Suppose that $\{\mathfrak{B}_\iota; \iota \in I\}$ is a complete $\mathfrak{m}$-system of open coverings of a space P. Then P is a $G(\mathfrak{m})$-space.*

Proof. Let R be an extension of P. For every open subset V of P choose and open subset V' of R such that $V' \cap P = V$. For each ι in I put

$$U_\iota = \bigcup \{V'; V \in \mathfrak{B}_\iota\}$$

and consider the set

$$G = \bigcap \{U_\iota; \iota \in I\}.$$

Since G is a $G(\mathfrak{m})$-subset of R if is sufficient to prove that $G = P$. Evidently $G \supset P$. Suppose the contrary, that there exists an element x in $G - P$. Denote by $\mathfrak{A}$ the family of all open neighborhoods of x. Since $x \in U_\iota$ for each ι, we may choose $V_\iota \in \mathfrak{B}_\iota$ such that x belongs to V'_ι, that is, $V'_\iota \in \mathfrak{A}$. The family $\mathfrak{A} \cap P$ has the finite intersection property since P is a dense subset of R. Evidently $\mathfrak{A} \cap P$ satisfies all the assumptions of 2.2.1, and hence, using the completness of $\{\mathfrak{B}_\iota\}$ we have $\bigcap \overline{\mathfrak{A} \cap P}^P \neq \emptyset$. Choose a point y in this intersection. Evidently $x \neq y$. R is a Hausdorff space and hence for some $A \in \mathfrak{A}$ the point y does not belong to the closure (in R) of the set A. On the other hand we have

$$y \in \overline{A \cap P} \subset \overline{A}^R .$$

This contradiction completes the proof of the theorem.

2.5. Theorem. *Suppose that a regular space R possesses a complete $\mathfrak{m}$-system $\{\mathfrak{V}_\iota;\ \iota \in I\}$ of open coverings. Then every non-void $G(\mathfrak{m})$-subset of R possesses a complete $\mathfrak{m}$-system of open coverings.*

Proof. Without loss of generality we may suppose that the potency of I is exactly $\mathfrak{m}$. Select open subsets U_ι of R such that

$$G = \bigcap\{U_\iota;\ \iota \in I\}\,.$$

Let $\iota \in I$ be fixed; for each x in G choose on open neighborhood U_x of x such that $\overline{U}_x \subset U_\iota$ and that some V in $\mathfrak{V}_\iota$ contains U_x. Denote by $\mathfrak{W}_\iota$ the family of all $W_x = U_x \cap G$, $x \in G$.

We shall prove that the system $\{\mathfrak{W}_\iota;\ \iota \in I\}$ is complete. Let $\mathfrak{A}$ be any open centered family in G such that $\mathfrak{A} \cap \mathfrak{W}_\iota \neq \emptyset$ for each ι in I. Consider the family $\mathfrak{A}_1$ of all open subsets A_1 of R such that A_1 contains some $A \in \mathfrak{A}$. Evidently the family $\mathfrak{A}_1$ has the finite intersection property. Let $\mathfrak{A}_2 \supset \mathfrak{A}_1$ be a maximal centered family of open subsets of R. It is easy to see that $\mathfrak{A}_2 \cap \mathfrak{V}_\iota \neq \emptyset$ for each ι in I. Using the completness of $\{\mathfrak{V}_\iota\}$ we have $\bigcap\overline{\mathfrak{A}}_2^R \neq \emptyset$. Since $\mathfrak{A}_2$ is a maximal family with the finite intersection property, this intersection contains only one point, namely x. First we show that x belongs to G. Indeed, if $W_x \in \mathfrak{A} \cap \mathfrak{W}_\iota$, then $U_x \in \mathfrak{A}_1 \subset \mathfrak{A}_2$ and $\overline{U}_x \subset U_\iota$. It follows that $x \in G$.

Now we shall prove that

2.5.1 $$y \in \bigcap\overline{\mathfrak{A}}^G\,.$$

Suppose this is not true. Then for some $A \in \mathfrak{A}$ we have y non $\in \overline{A}^G$, and consequently y does not belong to $\overline{A}^R$ since $y \in G$. The space R is regular and hence there exists a closed neighborhood F of the point y such that $F \cap \overline{A}^R = \emptyset$. But the set $R - F$ is open and contains $A \in \mathfrak{A}$, and consequently, according to the definition of $\mathfrak{A}_1$ we have that $R - F$ belongs to $\mathfrak{A}_1$. It follows that $y \in \overline{R - F}^R$. But this is impossible since F is a neighbourhood of y. This contradiction completes the proof.

2.6. *Suppose that a regular space R possesses a complete $\mathfrak{m}$-system $\{\mathfrak{V}_\iota;\ \iota \in I\}$ of open coverings. Every closed subspace F of R possesses a complete* $\mathfrak{m}$-system *of open coverings.*

Proof. We shal prove that the $\mathfrak{m}$-system

$$\{\mathfrak{V}_\iota \cap F\,;\ \iota \in I\}$$

is complete. Let $\mathfrak{A}$ be an open centered family of F such that $\mathfrak{A} \cap (\mathfrak{V}_\iota \cap F) \neq \emptyset$ for each ι in I. Consider any maximal centered family $\mathfrak{A}_1 \supset \mathfrak{A}$ of subsets of R. Clearly the intersection $\mathfrak{V}_\iota \cap \mathfrak{A}_1$ is non-void for each ι in I, and hence, using the completness of $\{\mathfrak{V}_\iota\}$ we have by 2.3 that $\bigcap \overline{\mathfrak{A}}_1^R \neq \emptyset$. Hence $\bigcup \overline{\mathfrak{A}}^R \neq \emptyset$. But F is closed in R and consequently $\overline{M}^R = \overline{M}^F$ for $M \subset F$. It follows that $\bigcap \overline{\mathfrak{A}}^R = \bigcap \overline{\mathfrak{A}}^F$. The proof is complete.

In 3.3 we shall show that the assumption of regularity of the space R in 2.5 and 2.6 is essential.

2.7. Theorem. *Suppose that a regular space R possesses a complete $\mathfrak{m}$-system of open coverings. The following properties of a subset M of R are equivalent:*

(1) *M is a $G(\mathfrak{m})$-space.*

(2) *M possesses a complete $\mathfrak{m}$-system of open coverings.*

(3) *M is a $G(\mathfrak{m})$-subset of $\overline{M}^R$ (that is, M is the intersection of a closed and of a $G(\mathfrak{m})$-subset of R).*

Proof. According to the definition 2.1 of $G(\mathfrak{m})$-spaces we have that (1) implies (3). Combining theorems 2.5 and 2.6 we obtain that (3) implies (2). By theorem 2.4 we have that (2) implies (1). The proof is complete.

2.8. Theorem. *The following properties of a completly regular space P are equivalent:*

(1) *P is a $G(\mathfrak{m})$-space.*

(2) *P possesses a complete $\mathfrak{m}$-system of open coverings.*

(3) *P is a $G(\mathfrak{m})$-subset of $\beta(P)$.*

(4) *P is a $G(\mathfrak{m})$-subset of some completely regular $G(\mathfrak{m})$-space.*

Proof. Since a compact space is a $G(0)$-space, by theorem 2.7 the conditions (1), (2) and (3) are equivalent. By 2.4 condition (2) implies (4). If P is a $G(\mathfrak{m})$--subset of some completely regular $G(\mathfrak{m})$-space R, then P is a $G(\mathfrak{m})$-subset of $\beta(R)$, and hence, by 2.7 (4) implies (2). The proof is complete.

