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THE TOPOLOGICAL PRODUCT OF TWO PSEUDOCOMPACT SPACES

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The present paper is concerned with the following three questions (all spaces are supposed completely regular):

(1) Supposing P and Q pseudocompact, under what conditions is the topological product $P \times Q$ a pseudocompact space? According to [3], $P \times Q$ is pseudocompact if any only if $\beta(P \times Q) = \beta(P) \times \beta(Q)$. We give a new and short proof of this proposition and in addition we deduce some further necessary and sufficient conditions (Section 2).

(2) Under what conditions on a space P is the topological product $P \times Q$ a pseudocompact space for every pseudocompact Q? The class (denoted by $\mathfrak{P}$) of all such spaces P is investigated in section 3.

(3) Under what conditions on a space P does every closed subspace of P belong to $\mathfrak{P}$? The class of all such spaces is studied in the section 4.

I

In this section we recall some results conserning pseudocompact spaces and prove a few lemmas which will be needed in the whole paper.

The terminology of J. KELLEY, *General Topology*, is used throughout. A subset of a space is always considered as a subspace. The closure of a subset M of a space P is denoted by $\overline{M}^P$, or merely by $\overline{M}$ if no confusion is possible. $\beta(P)$ denotes the Stone-Čech compactification of a (completely regular) space P. Function shall mean a real-valued function.

A space P is said to be pseudocompact if (and only if) every continuous function on P is bounded, or equivalently, if every continuous bounded function assumes its bounds. A completely regular space is pseudocompact if and only if every locally finite family of its open subsets is finite, or equivalently, if there exists no locally finite sequence of its non-void open subsets. A regular closed subset (i. e. a set of the form $M = \overline{\operatorname{int} \overline{M}}$) of a completely regular pseudocompact space is a pseudocompact space. For further information (in a more general situation) see [2].

1.1. Lemma. *Let X be a space and let K be a compact one. Let f be a continuous function on $X \times K$. For each x_1, x_2 and x in X put*

$$\varphi(x_1, x_2) = \sup_{k \in K} |f(x_1, k) - f(x_2, k)|$$

and

$$F(x) = \inf_{k \in K} f(x, k) \, .$$

Then φ is a continuous pseudometric in X and F is a continuous function on X.

Proof. The second statement of the lemma is an obvious consequence of the first. Since φ is a pseudometric (φ is finite since K is compact and hence pseudocompact), to prove the continuity of φ it is sufficient to show that for each x in X and every $\varepsilon > 0$ there exists a neighborhood U of x such that $\varphi(x, x') \leqq \varepsilon$ for each x' in U. By the continuity of f, for each k in K we may choose a canonical open neighborhood $W(k) = U(k) \times V(k)$ of (x, k) such that f varies $< \varepsilon$ on $W(k)$. The space K is compact and consequently for some finite subset M of K the family $\{V(k); k \in M\}$ covers K. Put $U = \bigcap \{U(k); k \in M\}$. For each x' in U and each k in K we have

$$|f(x, k) - f(x', k)| < \varepsilon \, .$$

Thus $\varphi(x, x') \leqq \varepsilon$ for each x' in U; this establishes continuity and completes the proof.

1.2. Lemma. *Let X and Y be completely regular infinite spaces. If the topological product $R = X \times Y$ is not pseudocompact then there exists a locally finite sequence $\{U_n \times V_n\}$ of non-void canonical open subsets of $X \times Y$ such that the sequences $\{U_n\}$ and $\{V_n\}$ are disjoint.*

Proof. If one of the spaces is not pseudocompact, for instance X, then there exists a locally finite disjoint sequence $\{U_n\}$ of non-void open subsets of X. Selecting a disjoint sequence $\{V_n\}$ of non-void subsets of Y (this is possible since Y in an infinite Hausdorff space), we obtain a sequence $\{U_n \times V_n)$ with the desired properties.

