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NORMS, SPECTRA AND COMBINATORIAL PROPERTIES OF MATRICES

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Dedicated to Professor Vladimír Kořínek on the occasion of his sixtieth birthday.

The main result of the present paper may be formulated as follows: Put $|A| = \max_i \sum_k |a_{ik}|$ for an arbitrary matrix $A = (a_{ik})$. Let n be a natural number. Then $n^2 - n + 1$ is the minimum of all numbers q with the following property: If A is a matrix of order n such that $|A| = \ldots = |A^q| = 1$ then $|A^r| = 1$ for all r.

Let $B = (b_{ik})$ be a complex matrix of order n such that $\det B \neq 0$ and $|b_{ii}| \geqq \sum_{k \neq i} |b_{ik}|$ for $i = 1, 2, \ldots, n$. Consider the system of equations $\sum_k b_{ik} x_k = y_i$ or, more briefly,

$$Bx = y \,. \tag{1}$$

Clearly we may suppose that $b_{ii} = 1$ for all i. If we put $|A| = \max_i \sum_k |a_{ik}|$ for an arbitrary matrix A with elements a_{ik}, we can instead of (1) write

$$x = Ax + y \,, \tag{2}$$

where $A = E - B$ and $|A| \leqq 1$. In order that the iterative method

$$x^{(r+1)} = Ax^{(r)} + y \tag{3}$$

be convergent for every initial vector $x^{(0)}$ and every y it is necessary and sufficient that the series $E + A + A^2 + \ldots$ be convergent. Now it is well-known that the following three properties of A are equivalent:

1° the series $E + A + A^2 + \ldots$ is convergent,
2° the powers A^r converge to the zero matrix,
3° the inequality $|\alpha| < 1$ holds for every proper value α of A.

Clearly these conditions are fulfilled if $|A| < 1$ or, more generally, if $|A^p| < 1$ for some p. We are thus led to the following problem. Consider a matrix A

with $|A| \leqq 1$ and construct the sequence $A, A^2, A^3, \ldots$ Clearly $|A| \geqq |A^2| \geqq \geqq |A^3| \geqq \ldots$ so that either $|A^r| = 1$ for every natural r or $|A^p| < 1$ for some p and, consequently, A^r converges to the zero matrix.

It is thus natural to ask how far it is necessary to go in the sequence $|A|$, $|A^2|, \ldots$ to decide which of the two preceding cases takes place. One of the main results of the present paper consists in proving the existence of a number $q(n)$ which depends on n only and which possesses the following properties: (1) if A is an arbitrary complex matrix of order n and $|A| = |A^{q(n)}| = 1$ then $|A^r| = 1$ for every natural r (2) there exists a matrix A of order n such that $|A| = |A^{q(n)-1}| = 1$ and $|A^{q(n)}| < 1$ (in other words, $q(n)$ is the minimum of all numbers possessing the first property). We prove that $q(n) = n^2 - n + 1$.

It turns out that the substance of this and other similar results consists in the investigation of the combinatorial structure of matrices.

The proof of the result mentioned above is divided into forty-eight propositions most of which are of independent interest. The purely combinatorial ones are collected in the first section; in a suitable interpretation, this section contains a large part of the theory of non-negative matrices. In the second section, we study the combinatorial properties of matrices and vectors and in the third section we obtain a series of theorems relating the distribution of zeros in the given matrix A and the norms of iterations of A. These theorems culminate in the result mentioned above. For matrices whose diagonal elements are different from zero, analogous theorems may be proved with n instead of $q(n)$. They form the subject matter of section four.

1. COMBINATORIAL THEOREMS

Throughout the present paper let n be a fixed natural number and let N be the set of numbers $1, 2, \ldots, n$. We shall denote by $\mathbf{F}$ the set of all mappings φ with the following two properties:

1° the mapping φ assigns to every set $S \subset N$ some set $\varphi(S) \subset N$,

2° the mapping φ is additive; we have $\varphi(0) = 0$ and $\varphi(S_1 \cup S_2) = \varphi(S_1) \cup \cup \varphi(S_2)$ for any two sets $S_1 \subset N$, $S_2 \subset N$.