2.9. Theorem. *Let $\{P_b; b \in B\}$ be a non-void system of completely regular spaces. Suppose that P_b is a $G(\mathfrak{m}_b)$-space. Then the topological product*

2.9.1 $$P = X\{P_b; b \in B\}$$

is a $G(\mathfrak{m})$-space for

2.9.2 $$\mathfrak{m} = \Sigma\{\mathfrak{m}_b; b \in B\}\,.$$

Proof. Select compact extensions K_b of P_b. Put

$$K = X\{K_b; b \in B\}\,.$$

For each b in B there exists a family $\mathfrak{A}_b$ of open subsets of K_b such that the potency of $\mathfrak{A}_b$ is $\leqq \mathfrak{m}_b$ and $\bigcap \mathfrak{A}_b = P_b$ (if $\mathfrak{m}_b = 0$, then $\mathfrak{A}_b = \emptyset$ and by the usual convention $\bigcap \mathfrak{A}_b = K_b = P_b$). Denote by π_b the projection of K onto K_b (that is, π_b is a map) and consider the space

2.9.3 $$P' = \bigcap_{b \in B} \bigcap \{\pi_b^{-1}[A]; A \in \mathfrak{A}_b\}\,.$$

Evidently $P' \supset P$. On the other hand, supposing that some $x = \{x_b\}$ belongs to $P' - P$ we obtain immediately that $x_{b_0} \in K_{b_0} - P_{b_0}$ for some b_0 in B. Thus $\mathfrak{m}_{b_0} \neq 0$ and there exists an $A \in \mathfrak{A}_{b_0}$ such that x_{b_0} non $\in A$. Thus x non $\in \pi_{b_0}^{-1}[A] \supset$

$\supset P'$. This contradiction proves $P = P'$. The intersection in 2.9.3 is taken over a set of potency $\leqq \mathfrak{m}$. Thus P is a $G(\mathfrak{m})$ subset of the compact space K. It follows that P is a G($\mathfrak{m}$)-space.

The preceding proof does not use the characterisation of completely regular $G(\mathfrak{m})$-spaces by complete $\mathfrak{m}$-systems open coverings. On the other hand this proof uses the existence of compactifications of completely regular spaces and the Tychonoff theorem on the topological product of compact spaces. We shall give a direct proof.

2.10. Theorem. *Let $\{P_b; b \in B\}$ be a non-void system of spaces. Suppose that the space P_b possesses a complete $\mathfrak{m}_b$-system of open coverings $\{\mathfrak{V}_\iota; \iota \in I_b\}$. (We assume that the system $\{I_b; b \in B\}$ is disjoint.) Then the topological product* 2.9.1 *possesses a complete $\mathfrak{m}$-system of open coverings with $\mathfrak{m}$ given by* 2.9.2.

Proof. Denote by I the union of all sets I_b. For each ι in I we define an open covering $\mathfrak{V}'_\iota$ of P as follows: there exists a b in B with $\iota \in I_b$; put $\mathfrak{V}'_\iota = \pi_b^{-1}[\mathfrak{V}_\iota]$. (Of course π_b is the projection of P onto P_b). We shall prove that $\{\mathfrak{V}'_\iota; \iota \in I\}$ is a complete $\mathfrak{m}$-system of open coverings of P. Clearly it is sufficient to prove completness only. Let $\mathfrak{A}$ be a maximal open centered family of P with $\mathfrak{A} \cap \mathfrak{V}'_\iota \neq \emptyset$ for each ι in I. We have to show that $\bigcap \overline{\mathfrak{A}} \neq \emptyset$. First notice that for every fixed $b \in B$ the set $\pi_b[\mathfrak{A}] = \mathfrak{A}_b$ is a maximal open centered family in P_b, and that $\mathfrak{V}_\iota \cap \mathfrak{A}_b \neq \emptyset$ for each ι in I_b. It follows that for each b in B we have $\bigcap \overline{\mathfrak{A}}_b^{P_b} \neq \emptyset$. Choose a point x_b in this intersection. It is easy to show that the point $x = \{x_b\}$ belongs to $\bigcap \overline{\mathfrak{A}}$. Indeed, if V is an arbitrary canonical open neighborhood of x, namely

$$V = \bigcap_{j=1}^{n} \pi_{b_j}^{-1}[V_j]$$

where V_j is an open set containing x_{b_j}, then $\pi_{b_j}^{-1}[V_j] \in \mathfrak{A}$ as we had noted above, and hence by maximality of $\mathfrak{A}$, V belongs to $\mathfrak{A}$. Thus V meets every A in $\mathfrak{A}$. The proof is complete.

2.11. Definition. A space is said to be an exact $G(\mathfrak{m})$-space if it is a $G(\mathfrak{m})$-space and it is not a $G(\mathfrak{n})$-space for any $\mathfrak{n} < \mathfrak{m}$.

2.12. Theorem. *If $\mathfrak{m}$ is either an infinite cardinal number or $\mathfrak{m} = 0, 1$, then there exists a completely regular exact $G(\mathfrak{m})$-space.* For $\mathfrak{m} = 2, 3, \dots$ there exist no exact $G(\mathfrak{m})$-spaces.

Proof. The second assertion is obvious since the intersection of a finite number of open sets is an open set. A space is an exactly $G(0)$-space if and only if it is a $G(0)$-space, and consequently, in the case of completely regular spaces, if and only if it is compact. A completely regular space is an exact $G(1)$-space if and only if it is locally compact and non-compact. Hence the existence of exact $G(1)$-spaces is obvious. If $\mathfrak{m}$ is a infinite cardinal number and if P is an exact $G(1)$-space, then the cube $P^{\mathfrak{m}}$ is an exact $G(\mathfrak{m})$-space.

In the remainder of this section we shall investigate the concept of a complete system of open coverings. We shall shaw that there holds a generalisation of the Cantor theorem for complete metric spaces.

2.13. Proposition. *Let $\{\mathfrak{V}_\iota;\ \iota \in I\}$ be a complete system of open coverings of a regular space R. Suppose that $\mathfrak{M}$ is a centred system of subsets of R such that for each ι in I there exist a $M \in \mathfrak{M}$ and a finite subfamily $\mathfrak{U}_\iota$ of $\mathfrak{V}_\iota$ which covers M. Then $\bigcap \overline{\mathfrak{M}} \neq \emptyset$.*

Proof. Without loss of generality we may assume that $\mathfrak{M}$ is a maximal family possessing the finite intersection property. By 2.3 it is sufficient to prove that for each ι in I the intersection $\mathfrak{V}_\iota \cap \mathfrak{M}$ is non-void. For a fixed $\iota \in I$ choose $M \in \mathfrak{M}$ and a finite $\mathfrak{U}_\iota \subset \mathfrak{V}_\iota$ such that $\mathfrak{U}_\iota$ covers M. We shall prove that some U in $\mathfrak{U}_\iota$ belongs to $\mathfrak{M}$. Suppose the contrary. According to maximality of $\mathfrak{M}$ the sets $R - U$ with $U \in \mathfrak{U}_\iota$ belong to $\mathfrak{M}$. It follows that the set $\bigcap\{R - U;\ U \in \mathfrak{U}_\iota\}$ belongs to $\mathfrak{M}$. But this is impossible since

$$M \cap \bigcap \{R - U;\ U \in \mathfrak{U}_\iota\} = M - \bigcup \mathfrak{U}_\iota = \emptyset$$

and $\mathfrak{M}$ has the finite intersection property. This contradiction completes the proof.

The preceding proposition may be stated in the following manner.

2.14. Theorem. *Suppose that $\{\mathfrak{V}_\iota;\ \iota \in I\}$ is a complete system of open coverings of a regular space R. For each ι in I let $\mathfrak{U}_\iota$ be the familly of all unions of finite subfamilies of $\mathfrak{V}_\iota$.* Then $\{\mathfrak{U}_\iota;\ \iota \in I\}$ is a complete *system of open coverings of R.*

2.14. Theorem. *Let $\{\mathfrak{V}_\iota;\ \iota \in I\}$ be a system of open coverings of P satisfying the following two conditions* 2.14.1 *and* 2.14.2.

2.14.1. If K is a closed subspace of P and if for each ι in I there exists a V_ι in $\mathfrak{V}_\iota$ with $K \subset V_\iota$, then K is a compact space.