Now let us suppose that X and Y are pseudocompact spaces. Let $\{U'_n \times V'_n; n \in N\}$ be a locally finite sequence of non-void open subsets of $X \times Y$. First we shall prove

1.2.1. Let $\{U'_n \times V'_n; n \in N'\}$ be a subsequence of the given sequence. For each x in X there exists an open neighborhood U of x such that $U \cap U'_n = \emptyset$ for an infinite number of $n \in N'$.

For proof, suppose the contrary. Selecting a cluster point y of $\{V'_x; n \in N'\}$, it is easy to see that (x, y) is a cluster point of $\{U'_n \times V'_n; n \in N'\}$. This contradiction completes the proof of 1.2.1.

According to 1.2.1 we can choose by induction a sequence $n_1, n_2, \ldots$ in N and open non-void $U_{n_i} \subset U'_{n_i}$ such that the sequence $\{U_{n_i}\}$ is disjoint. Applied the same argument to the sequence $\{U_{n_i} \times V'_{n_i}\}$, we obtain a subsequence $\{n_{i_k}\}$ of $\{n_i\}$ and open non-void $V_{n_{i_k}} \subset V'_{n_{i_k}}$ such that the sequence $\{V_{n_{i_k}}\}$ is disjoint. The sequence $\{U_{n_{i_k}} \times V_{n_{i_k}}\}$ possesses the desired properties.

1.3. Lemma. *Let f be a continuous function on a pseudocompact completely regular space $X \times Y$. For each x in X put*

$$F(x) = \inf_{y \in Y} f(x, y) \quad \text{and} \quad G(x) = \sup_{y \in Y} f(x, y) .$$

Then F and G are continuous functions on X.

Proof. Clearly it is sufficient to prove only the continuity of F. Since the function F is upper semi-continuous as an infimum of continuous functions $f(.\,, y)$, $y \in Y$, it is sufficient to show that F is lower semi-continuous. Suppose that F is not lower semi-continuous. There exists an x_0 in X and a real number $\varepsilon > 0$ such that for every neighborhood U of x_0 there exists an (x, y) in $U \times Y$ with $f(x, y) < F(x_0) - 3\varepsilon$. By introduction we can construct points (x_n, y_n) and open neighborhoods $W_n = U_n \times V_n$ and $W'_n = U'_n \times V_n$ of (x_n, y_n) and (x_0, y_n) such that

(i) f varies $< \varepsilon$ on both W_n and W'_x,

(ii) $U'_{n-1} \supset U_n$,

(iii) $f(x_n, y_n) < F(x_0) - 3\varepsilon$.

Having chosen x_i, y_i, W_i and W'_i for $i < n$, put $U = U'_{n-1}$ if $n > 1$ and $U = X$ if $n = 1$. By our assumption we can choose $(y_n, y_n) \in U \times Y$ such that (iii) holds. According to the continuity of f there exist U_n, V_n and U'_n such that (i) and (ii) holds. The space $X \times Y$ is pseudocompact and completely regular and hence there is an accumulation point $(\bar{x}, \bar{y})$ of the sequence $\{W_n\}$. Since $f(x) < F(x_0) - 2\varepsilon$ on W_n, by continuity of f we have

$$f(\bar{x}, \bar{y}) \leqq F(x_0) - 2\varepsilon .$$

On the other hand, from the condition (ii) we conclude that $(\bar{x}, \bar{y})$ is also an accumulation point of $\{W'_n\}$. Since $f(x) > F(x_0) - \varepsilon$ for x in W'_n, according to the continuity of f we have

$$f(\bar{x}, \bar{y}) \geqq F(x_0) - \varepsilon .$$

This contradiction completes the proof of the lemma.