If $\varphi_1 \in \mathbf{F}$ and $\varphi_2 \in \mathbf{F}$, we define the superposition $\varphi = \varphi_1\varphi_2$ in the following manner: $\varphi(S) = \varphi_2(\varphi_1(S))$ for every $S \subset N$. Evidently $\varphi \in \mathbf{F}$ as well.

In the whole paper, the symbol φ will denote an element of $\mathbf{F}$. If $r \in N$ and R is the set consisting of the point r, we shall frequently write $\varphi(r)$ for $\varphi(R)$.

A mapping φ is said to be *reducible* if either $\varphi(N) = 0$ or if there exists a set S different from 0 and N such that $\varphi(S) \subset S$. (The existence of such a set S is evident if $\varphi(N) = 0$ and $n > 1$.) A mapping φ is said to be *irreducible* if it is not reducible.

(1,1) *Let φ be irreducible. Then $\varphi(N) = N$ and $\varphi(T) \neq 0$ whenever $T \neq 0$.*

Proof. Since $\varphi(\varphi(N)) \subset \varphi(N)$ and $\varphi(N) \neq 0$, we have $\varphi(N) = N$. If $T \subset N$, $T \neq N$ and $\varphi(T) = 0$ then $\varphi(T) \subset T$, whence $T = 0$.

(1,2) *Let $S \subset N$ and let s be the number of elements of S. If r is a natural number such that the set $R = S \cup \varphi(S) \cup \ldots \cup \varphi^r(S)$ contains at most $s + r - 1$ elements then $\varphi(R) \subset R$.*

Proof. There exists a k such that $1 \leqq k \leqq r$ and $\varphi^k(S) \subset S \cup \ldots \cup \varphi^{k-1}(S)$; otherwise the set R would contain at least $s + r$ elements. Clearly we have $\varphi^j(S) \subset T$ for every $j \geqq k$, where $T = S \cup \ldots \cup \varphi^{k-1}(S)$. It follows that $R = T$ and $\varphi(R) \subset R$.

(1,3) *Let $S \subset N$ and let S have s elements. If $R = S \cup \varphi(S) \cup \ldots \cup \varphi^{n-s}(S)$ then $\varphi(R) \subset R$. If φ is irreducible and $S \neq 0$, then $\varphi(S) \cup \ldots \cup \varphi^n(S) = N$.*

Proof. This is an easy consequence of (1,2).

(1,4) Definition. Every disjoint system $\mathbf{R}$ of non-void sets whose union is N is called a *partition* of N; the number of elements of $\mathbf{R}$ is called the *length* of $\mathbf{R}$. Now let $\varphi \in \mathbf{F}$ and let $\mathbf{R}$ be a partition of length k. We say that $\mathbf{R}$ is a *cyclic partition* (of N with respect to φ) if there is a $R \subset N$ such that $\mathbf{R} = \{R, \varphi(R), \ldots$ $\ldots, \varphi^{k-1}(R)\}$ and $R = \varphi^k(R)$. The maximal possible length of a cyclic partition is called the *index of imprimitivity* of φ and is denoted by $d(\varphi)$. If $d(\varphi) = 1$ and if φ is irreducible, we say that φ is *primitive*.

(1,5) *Let φ be irreducible and let $\mathbf{T}$ be a non-void disjoint system of non-void subsets of N. Suppose that for each $T \in \mathbf{T}$ there exists a $V \in \mathbf{T}$ such that $\varphi(T) \subset V$. Then $\mathbf{T}$ is a cyclic partition.*