2.14.2. For every centered system $\{F_\iota;\ \iota \in I\}$ of closed subsets of P such that for each ι in I $F_\iota \subset V_\iota$ for some V_ι in $\mathfrak{V}_\iota$, the intersection $\bigcap \{F_\iota;\ \iota \in I\}$ is non-void.

Then the following condition 2.14.3 is satisfied:

2.14.3 *If $\mathfrak{F}$ is a centered family of closed subsets of P such that for each ι in I there exists $F_\iota \in \mathfrak{F}$ and $V_\iota \in \mathfrak{V}_\iota$ with $F_\iota \subset V_\iota$, then $\bigcap \mathfrak{F} \neq \emptyset$.*

Proof. Suppose that $\mathfrak{F}$, F_ι and V_ι satisfy the assumptions of 2.14.3. Without loss of generality we may assume that $\mathfrak{F}$ is multiplicative, that is, the intersection of every finite subfamily of $\mathfrak{F}$ belongs to $\mathfrak{F}$. Put

$$K = \bigcap\{F_\iota;\ \iota \in I\}\ .$$

According to condition 2.14.1 the space K is compact and by 2.14.2 it is non--void. In consequence it is sufficient to show that the family $\mathfrak{F} \cap K$ has the finite intersection property. Choosing F in $\mathfrak{F}$ we have by 2.14.3.

$$\bigcap\{F \cap F_\iota;\ \iota \in I\} \neq \emptyset\ .$$

But this intersection is contained in K. Thus $F \cap K \neq \emptyset$. Since $\mathfrak{F}$ is multiplicative, it follows that $\mathfrak{F} \cap K$ has the finite intersection property. The proof is complete.

Note. In 3.10 we shall show that neither 2.14.1 nor 2.14.2 alone implies 2.14.3.

3. G_δ-SPACES

In this section we shall study in detail $G(\aleph_0)$-spaces. In accordance with the usual terminology we shall call them G_δ-spaces.

3.1. Definition. A space P is said to be a F-hereditary G_δ-space if every closed subspace is a G_δ-space. A space is said to be a feeble G_δ-space if it is regular and is a G_δ-subset of every regular extension.

Examples. A regular R-closed space is a feeble G_δ-space. A hereditady H-closed space is a F-hereditary G_δ-space. Recall that a space P is H-closed if P is the unique extension of P. It is easy to see that P is a H-closed space if and only if every open covering of P contains a finite subfamily which covers a dense subset of P. Analoguously a regular space P is R-closed provided that P is the unique regular extension of P. For more detailed information see [2].

We shall prove that a closed (open) subspace of a G_δ-space may fail to be a G_δ-space. First we prove.

3.2. Lemma. *Suppose that (K, τ) is a H-closed space. Let M_1 and M_2 be disjoint dense subsets of K with $M_1 \cup M_2 = K$. There exists a topology τ_1 for the set K such that (K, τ_1) is a H-closed space, M_1 is closed in (K, τ_1) and the topologies τ and τ_1 agree on the sets M_1 and M_2.*

Proof. We define the topology τ_1 as follows: If $x \in M_1$, then $U \subset K$ is a τ_1--neighborhood of x if and only if U is a τ-neighborhood of x; if $x \in M_2$, then $U \subset K$ is a neighborhood of the point x if and only if $U \cap M_2$ is a neigborhood of x in the space (M_2, τ). Evidently τ_1 is a topology and it agrees with τ on both M_1 and M_2. The set M_2 is τ_1-open. It remains to show that (K, τ_1) is a H-closed space. Clearly

3.2.1. $x \in M_1\,,\quad N \subset K \Rightarrow [x \in \tau_1[N] \Leftrightarrow x \in \tau[N]]\,.$

Now we shall prove

3.2.2. *If U is τ_1-open, then $\tau[U] = \tau_1[U]$.*

By 3.2.1 it is sufficient to show that $x \in M_2 \cap \tau[M] \Rightarrow x \in \tau_1[U]$. Let x belongs to $M_2 \cap \tau[M]$. Choose a τ_1-open neighborhood $V' \subset M_2$ of x. According to the definition of τ_1, there exists a τ-open subset V such that

$$x \in V \cap M_2 \subset V'\,.$$

The set V is τ-open and hence τ_1-open. If follows that the set $V \cap U$ is τ_1-open.

This set is non-void, since $x \in \tau[U]$ and V is a τ-neighborhood of x. The set M_2 is dense in (K, τ_1), and consequently

$$V \cap U \cap M_2 \neq \emptyset .$$

Thus the set $V' \cap U$ containing $V \cap U \cap M_2$ is non-void. Hence $x \in \tau_1[U]$. The proof of 3.2.2 is complete.

We shall also need the following assertion:

(3.2.3.) *If U is τ_1-open, then* $\operatorname{int}_\tau \tau_1[U] \supset U$.

If $x \in U \cap M_1$, then U is a τ-neighborhood of x and hence $x \in \operatorname{int}_\tau U \subset \subset \operatorname{int}_\tau \tau_1[U]$. Now suppose that $x \in M_2 \cap U$. According to the definition of τ_1 there exists a τ-open set V such that $x \in V \cap M_2 \subset U$. We shall prove that $V \subset \tau_1[U]$. The set M_2 is dense in (K, τ) and hence the set $V \cap M_2$ is dense in V (in the topology τ). Since $V \cap M_2$ is τ_1-open we have, according to 3.2.2, $\tau_1[V \cap M_2] = \tau[V \cap M_2]$ and consequently

$$V \subset \tau[V \cap M_2] = \tau_1[V \cap M_2] \subset \tau_1[U] .$$

Since $x \in V$, we have $x \in \operatorname{int}_\tau \tau_1[U]$. The proof of 3.2.3 is complete.

Now it is easy to see that (K, τ_1) is a H-closed space. Let $\{U\}$ be any open covering of (K, τ_1). We have to show that some finite subfamily of $\{U\}$ covers a dense subspace of (K, τ_1). According to assertion 3.2.3 the family $\{\operatorname{int}_\tau \tau_1[U]\}$ is an open convering of (K, τ). (K, τ) is a H-closed space and consequently there exist $U_1, \ldots, U_n$ in $\{U\}$ such that

$$\bigcup_{k=1}^{n} \tau[\operatorname{int}_\tau \tau_1[U]] = K .$$

According to 3.2.2

$$\tau[\operatorname{int}_\tau \tau_1[U]] = \tau_1[\operatorname{int}_\tau \tau_1[U]] .$$

Since $\operatorname{int}_\tau \tau_1[U] \subset \tau_1[U]$, we have

$$\tau[\operatorname{int}_\tau \tau_1[U]] \subset \tau_1[\tau_1[U]] = \tau_1[U] .$$

In consequence $\bigcup_{k=1}^{n} \tau_1[U_k] = K$. The proof of the lemma is complete.

3.3. Examples. *An open (closed) subset of a G_δ-space may fail to be a G_δ-space.*

Proof. Denote by R the space of all rational numbers in the closed interval $\langle 0, 1\rangle$ (with its usual topology). It is well-known that R is not a G_δ-subsef of $\langle 0,1\rangle$. Thus R is not a G_δ-space. According to 3.2 the space R may be embedded as an open (closed) subspace of some H-closed space. The proof is complete.

Recall that a family $\mathfrak{A}$ of subsets of a space is said to be regular provided that for each A in $\mathfrak{A}$ there exists an $A_1 \in \mathfrak{A}$ such that $\overline{A}_1 \subset \operatorname{int} A$.