1.4. Lemma. *Let f be a continuous function on a pseudocompact completely regular space $X \times Y$. If K is a compactification of Y such that every function $f(x, .)$ has a continuous extension to K, then f has an continuous extension to $X \times K$.*

Proof. Extending continuously every function $f(x, .)$ on $(x) \times K$, we obtain a function f^* on $X \times K$. We shall prove that f^* is continuous. Given an $(x_0, y_0) \in X \times K$ and $\varepsilon > 0$, we have to find a neighborhood $W = U \times V$

of (x_0, y_0) such that $|f^*(x_0, y_0) - f(x, y)| < \varepsilon$ for each (x, y) in W. Choose an open neighborhood V of y_0 in K such that $|f^*(x_0, y) - f^*(x_0, y_0)| < \varepsilon$ for each y in V. The space $\overline{V \cap Y^Y}$ is pseudocompact and hence by lemma 1.5 we may choose an open neighborhood U of x_0 such that for each x in U both

$$\inf_{y \in V \cap Y} f(x, y) > f^*(x_0, y_0) - 2\varepsilon$$

and

$$\sup_{y \in V \cap Y} f(x, y) < f^*(x_0 \, y_0) + 2\varepsilon \, .$$

In consequence we have $|f^*(x, y) - f^*(x_0, y_0)| < 2\varepsilon$ for each (x, y) in $U \times V$.

II

2.1. Theorem. *The following conditions on infinite completely regular spaces X and Y are equivalent.*

(1) *The topological product $X \times Y$ is a pseudocompact space.*

(2) $\beta(X \times Y) = \beta(X) \times \beta(Y)$, *that is, very bounded continuous function on $X \times Y$ posesses a continuous extension to $\beta(X) \times \beta(Y)$.*

(3) *If f is a bounded continuous function on $X \times Y$, then for every $\varepsilon > 0$ there exists a finite cover $\mathfrak{A} = \{A_1, A_2, \ldots, A_n\}$ of $X \times Y$ consisting of canonical open sets A_i on each of which f varies $< \varepsilon$.*

Proof. Suppose that (1) holds and let f be a continuous function on $X \times Y$. By lemma 1.4 there exists a continuous extension of f to $X \times \beta(Y)$, and by the same lemma this extension possesses a continuous extension to $\beta(X) \times \times \beta(Y)$. Hence (1) implies (2).

Suppose (2) holds, and let f be a bounded continuous function on $X \times Y$. Denote by f^* the continuous extension of f to $\beta(X) \times \beta(Y)$. The family $\mathfrak{A}_1$ of all canonical open subsets of $\beta(X) \times \beta(Y)$ on which f^* varies $< \varepsilon$ is an open cover of the compact space $\beta(X) \times \beta(X)$; hence some finite subfamily $\mathfrak{A}_2$ of $\mathfrak{A}_1$ is also a cover. The family $\mathfrak{A}$ of all intersections $A \cap (X \times Y)$ with $A \in \mathfrak{A}_2$ has the desired properties.

To prove that (3) implies (1), suppose non (1). By lemma 1.2 there exists a locally finite sequence $\{W_n\} = \{U_n \times V_n\}$ of non-void canonical open subsets of $X \times Y$ such that the sequences $\{U_n\}$ and $\{V_n\}$ are disjoint. Choose points $Z_n \in W_n$ and continuous functions $f_n \leqq 1$ such that $f_n(z_n) = 1$ and $f_n(z) = 0$ for each z non $\in W_n$. Put $f = \sum_{n=1}^{\infty} f_n$; this is a bounded continuous function on $X \times Y$, and if $A = A_1 \times A_2$ is a subset of $X \times Y$ containing two points z_n and z_k with $n \neq k$, then f varies $\geqq 1$ on A. Indeed, if $z_i = (x_i, y_i)$, $i = 1, 2$, then the point (x_n, y_k) belongs to A and $f(x_n, y_k) = 0$. Hence $f(z_n) - f(x_n, y_k) = 1$. If follows that the condition (3) is not satisfied with our f and $\varepsilon = 1$.

2.2. Theorem. *The following conditions on completely regular infinite spaces X and Y are equivalent.*

(1) *$X \times Y$ is a pseudompact space.*

(2) *The spaces X and Y are pseudocompact and for every bounded continuous function on $X \times Y$ the mappings*

$$\Phi : y \to f(\cdot, y) \,\varepsilon\, C(X)$$

and

$$\Psi : x \to f(x, \cdot) \,\epsilon\, C(Y)$$

of Y to $C(Y)$ and X to $C(Y)$ respectively, are continuous.