Proof. For each $T \in \mathbf{T}$ there is exactly one $V \in \mathbf{T}$ such that $\varphi(T) \subset V$. Hence it is possible to define a mapping χ of $\mathbf{T}$ into $\mathbf{T}$ by the relation $\chi(T) = V$, where $V \supset \varphi(T)$. Let $0 \neq \mathbf{T}' = \{T_1, \ldots, T_r\} \subset \mathbf{T}$ and let $\chi(\mathbf{T}') \subset \mathbf{T}'$. Then $\varphi(\bigcup \mathbf{T}') = \bigcup \varphi(T_j) \subset \bigcup \chi(T_j) = \bigcup \chi(\mathbf{T}') \subset \bigcup \mathbf{T}'$ so that $\bigcup \mathbf{T}' = N$, $\mathbf{T}' = \mathbf{T}$. It follows that $\mathbf{T}$ is a partition of N. If we put $\mathbf{T}' = \chi(\mathbf{T})$, we see that χ is a permutation of $\mathbf{T}$; further we see that for an arbitrary $T \in \mathbf{T}$ we have $\mathbf{T} = \{T,$ $\chi(T), \ldots, \chi^{k-1}(T)\}$, $T = \chi^k(T)$ (where k is the length of $\mathbf{T}$). Evidently $\varphi^j(T) \subset$ $\subset \chi^j(T)$ for all j. Put $V = T \cup \varphi(T) \cup \ldots \cup \varphi^{k-1}(T)$. Since $\varphi^k(T) \subset \chi^k(T) = T$, we have $\varphi(V) \subset V$ whence $V = N$ and $\varphi(T) \cup \ldots \cup \varphi^k(T) = \chi(T) \cup \ldots$ $\ldots \cup \chi^k(T) = N$. It follows that $\varphi^j(T) = \chi^j(T)$ for $j = 1, \ldots, k$, so that $\mathbf{T} =$ $= \{T, \varphi(T), \ldots, \varphi^{k-1}(T)\}$, $\varphi^k(T) = \chi^k(T) = T$.

(1,6) Notation. If $\varphi \in \mathbf{F}$, let $\mathbf{V}(\varphi)$ be the system of all $V \subset N$ such that $\varphi^j(V) \subset V$ for some natural j. Further let $\mathbf{M}(\varphi)$ be the system of all minimal non-void elements of $\mathbf{V}(\varphi)$.

(1,7) *If V, $W \in \mathbf{V}(\varphi)$ then $V \cap W \in \mathbf{V}(\varphi)$.*

Proof. If $\varphi^j(V) \subset V$, $\varphi^k(W) \subset W$, then $\varphi^{jk}(V \cap W) \subset V \cap W$.

(1,8) *Suppose that* $\varphi(T) \neq 0$ *for every* $T \neq 0$. *Then* $\varphi(M) \in \mathbf{M}(\varphi)$ *for every* $M \in \mathbf{M}(\varphi)$.

Proof. If $\varphi^j(M) \subset M$, then evidently $\varphi^j(\varphi(M)) = \varphi(\varphi^j(M)) \subset \varphi(M)$ so that $\varphi(M) \in \mathbf{V}(\varphi)$. Let now $0 \neq L \subset \varphi(M)$, $\varphi^k(L) \subset L$ and put $p = jk$, $M_1 = \varphi^{p-1}(L)$. We have $\varphi^p(M) \subset M$, $\varphi^p(L) \subset L$, $\varphi^p(M_1) = \varphi^{p-1}(\varphi^p(L)) \subset \varphi^{p-1}(L) = M_1$ whence $M_1 \in \mathbf{V}(\varphi)$. Further $M_1 \subset \varphi^{p-1}(\varphi(M)) \subset M$ and $M_1 \neq 0$, whence $M_1 = M$, so that $\varphi(M) = \varphi(M_1) = \varphi^p(L) \subset L$. It follows that $L = \varphi(M)$ which completes the proof.

(1,9) Theorem. *Let* φ *be irreducible and let* m *be the number of elements of* $\mathbf{M}(\varphi)$. *Then*

1° $\mathbf{M}(\varphi)$ *is the finest cyclic partition,*

2° $m = d(\varphi)$,

3° $\varphi^m(M) = M$ *for every* $M \in \mathbf{M}(\varphi)$ *and* φ^m *is primitive on the set* M.