3.4. Definition. Let $\{\mathfrak{B}_n; n = 1, 2, \ldots)$ be a sequence of open coverings of a space P. The sequence $\{\mathfrak{B}_n\}$ is said to be complete in the strong sense if for every centered family $\mathfrak{M}$ of subsets of P such that $\mathfrak{M} \cap \mathfrak{V}_n \neq \emptyset$ for each $n = 1, 2, \ldots$ the intersection $\bigcap \overline{\mathfrak{M}}$ is non-void. The sequence $\{\mathfrak{V}_n\}$ is said to be feebly complete provided the following condition is satisfied:

If $\mathfrak{A}$ is a centered regular family of open subsets of P such that $\mathfrak{A} \cap \mathfrak{V}_n \neq \emptyset$ for each $n = 1, 2, \ldots$, then $\bigcap \overline{\mathfrak{A}} \neq \emptyset$.

Note. According to 2.3, every complete sequence of open coverings of a regular space is complete in the strong sense.

The proof of the following theorem is quite simple and may be left to the reader.

3.5. Lemma. *Suppose that $\{\mathfrak{V}_n\}$ is a complete in the strong sense sequence of open coverings of a space P. If F is a closed subset of P, then the sequence $\{\mathfrak{V}_n \cap F\}$ is also complete in the strong sense. Thus (by 2.4) P is a* F*-hereditary G_δ-space.*

3.6. Proposition. *Suppose that there exists a feebly complete sequence $\{\mathfrak{V}_n\}$ of open coverings of a regular space P. Then P is a feeble G_δ-space.*

Proof. The proof is quite analoguous to that of 2.4. Given a regular extension R of P we have to shaw that P is a G_δ-subset of R. For every open subset V of P choose an open subset V' of R such that $V' \cap P = V$. For $n = 1, 2, \ldots$ put

$$U_n = \bigcup\{V'; V \in \mathfrak{V}_n\} .$$

It is sufficient to show $P = \bigcap\{U_n; n = 1, 2, \ldots\}$. Denoting this intersection by G we have $P \subset G$. Suppose that there exists an element x of $G - P$. Denote by $\mathfrak{A}$ the family of all open neighborhoods of the point x. The space R is regular, and consequently $\mathfrak{A}$ is a regular family. Evidently $\mathfrak{A} \cap P$ is a regular centered family in P and $\mathfrak{A} \cap \mathfrak{V}_n \neq \emptyset$ for each $n = 1, 2, \ldots$. Hence $\bigcap \overline{\mathfrak{A} \cap P} \neq \emptyset$. Choose a point y in this intersection. Clearly $x \neq y$ and hence there exists an $A \in \mathfrak{A}$ such that y does not belong to $\overline{A}$; this is impossible since $y \in \overline{A}$. This contradiction completes the proof.

3.7. Proposition. *Suppose that a regular space P possesses a feebly complete sequence $\{\mathfrak{V}_n\}$ of open coverings. Then every open subspace U of P possesses a feebly complete sequence of open coverings.*

Proof. For every $n = 1, 2, \ldots$ denote by $\mathfrak{V}'_v$ the family of all open subsets V' of P such that for some $V \in \mathfrak{V}_n$ the inclusion $\overline{V}' \subset V \cap U$ holds. It is easy to see that $\{\mathfrak{V}'_n\}$ is a feebly complete sequence of open coverings of U.

Note. I do not know whether the assumption "U is open" may be replaced by the assumption that U is a G_δ-subset. In the case of a positive answer the following theorem is true: A regular space is a feeble G_δ-space if and only if it possesses a feebly complete sequence of open coverings.

3.8. Theorem. *Suppose that a regular space P possesses a complete sequence of open coverings. The following conditions on a subspace M of P are equivalent:*

3.8.1. *M possesses a feebly complete sequence of open coverings.*
3.8.2. *M possesses a complete sequence of open coverings.*
3.8.3. *M possesses a complete in the strong sense sequence of open coverings.*
3.8.4. *M is a G_δ-subset of $\overline{M}^P$.*
3.8.5. *M is a feeble G_δ-space.*
3.8.6. *M is a G_δ-space.*
3.8.7. *M is a F-hereditary G_δ-space.*

The proof is an immediate consequence of Theorem 2.7 and propositions 3.5 and 3.6.

3.9. Theorem. *A metrizable space P is a G_δ-space if and only if there exists a metric ϱ for the space P such that (P, ϱ) is a complete metric space.*

Proof. First suppose that (P, ϱ) is a complete metric space. For $n = 1, 2, \ldots$, let $\mathfrak{V}_n$ be an open covering of P such that the diameters of elements of $\mathfrak{V}_n$ are $\leqq \frac{1}{n}$. It is easy to see that the sequence $\{\mathfrak{V}_n\}$ is complete. Conversely, suppose that a metric space (P, φ) is a G_δ-space. Let (P^*, φ^*) be the completion of (P, φ). There exist open sets U_n, $n = 1, 2, \ldots$, such that

$$P = \bigcap \{U_n; n = 1, 2, \ldots\}.$$

Denote by $f_n(x)$ the distance of a point x of P to the set $P^* - U_n$. For x and y in P put

$$\varrho(x, y) = \varphi(x, y) + \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|[f_n(x)]^{-1} - [f_n(y)]^{-1}|}{1 + |[f_n(x)]^{-1} - [f_n(y)]^{-1}|}.$$

Evidently the metrics φ and ϱ are topologically equivalent and (P, ϱ) is a complete metric space. The proof is complete.

3.10. Examples relating to 2.14. No one of conditions 2.14.1 and 2.14.2 is sufficient for $\{\mathfrak{V}_\iota\}$ to be a complete system of open coverings.

The condition 2.14.1 is not sufficient. Let (P, φ) be a metric spaces which is not a G_δ-space. For $n = 1, 2, \ldots$ let $\mathfrak{V}_n$ be an open covering of P consisting of sets of diameters $\leqq \frac{1}{n}$. Evidently the condition 2.14.1 is satisfied since every such set contains at most one point. On the other hand the sequence $\{\mathfrak{V}_n\}$ is not complete since P is not a G_δ-space.

The condition 2.14.2 is not sufficient. If P is a countably compact space, then for every sequence $\{W_n\}$ of open coverings of P the condition 2.14.2 is satisfied in a trivial manner. Consequently, it is sufficient to prove the following proposition.

3.11. *There exists a completely regular countably compact space which is not a G_δ-space.* First we prove the following lemma:

3.12. Lemma. *A regular space P possesses no complete sequence of open coverings provided that it has the following two properties* 3.12.1 *and* 3.12.2.

3.12.1. *P contains no infinite compact subset.*

3.12.2. *P contains at least one point x which is not a G_δ-point* (*that means, the one-point set* (x) *is not a G_δ-set*).

Proof. Suppose that there exists a complete sequence $\{\mathfrak{V}_n\}$ of open covering of a regular space P possessing the properties 3.12.1 and 3.12.2. Choose a point x in P which is not a G_δ-point. For each $n = 1, 2, \ldots$ choose a set V_n in $\mathfrak{V}_n$ containing the point x. Since the space P is regular we may select neighborhoods U_n of x with $\overline{U}_n \subset V_n$, $n = 1, 2, \ldots$. By theorem 2.14 (condition 2.14.1) the intersection K of all the $\overline{U}_n$ is a compact space. K contains the G_δ-set $G = \bigcap_{n=1}^{\infty} U_n$ containing x. According to our choice of x, G is an infinite set. This contradicts the condition 3.12.1 and completes the proof of 3.12.

Proof of the proposition 3.11. By 3.12 it is sufficient to construct a countably compact completely regular space P possessing the properties 3.12.1 and 3.12.2. By [1], theorem 3.1.5 or [2], theorem, there exists a countably infinite compact space P, $N \subset P \subset \beta(N)$ with card $P \leqq 2^{\aleph_0}$, where N is the countable discrete space. Since every infinite closed subset of $\beta(N)$ has potency $2^{2^{\aleph_0}}$, the space P contains no infinite compact set. Hence 3.12.1 holds. No ideal point of P, (that is, no point in $P - N$) is a G_δ-set. Indeed, since P is countably compact, every G_δ-point is of countable character. But no ideal point is of countable character.*)

Another proof of the proposition 3.11. By [1], theorem 3.1.6 or [2], theorem 2.6, there exist disjoint countably compact dense subspaces P and Q of $\beta(N) - N$, where N denotes the countable infinite discrete space. From the following theorem we can conclude that at least one of the spaces P and Q is not a G_δ-subset of $\beta(N) - N$.