(3) *For every bounded continuous function f on $X \times Y$, the pseudometrics*

$$\varphi(x_1, x_2) = \sup_{y \epsilon Y} |f(x_1, y) - f(x_2, y)|$$

and

$$\psi(y_1, y_2) = \sup_{x \epsilon X} |f(x, y_1) - f(x, y_2)|$$

are continuous and totally bounded in X and Y, respectively.

Proof. Denoting by $\| \ \|$ the norms in $C(X)$ and $C(Y)$, we have

$$\varphi(x_1, x_2) = \|f(x_1, \cdot) - f(x_2, \cdot)\|$$

and

$$\psi(y_1, y_2) = \|f(\ , y_1) - f(\cdot, y_2)\| .$$

It follows that the mappings Φ and Ψ are continuous if and only if the pseudometrics φ and ψ are continuous.

Suppose (1) holds, and let f be a bounded continuous function on $X \times Y$. By theorem 2.1 there exists a continuous extension f^* of f to $\beta(X) \times \beta(Y)$. By lemma 1.1 the pseudometrics

$$\varphi^*(x_1, x_2) = \sup_{y \epsilon \beta(Y)} |f^*(x_1, y) - f^*(x_2, y)|$$

and

$$\psi^*(y_1, y_2) = \sup_{y \epsilon \beta(X)} |f^*(x, y_1) - f^*(x, y_2)|$$

are continuous in $\beta(X)$ and $\beta(Y)$ respectively. Evidently φ and ψ are restrictions of φ^* and ψ^*. Thus φ and ψ are continuous. X and Y are pseudocompact since $X \times Y$ is a pseudocompact space. Hence (1) implies (2). Clearly (2) implies (3).

It remains to prove that (3) implies (1). Let f be any bounded continuous function on $X \times Y$. By (3), the pseudometrics φ and ψ are totally bounded and consequently we can find finite open coverings $\|A_i\|$ and $\|B_j\|$ of X and Y respectively, such that the diameters of A_i in φ and B_j in ψ are less then ε. Then $\{A_i \times B_j\}$ is a covering of $X \times Y$ by canonical open sets, and clearly, f varies on every $A_i \times B_j < 2\varepsilon$. By theorem 2.1 the space $X \times Y$ is pseudocompact. The proof is complete.

Note. Analogous theorems hold for infinite numbers of factors. To extend these theorems it suffices to prove that every bounded continuous function on a pseudocompact space $X\{P_a; a \in A\}$ can be aproximated uniformly by continuous functions depending only on a finite number of coordinates.

III. THE CLASS $\mathfrak{P}$

3.1. Definition. Let $\mathfrak{P}$ be the class of all completely regular spaces X such that for every pseudocompact completely regular space Y the topological product $X \times Y$ is pseudocompact space.

Evidently:

3.2. If a completely regular space X is a continuous image of a space belonging to $\mathfrak{P}$, then X belongs to $\mathfrak{P}$. If X and Y belong to $\mathfrak{P}$, then the topological product $X \times Y$ belongs to $\mathfrak{P}$. If $X \times Y$ belongs to $\mathfrak{P}$ then both X and Y belong to $\mathfrak{P}$. If F is a regularly closed subspace (that is, $F = \overline{\text{int } F}$), of a space $X \in \mathfrak{P}$ then F belongs to $\mathfrak{P}$.

3.3. Theorem. *Let X be a pseudocompact space such that each point of X has a neighborhood belonging to $\mathfrak{P}$. Then X belongs to $\mathfrak{P}$.*

Proof. Suppose the contrary, that for some pseudocompact completely regular space Y the topological product $X \times Y$ is not pseudocompact. Let $\{U_n \times V_n\}$ be a locally finite sequence of non-void open subsets of $X \times Y$. We shall prove that $\{U_n\}$ is a locally finite sequence, in contradiction with the pseudocompactness of X. Given an $x \in X$ we choose a neighborhood U of x belonging to $\mathfrak{P}$. Henge $U \times Y$ is a pseudocompact space. In consequence the intersection $(U_n \times V_n) \cap (U \times Y)$ is non-void for only a finite number of n's and hence $U_n \cap U \neq \emptyset$ only for a finite number of n's. This establishes the local finiteness of $\{U_n\}$ and completes the proof of 3.3.