Proof. According to (1,7), $\mathbf{M}(\varphi)$ is a disjoint system; lemmas (1,8) and (1,5) imply that $\mathbf{M}(\varphi)$ is a cyclic partition. Now let $\mathbf{R}$ be an arbitrary cyclic partition. Since we have evidently $\mathbf{R} \subset \mathbf{V}(\varphi)$, it follows from (1,7) that $\mathbf{M}(\varphi)$ is finer than $\mathbf{R}$. At the same time we see that $m = d(\varphi)$. For each $M \in \mathbf{M}(\varphi)$ we have obviously $\varphi^m(M) = M$ and φ^m is irreducible on M. Now let $\mathbf{T}$ be a cyclic partition of the set $M \in \mathbf{M}(\varphi)$ with respect to φ^m; let t be the length of $\mathbf{T}$ and let $M_1 \in \mathbf{T}$. Then we have $\varphi^{tm}(M_1) = M_1 \subset M$, whence $M_1 = M$, $t = 1$; the proof is complete.

(1,10) Theorem. *Let* $\varphi \in \mathbf{F}$. *Then the following three conditions are equivalent:*

1° φ^k *is irreducible for* $k = 1, \ldots, n$,

2° φ *is primitive,*

3° φ^k *is irreducible for all natural* k.

Proof. Let φ be irreducible. If $\varphi^k(B) \subset B$, $0 \neq B \neq N$, there is a $M \in \mathbf{M}(\varphi)$ such that $M \subset B$; since, according to theorem (1,9), $\mathbf{M}(\varphi)$ is a cyclic partition, we have $d(\varphi) > 1$. We see that 2° implies 3°. If $d(\varphi) > 1$ then $\varphi^{d(\varphi)}$ is reducible so that 1° implies 2°. Condition 1° being a trivial consequence of 3°, the proof is complete.

(1,11) *Let* φ *be primitive and let.* $n > 1$. *Then there exists a natural* $k < n$ *and an* $i \in N$ *such that* $i \in \varphi^k(i)$.

Proof. If $\varphi(x)$ contains exactly one point for every $x \in N$, then the sets $\{1\}, \ldots, \{n\}$ form a cyclic partition of N according to (1,5), so that φ is not primitive. It follows that there exists an $i \in N$ such that $\varphi(i)$ contains at least two elements. Further, lemma (1,3) implies that $i \in \varphi^m(i)$ for some natural $m \leqq n$. If $m < n$, the proof is finished; if $m = n$, put $i_0 = i_n = i$ and choose $i_{n-1} \in \varphi^{n-1}(i), \ldots, i_1 \in \varphi(i)$ in such a way that $i_{r+1} \in \varphi(i_r)$ for $r = 0, 1, \ldots, n-1$. Since $i_k \in \varphi^{k-j}(i_j)$ we can suppose that $i_j \neq i_k$ for $1 \leqq j < k \leqq n$. Then $\{i_1, \ldots, i_n\} = N$ so that there exists a j such that $1 < j \leqq n$ and $i_j \in \varphi(i)$. Then $i = i_n \in \varphi^{n-j}(i_j) \subset \varphi^{n-j+1}(i)$ and $n - j + 1 < n$.

(1,12) *Let φ be primitive. Let $i \in N$, $1 \leqq k \leqq n$ and $i \in \varphi^k(i)$. Then $\varphi^{(n-1)k}(i) = N$.*

Proof. Let S_0 be the set consisting of the point i. Put $S_j = \varphi^{jk}(S_0)$. Since $S_j \subset \varphi^k(S_j)$ and φ^k is irreducible by (1,10), we cannot have equality here unless $S_j = N$. Hence $S_{n-1} = N$, or, in other words, $\varphi^{(n-1)k}(i) = N$.