3.13. Theorem. *Suppose that a regular space P possesses a complete sequence $\{\mathfrak{V}_n\}$ of open coverings. Then the intersection of every countable family of open dense subsets of P is a dense subset of P. It follows that P contains no two disjoint dense G_δ-subsets.*

Proof. Let $\{U_n\}$ be a sequence of dense open subsets of P. Given a non-void

*) Suppose that a point $x \in P - N$ is of countably character. It is easy to see that there exists a continuous function f on P with $f(x) = 0$ and $f(y) > 0$ for $y \in P - (x)$. Evidently the continuous function $g(y) = \sin (f(y))^{-1}$ on $P - (x)$ has no continuous extension to P; but this is impossible.

open subset W of P, we can construct by induction sequences $\{V_n\}$ and $\{W_n\}$ of non-void open subsets of P such that $V_n \in \mathfrak{B}_n$, $V_n \supset W_n$ and

$$\overline{W}_{n+1} \subset W_n \subset W \cap U_n .$$

The sequence $\{\mathfrak{B}_n\}$ is complete and hence $\bigcap_{n=1}^{\infty} \overline{W}_n = \bigcap_{n=1}^{\infty} W_n \neq \emptyset$. Evidently $W \cap \bigcap_{n=1}^{\infty} U_n \supset \bigcap_{n=1}^{\infty} W_n$. The proof is complete.

3.14. Theorem. *Let $\{P_b; b \in B\}$ be a non-void system of non-void completely regular spaces. Suppose that P_b is an exact $G(\mathfrak{m}_b)$-space with $\mathfrak{m}_b \leqq \aleph_0$. Put*

$$P = X\{P_b; b \in B\} \quad \text{and} \quad \mathfrak{m} = \Sigma\{\mathfrak{m}_b; b \in B\} .$$

If $\mathfrak{m} \neq 0$ is a finite cardinal number, then P is an exact $G(1)$-space. In the other case the space P is an exact $G(\mathfrak{m})$-space.

Proof. By Theorem 2.9 the space P is a $G(\mathfrak{m})$-space. Put

$$B_1 = \{b; b \in B, \mathfrak{m}_b = 0\} , \qquad B_2 = B - B_1 ;$$
$$P_i = X\{P_b; b \in B_i\} \quad (i = 1, 2) .$$

First let us notice that in the following special cases the theorem holds.

3.14.1. If K is a compact space and R is an exact $G(\mathfrak{m})$-space, then $K \times R$ is an exact $G(\mathfrak{m})$-space.

3.14.2. Suppose that Q is a space and R is an exact $G(\mathfrak{m})$-space. If $Q \times R \neq \emptyset$ is a $G(\mathfrak{n})$-space, then $\mathfrak{n} \geqq \mathfrak{m}$.

From 3.14.1 a 3.14.2 and from the evident equality $\mathfrak{m} = \Sigma\{\mathfrak{m}_b; b \in B_2\}$ we see that $B_1 = \emptyset$ may be assumed without loss of generality. First suppose that the set $B_2 = B$ is finite. Either $\mathfrak{m}_b = 1$ for each b in B or there exists a $b \in B$ with $\mathfrak{m}_b = \aleph_0$. In the first case according to Theorem 2 P is a $G(\mathfrak{n})$-space with $n \neq 0$ a finite cardinal. By 3.14.2 P is an exact $G(1)$-space. In the other case by 3.14.2 P is an exact $G(\aleph_0)$-space.

Now suppose that B is infinite. Since $0 < \mathfrak{m}_b \leqq \aleph_0$, we have $\mathfrak{m} = \operatorname{card} B$. The following lemma completes the proof.

3.15. Lemma. *Let $\{R_b; b \in B\}$ be a system of topological spaces, $P_b \subset R_b$.*

$$R = X\{R_b; b \in B\} \quad \textit{and} \quad P = X\{P_b; b \in B\} .$$

Suppose that P is non-void and $R_b - P_b \neq \emptyset$ for each b in B. Let G be a $G(\mathfrak{n})$-subset of R containing P with $0 < \mathfrak{n} < \mathfrak{m} = \operatorname{card} B \geqq \aleph_0$. Then $G - P \neq \emptyset$.

Proof. There exists an open $\mathfrak{n}$-system $\{U_a; a \in A\}$ in R with

$$G = \bigcap \{U_a; a \in A\} .$$

Choose a point $x = \{x_b\}$ in P. For each a in A we can choose a finite subset B_a of B such that

$$x \in H_a = X\{H_b(a); b \in B\} \subset U_a$$

where $H_b(a)$ is open in R_b, $b \in B$, and $H_b(a) = R_b$ for b non $\in B_a$. Put

$$H = \bigcap \{H_a;\ a \in A\}\ .$$

Since $H \subset G$ it is sufficient to show that $H - P \neq \emptyset$. Put $B_1 = U\{B_a;\ a \in A\}$. Since $\mathfrak{n} < \mathfrak{m}$ we have $B - B_1 \neq \emptyset$. Choosing $y_b \in R_b - P_b$, consider the point $z = \{z_b\}$ where

$$z_b = \left\langle \begin{array}{l} x_b \text{ for } b \in B_1\ , \\ y_b \text{ for } b \in B - B_1\ . \end{array}\right.$$

Clearly $z \in H - P$. The proof of the lemma is complete.

4. SPACES CONTAINING A DENSE $G(\mathfrak{m})$-SPACES

In this section we introduce the concept of a complete system of families of subsets of a topological space.

4.1. Definition. A system $\{\mathfrak{B}_\iota;\ \iota \in I\}$ of open families of a topological space P is said to be complete if for every centered open family $\mathfrak{A}$ in P with $\mathfrak{A} \cap \mathfrak{B}_\iota \neq \emptyset$, the intersection $\bigcap \overline{\mathfrak{A}}$ is non-void for each ι in I.

4.2. Proposition. *Let $\{\mathfrak{B}_\iota;\ \iota \in I\}$ be a complete $\mathfrak{m}$-system of open families of a regular space P. Denoting the union $\bigcup \mathfrak{B}_\iota$ of the family $\mathfrak{B}_\iota$ by U_ι, suppose that the set*

$$R = \bigcap \{U_\iota;\ \iota \in I\}$$

is dense in P. Then R possesses a complete $\mathfrak{m}$-system of open coverings (and hence, R is a *$G(\mathfrak{m})$*-space).

Proof. As P is regular for each ι in I there exists a refinement $\mathfrak{B}'_\iota$ of the covering $\mathfrak{B}_\iota$ of the space U_ι, such that $\overline{V} \subset U_\iota$ for each V in $\mathfrak{B}'_\iota$. For each ι in I put $\mathfrak{W}_\iota = \mathfrak{B}'_\iota \cap R$. We shall prove that $\{\mathfrak{W}_\iota;\ i \in I\}$ is a complete system of open coverings of the space R. Let $\mathfrak{A}$ be an open centered family in R such that $\mathfrak{A} \cap \mathfrak{W}_\iota \neq \emptyset$ for each ι in I. Denote by $\mathfrak{A}_1$ the family of all open subsets A of P such that $A_1 \supset A$ for some A in $\mathfrak{A}$. Finally let $\mathfrak{A}_2$ be a maximal open (in P) centered family containing $\mathfrak{A}_1$. Evidently $\mathfrak{A}_1 \cap \mathfrak{B}_\iota \neq \emptyset$ for each $\iota \in I$. Thus the intersection $\bigcap \overline{\mathfrak{A}}_2$ is non-void. Since $\mathfrak{A}_2$ is a maximal family, this intersection is a one-point set e. g. (x). We shall prove that x belongs to $\bigcap \overline{\mathfrak{A}}^R$. Suppose the contrary, that for some A in $\mathfrak{A}$ the point x does not belong to $\overline{A}^R$. First let us notice that $x \in R$. Indeed, choosing V_ι in $\mathfrak{A}_1 \cap \mathfrak{B}'_\iota$, we have

$$x \in \bigcap \{\overline{V}_\iota^P;\ \iota \in I\} \subset \bigcap \{U_\iota;\ \iota \in I\} = R\ .$$

Since x non $\in \overline{A}^R$ and $x \in P$ we have x non $\in \overline{A}^P$. There exists a closed neighborhood F of x in P with $F \cap \overline{A}^P = \emptyset$. Hence, according to the definition of $\mathfrak{A}_1$ the set $P - F$ belongs to $\mathfrak{A}_1$. Thus $x \in \overline{P - F}$, which is impossible since F is a neighborhood of x. This contradiction completes the proof.