3.4. *Compact spaces belong to $\mathfrak{P}$.*

Proof. Let X be a pseudocompact space and let K be a compact space. To prove that $X \times K$ is pseudocompact, it is sufficient to show that every bounded continuous function f on $X \times K$ assumes its lower bound. But this is an obvious consequence of lemma 1.1. Consider the function F of this lemma. X is a pseudocompact space and hence assumes its lower bound, at a point x. The function $f(x, .)$ on $(x) \times K$ assumes its lower bound at a point y. Evidently f assumes its lower bound at (x, y).

3.5. Theorem. *A space X belongs to $\mathfrak{P}$ if it satisfies the following condition:*

3.5.1. *If $\mathfrak{A}$ is an infinite disjoint family of non-void open subsets of X, then for some compact subset K of X the intersection $K \cap A$ is non-void for an infinite number of sets A belonging to $\mathfrak{A}$.*

Proof. Suppose the contrary, that X does not belong to $\mathfrak{P}$. Hence, for some pseudocompact completely regular space Y, the topological product $X \times Y$ is not a pseudocompact space. By lemma 1.2 there exists a locally finite sequence $\{U_n \times V_n\}$ of non-void open subsets of $X \times Y$ such that the sequence $\{U_n\}$ is disjoint. By condition 3.5.1 there is a compact space K meeting an infinite number of sets U_n. Consider the space $K \times Y$. By 3.4 the space $K \times Y$ is pseudocompact. But $\{(U_n \times V_n) \cap (K \times Y); U_n \cap K \neq \emptyset\}$ is a locally finite sequence of non-void open subsets of $K \times Y$. This contradiction completes the proof.

The simple condition 3.5.1 is not necessary for X to be an element of $\mathfrak{P}$. First we prove a necessary and sufficient condition and then we shall show that there exists a space satisfying this condition which does not satisfy the condition 3.5.1.

3.6. Theorem. *A completely regular space X belongs to $\mathfrak{P}$ if and only if it satisfies the following condition:*

3.6.1. *If $\mathfrak{A}$ is an infinite disjoint family of non-void open subsets of X then there exists a disjoint sequence $\{U_n\}$ in $\mathfrak{A}$ such that for every filter $\mathfrak{N}$ of infinite subsets of $N = \{n\}$ we have*

$$(*) \qquad \bigcap_{N_1 \in \mathfrak{N}} \overline{\bigcup_{n \in N_1} U_n} \neq \emptyset .$$

Proof. First suppose 3.6.1 and let Y be a pseudocompact space. We have to prove that $X \times Y$ is a pseudocompact space. Suppose that $X \times Y$ is not pseudocompact.

Evidently the spaces X and Y are infinite. By lemma 1.2 there exists a sequence $\{U_n \times V_n\}$ of non-void open subsets of $X \times Y$ such that the sequence $\{U_n\}$ is disjoint. By condition 3.6.1 there exists a subsequence $\{U_{n'}\}$ of $\{U_n\}$ such that (*) holds for every filter $\mathfrak{N}$ in the set N of all the integers n'. Consider the locally finite sequence $\{U_n \times V_n; n \in N\}$. Let y be a cluster point of $\{V_n; n \in N\}$. Let $\mathfrak{B}$ be the family of all neighborhoods of the point y. For each $B \in \mathfrak{B}$ put $N(B) = \{n; n \in N, B \cap V_n \neq \emptyset\}$. Evidently $\mathfrak{N} = \{N(B); B \in \mathfrak{B}\}$ is a filter in N. By our assumption we may choose an x in $\bigcap_{N_1 \in \mathfrak{N}} \overline{\bigcup_{n \in N_1} U_n}$. It is easy to see that (x, y) is a cluster point of the sequence $\{U_n \times V_n; n \in N\}$. This contradiction completes the proof of sufficiency of the condition.