(1,13) *Let φ be primitive and let $n > 1$. If $i \in \varphi^k(i)$ for some $i \in N$ and some natural k, then $\varphi^{(n-2)k+n}(j) = N$ for every $j \in N$.*

Proof. Let d be the least natural number such that $i \in \varphi^d(i)$. Put $i_0 = i_d = i$ and choose $i_{d-1}, \ldots, i_1$ so that $i_{r+1} \in \varphi(i_r)$ for $r = 0, 1, \ldots, d - 1$. Suppose that $i_p = i_q$ for some p, q such that $1 \leqq p < q \leqq d$. Then $i = i_d \in \varphi^{d-q}(i_q) = \varphi^{d-q}(i_p) \subset \varphi^{d-(q-p)}(i)$ which is impossible. The set C consisting of $i_1, \ldots, i_d$ has, consequently, exactly d elements. Clearly $m \in \varphi^d(m)$ for every $m \in C$. Now let j be an arbitrary element of N. The set $R = j \cup \varphi(j) \cup \ldots \cup \varphi^{n-d}(j)$ contains at least $n - d + 1$ elements by (1,2) so that $R \cap C \neq 0$. It follows that there exists an $m \in C$ and a non-negative integer $s \leqq n - d$ such that $m \in \varphi^s(j)$. Since $m \in \varphi^d(m)$, we have $\varphi^{(n-1)d}(m) = N$ according to (1,12). Hence $\varphi^{s+(n-1)d}(j) \supset \varphi^{(n-1)d}(m) = N$. Now $(n - 1)\, d + s \leqq (n - 1)\, d + n - d = (n - 2)\, d + n \leqq (n - 2\ )\, k + n$ which completes the proof.

(1,14) Theorem. *If φ is primitive, then $\varphi^{(n-1)^2+1}(j) = N$ for every $j \in N$.*

Proof. We can suppose that $n > 1$. According to (1,11), there exists an $i \in N$ and a natural $k < n$ such that $i \in \varphi^k(i)$. It is sufficient now to apply lemma (1,13) and to note that $(n - 2)\, k + n \leqq (n - 2)\,(n - 1) + n = (n - 1)^2 + 1$.

(1,15) Theorem. *Let $n > 2$ and let φ be primitive. Let $j \in N$ and $\varphi^{(n-1)^2}(j) \neq N$. Then it is possible to arrange the elements of N into a sequence $j_1, \ldots, j_n$ in such a way that $j = j_1$, $\varphi(j_r) = \{j_{r+1}\}$ for $r = 1, 2, \ldots, n - 1$ and $\varphi(j_n) = \{j_1, j_2\}$. Further, $\varphi^{(n-1)^2}(j) = N - \{j\}$ and $\varphi^{(n-1)^2}(x) = N$ for every $x \in N$ different from j.*

Proof. Suppose first that $i \in \varphi^k(i)$ for some $i \in N$ and some $k \leqq n - 2$. Then $\varphi^{(n-2)^2+n}(j) = N$ according to lemma (1,13). This, however, is impossible since $(n - 2)^2 + n \leqq (n - 1)^2$ for $n \geqq 3$. It follows that an inclusion $i \in \varphi^k(i)$ cannot hold unless $k \geqq n - 1$. The inclusion $j \in \varphi^{n-1}(j)$ is impossible by lemma (1,12). Hence j is not contained in $\varphi(j) \cup \ldots \cup \varphi^{n-1}(j)$. Put $F_r = \varphi(j) \cup \ldots \cup \varphi^r(j)$ for $r = 1, 2, \ldots, n$. It is easy to see that F_r contains exactly r elements (otherwise F_{n-1} would be equal to N which is a contradiction since $j \notin F_{n-1}$). Hence it is possible to arrange the elements of N into a sequence $i_1, \ldots, i_n$ so that F_r consists exactly of $i_1, \ldots, i_r$ for $r = 1, 2, \ldots, n$. Since $j \notin F_{n-1}$, we have $i_n = j$. Further, $\varphi(j) = F_1 = \{i_1\}$. Let $1 \leqq r \leqq n - 1$; then $i_{r+1} \in \varphi^{r+1}(j) \cup \ldots \cup \varphi^2(j) = \varphi(F_r)$ so that $i_{r+1} \in \varphi(i_k)$ for some k, $1 \leqq k \leqq r$. Since $i_{r+1} \notin F_r$ and $\varphi(i_k) \subset F_{k+1} \subset F_r$ for $k < r$, we have $k = r$, whence $i_{r+1} \in \varphi(i_r)$ for $r = 1, 2, \ldots, n - 1$. Suppose that $\varphi(i_r) = \{i_{r+1}\}$ for