4.3. Proposition. *Let $\{\mathfrak{B}_n\}$ be a complete sequence of open families in a regular space P such that*

4.3.1 $$\overline{\bigcup \mathfrak{B}_n} = P\,, \quad n = 1, 2, \ldots$$

The space $R = \bigcap\{\bigcup \mathfrak{B}_n;\ n = 1, 2, \ldots\}$ is dense in P and possesses a complete sequence of open coverings (*and hence, R is a dense G_δ-subspace of P*).

Proof. According to 4.1 it is sufficient to prove that R is a dense subset of P. Given a non-void open subset U of P, we shall prove that $R \cap U \neq \emptyset$. By 4.2.1 we may choose V_1 in $\mathfrak{B}_1$ with $V_1 \cap U \neq \emptyset$. Using the regularity of P we may choose a non-void open set H_1 such that $\overline{H}_1 \subset V_1 \cap U$. Proceeding by induction we can construct non-void open sets H_n and V_n in $\mathfrak{B}_n$ such that for $n = 2, 3, \ldots$.

$$\overline{H}_n \subset U \cap \bigcap_{k=1}^{n-1} V_k \cap \bigcap_{k=1}^{n-1} H_k\,.$$

Since $\{\mathfrak{B}_n\}$ is a complete sequence and $\emptyset \neq \overline{H}_{n+1} \subset H_n \subset V_n \in \mathfrak{B}_n$ we have

$$H = \bigcap_{n=1}^{\infty} \overline{H}_n = \bigcap_{n=1}^{\infty} H_n \neq \emptyset\,.$$

But $H \subset \bigcap_{n=1}^{\infty} V_n \subset R$. This establishes $U \cap R \neq \emptyset$ and completes the proof.

4.4. Proposition. *Suppose that a dense subspace R of a regular space P possesses a complete $\mathfrak{m}$-system $\{\mathfrak{B}_\iota;\ \iota \in I\}$ of open coverings. For every open subset U of R choose an open subset U' of P such that $U' \cap R = U$. For each ι in I let $\mathfrak{B}'_\iota$ be the family of all U' with $U \in \mathfrak{B}_\iota$. For ι in I put $U_\iota = \bigcup \mathfrak{B}'_\iota$. Then $\{\mathfrak{B}'_\iota;\ \iota \in I\}$ is a complete system of open families in P and*

$$\bigcap\{U_\iota;\ \iota \in I\} = R\,.$$

Proof. Let $\mathfrak{A}$ be an open centered family in P such that for some $V_\iota \in \mathfrak{B}_\iota$ the set V'_ι belongs to $\mathfrak{A}$. Since R is a dense subspace of P, the family $R \cap \mathfrak{A}$ is centered and hence $\bigcap \overline{\mathfrak{A} \cap R}^R \neq \emptyset$, since $V_\iota \in (\mathfrak{A} \cap R) \cap \mathfrak{B}_\iota$. Thus $\bigcap \overline{\mathfrak{A}}^P \neq \emptyset$. The proof of the second assertion is quite analoguous to that of 2.4 and may thus be omitted.

Recall that a family $\mathfrak{A}$ of subsets of a topological space P is said to be an *almost-cover* of P if the union of the family $\mathfrak{A}$ is a dense subset of P. As corollaries of 4.2, 4.3 and 4.4 we have the following three theorems.

4.5. Theorem. *A regular space P contains a dense subspace possessing a complete sequence of open coverings if and only if there exists a complete sequence of open almost-covers of P.*

4.6. Theorem. *A completely regular space P contains a dense G_δ-space if and only if there exists a complete sequence of open almost-covers of P.*

4.7. Theorem. *A dense subspace R of a completely regular space P is a $G(\mathfrak{m})$-space if and only if there exists a complete $\mathfrak{m}$-system $\{\mathfrak{V}_\iota;\ \iota \in I\}$ of open families in P such that*

$$\bigcap\{\bigcup\mathfrak{V}_\iota;\ \iota \in I\} = R\ .$$

5. EXTENSIONS OF CONTINUOUS MAPPINGS

In this section we give a generalisation of the theorem of Lavrentev concerning extensions of a continuous mapping from a subset of a metrizable space to a complete metric space (in our terminology, to a metrizable G_δ-space). With this end in view we introduce a generalisation of "uniform topology".

Let $\mathfrak{A}$ be a family of subsets of a given set N. Recall that for each $N \supset M$ the set

$$S(M, \mathfrak{A}) = \bigcup\{A;\ A \in \mathfrak{A},\ A \cap M \neq \emptyset\}$$

is said to be the star of M in $\mathfrak{A}$. The star of a point x in $\mathfrak{A}$ is defined as the star of the one-point set (x), in symbols,

$$S(x, \mathfrak{A}) = S((x), \mathfrak{A})\ .$$

5.1. Definition. A space P is said to be a $\mathfrak{m}$-space if there exists a complete $\mathfrak{m}$-system $\{\mathfrak{V}_\iota;\ \iota \in I\}$ of open coverings of P such that

5.1.1. For each x in P the family $\{S(x, \overline{\mathfrak{V}}_\iota);\ \iota \in I\}$ is a local base at x.

5.2. Evidently, every space possessing a system $\{\mathfrak{V}_\iota\}$ of open coverings such that the condition 5.1.1 holds is a regular space. If a space possesses a complete $\mathfrak{m}$-system of open coverings and a $\mathfrak{m}$-system of coverings satisfying 5.1.1, then it is a $\mathfrak{m}$-space. Thus the concept of a $\mathfrak{m}$-space is assembled from the concept of space possessing a complete $\mathfrak{m}$-system of open coverings and of the concept of a space possessing a $\mathfrak{m}$-system of open coverings satisfying 5.1.1. If a space P possesses a $\mathfrak{m}$-system $\{\mathfrak{V}_\iota\}$ of open coverings such that 5.1.1 holds, then for each $M \subset P$ the system $\{\mathfrak{V}_\iota \cap M\}$ satisfies 5.1.1. Suppose that $\{\mathfrak{V}_\iota;\ \iota \in I\}$ is a system of open coverings of a space P such that 5.1.1 holds. The system is complete if and only if for every centered system $\{F_\iota;\ \iota \in I\}$ of closed subsets of P such that some $V_\iota \in \mathfrak{V}_\iota$ contains F_ι, the intersection $\bigcap\{F_\iota;\ \iota \in I\}$ is non-void. This is an immediate consequence of Theorem 2.14, since the condition 2.14.1 is evidently satisfied.