For the proof of necessity, suppose that a completely regular space X does not satisfy the condition 3.6.1. Then there exists a countably infinite disjoint family $\{U_n; n \in N\}$ of non-void open subsets of X such that if N_0 is any infinite subset of N, then

$$(**) \qquad \bigcap_{N_1 \in \mathfrak{N}} \overline{\bigcup_{n \in N_1} U_n} = \emptyset$$

for some filter $\mathfrak{N}$ in N_0. Selecting points z_n in U_n and denoting the set of all these z_n by Z, consider the space $Y = \overline{Z}^{\beta X} - (X - Z)$. By our assumption, the space Y is pseudocompact since every infinite subset Z_0 of Z has an accumulation point in Y. Indeed, if we denote by N_0 the set of all the n's with $z_n \in Z_0$, there exists a filter $\mathfrak{N}$ in N_0 such that (**) holds. But every point of the non-void set

$$\bigcap_{N_1 \in \mathfrak{N}} \overline{\bigcup_{n \in N_1} (U_n \cap Z)}^{\beta P}$$

is an accumulation point of Z_0 in Y. We proceed to prove that the space $X \times Y$ is not pseudocompact. First note that the one-point sets (z_n) are open in Y. Consider the family $\mathfrak{A} = \{U_n \times (z_n); n \in N\}$ of non-void open subsets of $X \times Y$. We shall prove that $\mathfrak{A}$ is locally finite. Let z be a cluster point of Z in Y. Let $\mathfrak{A}$ be the family of all intersections $Z \cap W$, where W is a neighborhood of z. Then by our construction

$$\bigcap_{Z_1 \in \mathfrak{N}} \overline{\bigcup_{z_n \in Z_1} U_n}^X = \emptyset$$

and hence for each x in X we may choose a neighborhood U of x and a set Z' in $\mathfrak{N}$ such that

$$\overline{\bigcup_{z_n \in Z'} U_n} \cap U = \emptyset .$$

Selecting a neighborhood V of z in Y such that $Z \cap V = Z'$, it is easy to show that $U \times V$ is a neighborhood of (x, z) meeting no set $U_n \times (z_n)$, $n \in N$. The proof of theorem 3.6 is thus complete.

3.7. Example. Let K_n $(n = 1, 2, \ldots)$ denote the closed interval $\langle 2n, 2n + 1\rangle$ of real numbers. Let R_n and I_n denote the set of all the rational and all the irrational numbers of K_n, respectively. Denote the set $\bigcup_{n=1}^{\infty} K_n$ by R. There exists a subspace X of $\beta(R)$ containing R and satisfying the following two conditions:

3.7.1. If U_n $(n = 1, 2, \ldots)$ are non-void open subsets of K_n, then

$$\bigcup_{N_1 \in \mathfrak{N}_1} \overline{\bigcup_{n \in N_1} U_n}^X \neq \emptyset$$

for every filter $\mathfrak{N}$ in $N = \{1, 2, \ldots\}$.

3.7.2. There exists no compact subset K of X meeting an infinite number of sets K_n.

Construction. Let $\mathfrak{M}_1$ be the family of all sets $M_1 \subset R$ of rational numbers such that the sets $M_1 \cap K_n$ consist of one point analogously, denote by $\mathfrak{M}_2$ the family of all sets $M_2 \subset R$ such that for each n the one-point set $M_2 \cap K_n$ contains an irrational number. Let $\beta(N)$ be the Stone-Čech compactification of the discrete space N of all positive integers. Choose disjoint dense subspaces P_1 and P_2 of $\beta(N) - N$ such that $P_1 \cup P_2 = \beta(N) - N$. If now $M \in \mathfrak{M}_1 \cup \mathfrak{M}_2$, then the mapping $n \to y_n \in M \cap K_n$ is homeomorphic. Clearly this mapping has a (homeomorphic) extension φ_M to $\beta(N)$. Put

$$X = R \cup U\{\varphi_M[P_1]; M \in \mathfrak{M}_1\} \cup U\{\varphi_M[P_2]; M \in \mathfrak{M}_2\} .$$

It is easy to see that the space X possesses the properties 3.7.1 and 3.7.2.