$r = 1, 2, \ldots, n - 1$. It follows then that $\varphi^n(x) = \{x\}$ for every $x \in N$ which is impossible. Since $\varphi(i_r) \subset \varphi(F_r) \subset \{i_1, \ldots, i_{r+1}\}$, there exist natural numbers $p \leqq q \leqq n - 1$ such that $i_p \in \varphi(i_q)$. Hence $i_q \in \varphi^{q-p}(i_p) \subset \varphi^{q-p+1}(i_q)$. It follows that $q - p + 1 \geqq n - 1$ whence $q = n - 1$, $p = 1$. We have thus $\varphi(i_r) = \{i_{r+1}\}$ for $r = 1, 2, \ldots, n - 2$, $\varphi(i_n) = \{i_1\}$, $\varphi(i_{n-1}) = \{i_n, i_1\}$. Now it is sufficient to put $j_1 = i_n = j$ and $j_r = i_{r-1}$ for $r = 2, \ldots, n$.

Put $S = \varphi^{(n-1)^2}(j)$ so that $S \neq N$. At the same time, $\varphi(S) = N$ by (1,14). Since $j_{r+1} \in \varphi(S)$ for $r = 2, \ldots, n - 1$, the inclusion $j_r \in S$ holds for $r = 2, \ldots$ $\ldots, n - 1$. Since $j \in \varphi(S)$, we have $j_n \in S$ as well. It follows that $S = N - \{j\}$. Finally, for every $x \in N$ different from j we have evidently $x \in \varphi^{n-1}(x)$, whence $\varphi^{(n-1)^2}(x) = N$ by (1,12).

(1,16) Definition. Let $\mathbf{F}_0$ be the set of all $\varphi \in \mathbf{F}$ such that $\varphi(N) = N$ and that $\varphi(P) \neq 0$ whenever $0 \neq P \subset N$. For each $\varphi \in \mathbf{F}_0$ let us define a relation $\mathbf{E}(\varphi)$ on N in the following manner: we put $j\mathbf{E}(\varphi)k$ if and only if there exist $r_0, \ldots, r_p$ and $q_1, \ldots, q_p$ such that $r_0 = j$, $r_p = k$ and $r_{i-1}, r_i \in \varphi(q_i)$ for $i = 1, 2, \ldots$ $\ldots, p$. The relation $\mathbf{E}(\varphi)$ is clearly symmetrical and transitive; since $\varphi(N) = N$, it is also reflexive so that $\mathbf{E}(\varphi)$ is an equivalence on N. If $\varphi \in \mathbf{F}_0$ and if m is a non-negative integer, we have $\varphi^m \in \mathbf{F}_0$ as well. Evidently $\varphi \in \mathbf{F}_0$ for each irreducible $\varphi \in \mathbf{F}$.

(1,17) *Let $\varphi \in \mathbf{F}_0$; put $\mathbf{E}_r = \mathbf{E}(\varphi^r)$. Let $j\mathbf{E}_m k$, $x \in \varphi(j)$, $y \in \varphi(k)$. Then $j\mathbf{E}_{m+1}k$ and $x\mathbf{E}_{m+1}y$.*

Proof. Let $j = r_0$, $k = r_p$ and let $r_{i-1}, r_i \in \varphi^m(q_i)$ for $i = 1, 2, \ldots, p$. There exist $s_i \in N$ such that $q_i \in \varphi(s_i)$. Then $r_{i-1}, r_i \in \varphi^{m+1}(s_i)$ so that $j\mathbf{E}_{m+1}k$. Choose now $t_i \in \varphi(r_i)$ for $i = 2, \ldots, p - 1$ and put $t_0 = x$ and $t_p = y$. It follows that $t_{i-1}, t_i \in \varphi^{m+1}(q_i)$ for $i = 1, 2, \ldots, p$ whence $x\mathbf{E}_{m+1}y$.