5.3. *If a space P possesses a* $\mathfrak{m}$-*system* $\{\mathfrak{V}_\iota;\ \iota \in I\}$ *of open coverings satisfying* 5.1.1, *then every closed subset of P is a $G(\mathfrak{m})$-subset.*

Proof. Supposing that F is a closed subset of P, it is easy to show that

$$F = \bigcap\{S(F, \mathfrak{V}_\iota);\ \iota \in I\}\ .$$

Indeed, if $x \in (P - F)$, then by 5.1.1 there exists an ι in I such that $S(x, \overline{\mathfrak{V}}_\iota) \subset$

$\subset (P - F)$. It follows that x non $\in S\ (F, \mathfrak{B}_\iota)$. It may be noticed that it was necessary to prove that

$$F = \bigcap\{S(F, \overline{\mathfrak{B}}_\iota);\ \iota \in I\}\,.$$

5.4. Theorem. *A subspace R of a $\mathfrak{m}$-space P is a $\mathfrak{m}$-space if and only if it is a $G(\mathfrak{m})$-subset of P.*

Proof. By Theorem 2.7 the space R is a $G(\mathfrak{m})$-space if and only if R is a $G(\mathfrak{m})$-subset of $\overline{R}^P$. By 5.3 every closed subset of a $\mathfrak{m}$-space is a $G(\mathfrak{m})$-subset.

5.5. Theorem. *Suppose that $\{P_a;\ a \in A\}$ is a system of $\mathfrak{m}$-spaces such that every P_a contains at least two points. Then the topological product*

$$P = \mathsf{X}\{P_a;\ a \in A\}$$

is a $\mathfrak{m}$-space if and only if the potency of A is at most $\mathfrak{m}$.

Proof. If the potency of B is $> \mathfrak{m}$ then the character of every point of P is $> \mathfrak{m}$ and hence the condition 5.1.1 is satisfied by no $\mathfrak{m}$-system of open coverings. Thus P is not a $\mathfrak{m}$-space.

Conversely, suppose that the potency of A is at most $\mathfrak{m}$. By Theorem 2.10 the space P possesses a complete $\mathfrak{m}$-system of open coverings. In consequence it is sufficient to show that some $\mathfrak{m}$-system $\{\mathfrak{B}_\iota;\ \iota \in I\}$ of P has the property 5.1.1. By our assumption for each $a \in A$ there exists a $\mathfrak{m}$-system

$$\{\mathfrak{B}_{a,c};\ c \in C_a\}$$

of open coverings of P_a such that 5.1.1 holds. Without loss of generality we may assume that the system $\{C_a;\ a \in A\}$ is disjoint. Denote by I the family of all finite subsets of the set

$$\bigcup\{(a) \times C_a;\ a \in A\}\,.$$

Denote by π_a the projection of P onto P_a. For each ι in I, $i = \{(a_1, c_1), \ldots, \ldots, (a_k, c_k)\}$, let $\mathfrak{W}_\iota$ be an open refinement of the coverings

$$\pi_{a_i}^{-1}[\mathfrak{B}_{a_i,a_i}]\,,\quad i = 1, 2, \ldots, k\,.$$

Clearly $\{\mathfrak{W}_\iota;\ \iota \in I\}$ is a $\mathfrak{m}$-system of open coverings of P and for each x in P the family

$$\{S(x, \overline{\mathfrak{W}}_\iota)\,;\quad \iota \in I\}$$

is a local base at x. The proof is complete.

5.6. Theorem on extensions of continuous mappings. *Let P be a $\mathfrak{m}$-space. Let Q be a dense subset of a space R. Let f be a continuous mapping from Q to P. Then there exist a $G(\mathfrak{m})$-subset S of R containing Q and a continuous mapping F from S to P such that f is a restriction of F.*

Proof. Let $\{\mathfrak{B}_\iota;\ \iota \in I\}$ be a complete $\mathfrak{m}$-system of open coverings of the space P satisfying the condition 5.1.1. For each ι in I denote by $\mathfrak{W}_\iota$ the family of all open subsets W of R such that $f[W \cap Q] \subset V$ for some V in $\mathfrak{V}_\iota$. For each ι in I denote by U_ι the union of the family $\mathfrak{W}_\iota$. Consider the space $S = \bigcap\{U_\iota;\ \iota \in I\}$. By continuity of f, the set Q is contained in every U_ι, and consequently Q is contained in S. The sets U_ι are open and hence S is a $G(\mathfrak{m})$-subset of R.

We shall now construct the mapping F. For each x in S denote by $\mathfrak{F}(x)$ the family of all $V \in \bigcup\{\mathfrak{V}_\iota;\ \iota \in I\}$ such that for some neighborhood W of the point x we have $f[W \cap Q] \subset V$. $\mathfrak{F}(x)$ is an open centred family since x is an accumulation point of Q. Indeed, choosing $V_1, V_2, \ldots, V_n$ in $\mathfrak{F}(x)$ we may select open neighborhoods $W_1, W_2, \ldots, W_n$ of x with

$$f[V_i \cap Q] \subset W_i \quad (i = 1, 2, \ldots, n)\,.$$

The intersection $W = \bigcap_{i=1}^{n} W_i$ is a neighborhood of x and hence W meets Q. Clearly

$$f[Q \cap W] \subset \bigcap_{i=1}^{n} V_i\,.$$

It follows that $\bigcap_{i=1}^{n} V_i$ is non-void. By the construction of S, for each ι in I we can select an open neighborhood $W_\iota(x)$ of x and a $V_\iota(x) \in \mathfrak{V}_i$ such that

$$f[Q \cap W_\iota(x)] \subset V_\iota(x)\,.$$

Thus $V_\iota(x) \in \mathfrak{F}(x)$ for every ι, that is, $\mathfrak{F}(x) \cap \mathfrak{V}_\iota \neq \emptyset$ for every ι. The system $\{\mathfrak{V}_\iota\}$ is complete and hence the intersection $\bigcap \overline{\mathfrak{F}(x)}$ is non-void. Since $\{\mathfrak{V}_\iota\}$ satisfies condition 5.1.1, this intersection contains one point only. Denote this point by $F(x)$. If $x \in Q$, then $f(x) \in V_\iota(x)$ for every ι and consequently $F(x) = f(x)$. We have defined a mapping F from S to P such that f is a restriction of F to Q.

It remains to prove that F is a continuous mapping. First we show

5.6.1. For every open subset U of S we have

$$F[U] \subset \overline{f[U \cap Q]}\,.$$

Choosing x in U, for every ι

$$V_\iota(x) \supset f[W_\iota(x) \cap U \cap Q]\,.$$

Thus

$$f[U \cap Q] \cap V_\iota(x) \neq \emptyset\,.$$

Since $F(x) \in \overline{V_\iota(x)}$ for every ι, by condition 5.1.1 we have

$$F(x) \in \overline{f[U \cap Q]}$$

which completes the proof of 5.6.1.

To prove the continuity of F, given a x in S and an neighborhood V of $F(x)$ we choose ι in I with

5.6.2. $$S(F(x), \overline{\mathfrak{V}}_\iota) \subset V .$$

By definition of $V_\iota(x)$ we have $f[W_\iota(x) \cap Q] \subset V_\iota(x)$.

Applying 5.6.1, we obtain $F[W_\iota(x)] \subset \overline{f[W_\iota(x) \cap Q]}$ and hence $F[W_\iota(x)] \subset \subset \overline{V_\iota(x)}$. Since $F(x) \in \overline{V_\iota(x)}$, by 5.6.2 we have $F[W_\iota(x)] \subset V$. This establishes continuity and completes the proof of the theorem.

5.7. Theorem. *Suppose that f is a homeomorphic maping from a subspace of a $\mathfrak{m}$-space R onto a subspace N of a $\mathfrak{m}$-space P. There exists a homeomorphic mapping F of a $G(\mathfrak{m})$-subset $M_1 \supset M$ onto a $G(\mathfrak{m})$-subset $N_1 \supset N$ such that f is a restriction of F.*

Using theorems 5.6 and 5.3, the proof of this theorem could be led made analogous to that of the parallel theorem on complete metric spaces, see [4], pp. 335.

As an example of $\mathfrak{m}$-spaces we introduce the concept of complete $\mathfrak{m}$-metric spaces.