4.1. Definition. Denote by $\mathfrak{P}_F$ the class of all spaces X such that every closed subspace of X belongs to $\mathfrak{P}$.

4.2. Theorem. *A completely regular space X belongs to $\mathfrak{P}_F$ if and only if it satisfies the following condition:*

4.2.1. *If M is an infinite subset of X, then for some compact subspace K of X the intersection of M and K is an infinite set.*

Note. It is easy to see that 4.2.1 is equivalent to the condition

4.2.2. *Every infinite subset of X contains an infinite subset with a compact closure in X.*

Proof of theorem 4.2. Sufficiency is a quite simple consequence of 3.5. First notice that the condition 4.2.1 is F-hereditary, that is, if a space X satisfies the condition 4.2.1 then every closed subspace of X also satisfies this condition. In consequence it is sufficient to show that any space X satisfying the condition 4.2.1 belongs to $\mathfrak{P}$. But condition 4.2.1 implies condition 3.5.1; hence by 3.5, X belongs to $\mathfrak{P}$.

For the proof of necessity, suppose that a completly regular space X does not satisfy the condition 4.2.1. Since X is a Hausdorff space, there exists an infinite discrete subset N of X such that no infinite subset of N has a compact closure in X. Denote by F the closure of the set N in X. Put

$$Y = N \cup (\beta(F) - F)\,.$$

By our hypothesis every subset of N has an accumulation point in Y, and consequently, the space Y is pseudocompact. We shall prove that the space $R = X \times Y$ is not pseudocompact so that F does not belong to $\mathfrak{P}$. Consider the subset

$$M = \{(n, n);\, n \in N\}$$

of R. Clearly every points $n \in N$ (considered as a subset) is open in both X and Y. Hence the set M is open in R. On the other hand, the set M is closed in R (see [2], lemma 1.5). The proof is complete.

4.3. Theorem. *The topological product of any countable subfamily od $\mathfrak{P}_F$ belongs to $\mathfrak{P}_F$.*

Proof. Evidently a T_1-space X satisfies 4.2.2. if and only if it satisfies the following condition:

4.2.3. If $\{x_n\}$ is a sequence in X, then for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$ the set of all the x_{n_k}'s has a compact closure in X.

Let x_1, $i = 1, 2, \ldots$, belong to $\mathfrak{P}_F$. Put $X = \mathop{\times}\limits_{i=1}^{\infty} X_i$. We have to prove that the product space X belongs to $\mathfrak{P}_F$. It is sufficient to show that X satisfies the condition 4.2.3. Let $\{x_n; n \in N\}$ be a sequence in X, $x_n = \{x_n^i; i \in N\}$. X_1 satisfies 4.2.3. and thus there exists an infinite subset N_1 of N such that the set of all the x_n^i, $n \in N_1$, has a compact closure in X_1. Proceeding by induction, we obtain a sequence $\{N_n; n \in N\}$ of infinite subsets of N such that $N_n \supset N_{n+1}$ and that the set of all x_i^n, $i \in N_n$, has a compact closure in X_n. Now choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $n_k \in N_k$. Denote by Y_i, $i = 1, 2, \ldots$, the closure in X_i of the set of all the $x_{n_k}^i$, $k = 1, 2, \ldots$. The spaces Y_i are compact by construction. In consequence the product space $Y = \mathsf{X}_{i=1}^{\infty} Y_i$ is compact. The closure of the set of the x_{n_k}, $k = 1, 2, \ldots$, as contained in Y, is a compact space. The proof is complete.

The following example shows that the topological product of an uncountable subfamily of $\mathfrak{P}_F$ need not belong to $\mathfrak{P}_F$.

4.4. Example. Let N be the contably infinite discrete space. For each x in $\beta(N) - N$ denote by K_x the subspace $\beta(N) - (x)$ of $\beta(N)$. Let M be a subset of $\beta(N) - N$ such that every infinite subset of N has an accumulation point in M (it is well known that there exists such a subset with potency $2^{\aleph_0}$). The space $K = \mathsf{X}\{K_x; x \in M\}$ does not belong to $\mathfrak{P}_F$.

Proof. We shall prove that K does not satisfy the condition 4.2.2.

For each y in $\beta(N) - M$ denote by $g(y)$ the point of K having all coordinates equal to y. Evidently g is a homeomorphic mapping from $\beta(N) - M$ to K. Consider the subspace $g[N]$ of K. Since $\overline{g[N]}^K = g[\beta(N) - M]$ (see [2], Lemma 1.4), the subset $g[N]$ of K contains no subset with a compact closure in K. The proof is complete.

References

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[2] *Z. Frolík*: The topological product of two countably compact spaces. Czech. Math. J., *10 (85)*, 1960, 329–338.

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Резюме

ТОПОЛОГИЧЕСКОЕ ПРОИЗВЕДЕНИЕ ДВУХ ПСЕВДОКОМПАКТНЫХ ПРОСТРАНСТВ

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Основным результатом второй части является следующая теорема:

Теорема. *Следующие свойства бесконечных вполне регулярных пространств X и Y эквивалентны:*

(1) *Топологическое произведение* $X \times Y$ *является псевдокомпактным пространством.*

(2) $\beta(X \times Y) = \beta(X) \times \beta(X)$, *т. е. всякую ограниченную непрерывную функцию на* $X \times Y$ *можно непрерывно продолжить на* $\beta(X) \times \beta(X)$.

(3) *Если* f*-ограниченная непрерывная функция на* $X \times Y$, *то для всякого* $\varepsilon > 0$ *существует конечное открытое покрытие* $\{A_1 \times B_1, \ldots, A_n \times \times B_n\}$ *пространства* $X \times Y$ *так, что для* $i = 1, 2, \ldots$

$$x \in A_i \times B_i\,, \quad y \in A_i \times B_i \Rightarrow |f(x) - f(y)| < \varepsilon\,.$$

(4) *Пространства* X *и* Y *псевдокомпактны и для всякой ограниченной и непрерывной функции* f *на* $X \times Y$ *отображения*

$$\varphi:\ y \to f(\cdot\,,\, y) \in C(X) \qquad и \qquad \psi : x \to f(x,\ \cdot) \in C(X)$$

пространства Y *в* $C(X)$ *и* X *в* $C(Y)$ *непрерывны.*

(5) *Для всякой ограниченной непрерывной функции* f *на* $X \times Y$ *функции*

$$\varphi(x_1, x_2) = \sup_{y \in Y} |f(x_1, y) - f(x_2, y)|$$

и

$$\psi(y_1, y_2) = \sup_{y \in X} |f(x, y_1) - f(x, y_2)|$$

являются вполне ограниченным и непрерывными псевдометриками.

В третьей части рассматривается класс $\mathfrak{P}$ всех вполне регулярных пространств P, для которых топологическое произведение $P \times Q$ псевдокомпактно для всякого псевдокомпактного пространства Q. Компактные пространства принадлежат классу $\mathfrak{P}$. Далее,

$$P_1 \times P_2 \in \mathfrak{P} \Leftrightarrow P_1 \in \mathfrak{P}\,,\ P_2 \in \mathfrak{P}_2\,.$$

Дается достаточное и необходимое условие для того, чтобы пволне регулярное пространство принадлежало классу $\mathfrak{P}$.

В четвертой части рассматривается класс $\mathfrak{P}_F$. Пространство P принадлежит классу $\mathfrak{P}_F$ тогда и только тогда, если всякое замкнутое подпространства P принадлежит классу $\mathfrak{P}$.

Теорема. *Вполне регулярное пространство* P *принадлежит* $\mathfrak{P}_F$ *тогда и только тогда, если для всякого бесконечного* $M \subset P$ *существует компактное* $K \subset P$ *так, что* $K \cap M$ *бесконечно. Топологическое произведение счетного числа пространств из класса* $\mathfrak{P}_F$ *принадлежит* $\mathfrak{P}_F$.