5.8. Definition. A space P is said to be $\mathfrak{m}$-metrizable if there exist a $\mathfrak{m}$-system $\{\varphi_\iota\}$ of pseudometries in P such that for each x in P and every subset M of P P $x \in \overline{M}$ if and only if the φ_ι-distance of x and M is zero for every ι. The pair $(P, \{\varphi_\iota\})$ is said to be a $\mathfrak{m}$-metric space. A $\mathfrak{m}$-metric space $(P, \{\varphi_\iota\})$ is said to be complete if for every closed centered family $\mathfrak{F}$ in P with

5.8.1 $$\inf \{d_{\varphi_\iota}(F); F \in \mathfrak{F}\} = 0$$

for every φ_ι the intersection $\bigcap \mathfrak{F}$ is non-void. A space P is said to be a complete $\mathfrak{m}$-metrizable space if there exists a complete $\mathfrak{m}$-metric space $(P, \{\varphi_\iota\})$ such that the family $\{\varphi_\iota\}$ generates the topology of P, in the sense that $x \in \overline{M}$ if and only if the φ_ι-distance of x and M is zero for every ι.

5.9. If $1 \leqq \mathfrak{m} \leqq \aleph_0$, then every $\mathfrak{m}$-metrizable space is a metrizable space. If $\mathfrak{m}$ is an infinite cardinal number, then every complete $\mathfrak{m}$-metrizable space is a $\mathfrak{m}$-space. Conversely, if $\mathfrak{m}$ is an infinite cardinal number, then every fully normal (= paracompact) $\mathfrak{m}$-space is a complete $\mathfrak{m}$-metrizable space. The last assertion assertion is a consequence of the theorem asserting that for every open covering $\mathfrak{V}$ of a fully normal space P there exists a pseudometric φ in P such that the family of all spheres of radius 1 is a rafinement of $\mathfrak{V}$.

5.10. Theorem. *The topological product of a $\mathfrak{m}$-system of metrizable G_δ-spaces is a complete $\mathfrak{m}$-metrizable space.*

Proof. Let $\{P_a; a \in A\}$ be a $\mathfrak{m}$-system of metrizable G_δ-spaces. Denote by P its topological product. By Theorem 2, for every ι we can choose a metric ψ_a for P such that $(P_a; \psi_a)$ is a complete metric space. By a well-known theorem, the product of complete uniform spaces is a complete uniform space. But the product uniformity for P is generated by the $\mathfrak{m}$-system $\{\psi_a^*\}$ of pseudometrics in P defined as follows: For each $x = \{x_a\}$ and $x = \{y_a\}$ in P set $\psi_a^*(x, y) = = \psi_a(x_a, y_a)$. Denote by I the set of all finite subsets of A and for each $\iota \in I$ put

$$\varphi_\iota(x, y) = \sum_{a \in \iota} \psi_a^*(x, y) .$$

Evidently every centered family satisfied 5.8.1 is a Cauchy family. It follows that $\{P, \{\varphi_\iota\}\}$ is a $\mathfrak{m}$-complete metric space. The proof is complete.

5.11. *For every infinite cardinal number* $\mathfrak{m}$ *there exists a complete* $\mathfrak{m}$*-metrizable space which is not a* $G(\mathfrak{n})$*-space with* $\mathfrak{n} < \mathfrak{m}$.

Proof. Evidently the space Z of irrational numbers is a metrizable G_δ-space which is a $G(\mathfrak{n})$- space for no $\mathfrak{n} < \aleph_0$. By theorem 5.10 the cube $Z^{\mathfrak{m}}$ is a complete m-metrizable space and by theorem 5.5 $Z^{\mathfrak{m}}$ is $G(\mathfrak{n})$-space for no $\mathfrak{n} < \mathfrak{m}$.

Recall the following simple lemma.

5.12. Lemma. *Let F be a continuous mapping from P to Q such that for some dense subset P_1 of P the restriction F/P_1 of F to P_1 is a homeomorphic mapping. Then*

$$F[P_1] \cap F[P - P_1] = \emptyset .$$

5.13. Consider the identity mapping f of a $\mathfrak{m}$-space P which is not a $\mathfrak{n}$-space for any $\mathfrak{n} < \mathfrak{m}$. Let R be a extension of P. By lemma 5.12 there exists no continuous mapping F from R to P such that $F/P = f$. It follows that in theorem 5.6 the assumption "P is a $G(\mathfrak{m})$-space" is essential.

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Резюме

ОБОБЩЕНИЯ G_δ-СВОЙСТВА ПОЛНЫХ МЕТРИЧЕСКИХ ПРОСТРАНСТВ

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Пусть $\mathfrak{m}$-кардинальное число. Топологическое пространство Хаусдорфа P называется $G(\mathfrak{m})$-пространством, если выполнено следующее условие: если P является плотным подпространсвом пространства Хаусдорфа R, то P является пересечением $\mathfrak{m}$ открытых множеств пространства R, т.е. P является $G(\mathfrak{m})$-множеством в R. Следуя Э. Чеху, вполне регулярные $G(\aleph_0)$ -пространства называют топологически полными простpнаствами. Известно, что метризуемое пространство P является G_δ-пространством тогда и только тогда, если для некоторой метрики ϱ метрическое пространство (P, ϱ) полно.

Во второй части дается ,,внутренная" характеризация вполне регулярных $G(\mathfrak{m})$-пространств. Оказывается, что пользуясь этой характеризацией, освноные свойства $G(\mathfrak{m})$-пространств очевидны. Семейство $\{\mathfrak{W}_\iota;\ \iota \in I\}$ открытых покрытий пространства P называется полным, если для всякой центрированной системы множеств $\mathfrak{A}$, содержащей множества из каждого покрытия $\mathfrak{W}_\iota$, пересечение замыканий множеств из $\mathfrak{A}$ не пусто. Оказывается, что вполне регулярное пространство P является $G(\mathfrak{m})$-пространством тогда и только тогда, если существует полное семейство $\{\mathfrak{W}_\iota;\ \iota \in I\}$ отркытых покрытий пространства P так, что мощность множества индексов I равна $\mathfrak{m}$. Аналогично внутренне характеризуются вполне регулярные пространства, содержащие плотное $G(\mathfrak{m})$-пространство.

В третьей части рассматриваются дальнейшие свойства $G(\aleph_0)$-пространств, и дается несколько примеров.

В последней части рассматриваются $\mathfrak{m}$-пространства. Пространство P называется $\mathfrak{m}$-пространством, если существует полное семейство $\{\mathfrak{W}_\iota;\ \iota \in I\}$ открытых покрытий пространства P так, что мощность множества I равна $\mathfrak{m}$ и

$$\{S(x, \overline{\mathfrak{W}}_\iota)\ ;\quad \iota \in I\}$$

является базисом окрестностей для каждой точки $x \in P$, где $S(x, \overline{\mathfrak{W}}_\iota)$ обозначает звезду точки x в множестве замыканий элементов из $\mathfrak{W}$. Оказывается, что известная теорема о расширении непрерывного отображения до польного метрического пространства имеет некоторое обобщение на $\mathfrak{m}$-пространства.

Теорема. *Пусть P — $\mathfrak{m}$-пространство. Пусть Q — плотное подпространство пространства R. Наконец, пусть f — непрерывное отображерие пространства Q в P. Существует $G(\mathfrak{m})$-множество S в R и непрерывное отображение F пространства S в P так, что $f(x) = F(x)$ для $x \in Q$.*

Известная теорема о расширении гомеоморфизма имеет следующее обобщение:

Теорема. *Пусть f — гомеоморфное отображение подмножества M $\mathfrak{m}$-пространства P на подмножество N $\mathfrak{m}$-пространства Q. Существует гомеоморфное отображение F $G(\mathfrak{m})$-множества $M_1 \subset P$, $M \subset M_1$ на $G(\mathfrak{m})$-множество $N_1 \subset Q$, $N \subset N_1$ так, что $F(x) = f(x)$ для $x \in M$.*