(1,18) *Let $\varphi \in \mathbf{F}_0$. If $\mathbf{E}_m = \mathbf{E}_{m+1}$ then $\mathbf{E}_{m+1} = \mathbf{E}_{m+2}$.*

Proof. Let $j, k \in \varphi^{m+2}(q)$. We have $j \in \varphi(q_1)$, $k \in \varphi(q_2)$, where $q_1, q_2 \in \varphi^{m+1}(q)$. Since $\mathbf{E}_{m+1} = \mathbf{E}_m$, we have $q_1\mathbf{E}_m q_2$ whence $j\mathbf{E}_{m+1}k$ by the preceding result. Hence $x\mathbf{E}_{m+2}y$ implies $x\mathbf{E}_{m+1}y$. The other implication being a consequence of (1,17), we have $\mathbf{E}_{m+2} = \mathbf{E}_{m+1}$.

(1,19) *If $\varphi \in \mathbf{F}_0$, then for every m the equivalence $\mathbf{E}_{m+1}$ is coarser than $\mathbf{E}_m$. Further, $\mathbf{E}_{n-1} = \mathbf{E}_n$.*

Proof. This is an immediate consequence of the two preceding results.

(1,20) *Let $\varphi \in \mathbf{F}_0$. Let us denote by $\mathbf{C}$ the equivalence $\mathbf{E}_{n-1}$. Let $\{T_1, \ldots, T_p\}$ be the partition of N corresponding to $\mathbf{C}$. Then every $\varphi(T_j)$ is contained in some T_k.*

Proof. Since $\mathbf{C} = \mathbf{E}_{n-1} = \mathbf{E}_n$ by (1,19), the conclusion follows from lemma (1,17).

(1,21) *Let $\varphi \in \mathbf{F}_0$. Let $\mathbf{T} = \{T_1, \ldots, T_p\}$ be a partition of the set N such that every $\varphi(T_j)$ is contained in some T_k. Let m be a non-negative integer. Then $\mathbf{E}_m$ is finer than the equivalence on N generated by the partition $\mathbf{T}$.*

Proof. It is easy to see that every $\varphi^m(T_j)$ is contained in some T_k. It is thus sufficient to limit ourselves to the case $m = 1$. If $j, k \in \varphi(q)$ and $j \in T_s$, then the whole of $\varphi(q)$ is contained in T_s so that $k \in T_s$ as well. The rest is easy.

(1,22) Theorem. *Let φ be irreducible. Then the classes of the equivalence* $\mathbf{C}$ *constitute the finest cyclic partition with respect to φ.*

Proof. According to (1,20) and (1,5) the classes of $\mathbf{C}$ form a cyclic partition. By (1,21), the partition corresponding to $\mathbf{C}$ is finer than any cyclic partition of N.

(1,23) Notation. If $\varphi \in \mathbf{F}$ and $S \subset N$, let us define $\varphi^*(S)$ as the set of those $x \in N$ for which $\varphi(x) \cap S \neq 0$. It is easy to verify that $\varphi^* \in \mathbf{F}$ as well. Clearly φ^* may also be defined by the requirement that $\varphi^* \in \mathbf{F}$ and that $i \in \varphi^*(j)$ if and only if $j \in \varphi(i)$.

(1,24) *We have $\varphi^{**} = \varphi$ for every $\varphi \in$ **F** and $(\alpha\beta)^* = \beta^*\alpha^*$ for any two $\alpha, \beta \in$ **F**.*

Proof. Immediate.

(1,25) *Let $S \subset N$. Then $\varphi(N - \varphi^*(S)) \subset N - S$.*

Proof. Suppose that $S \cap \varphi(N - \varphi^*(S)) \neq 0$. Then there exists an $x \in N - \varphi^*(S)$ such that $S \cap \varphi(x) \neq 0$ whence $x \in \varphi^*(S)$ which is impossible.

(1,26) *We have $\varphi^*(S) \subset T$ if and only if $\varphi(N - T) \subset N - S$.*

Proof. If $\varphi^*(S) \subset T$, we have according to (1,25) $\varphi(N - T) \subset \varphi(N - \varphi^*(S)) \subset \subset N - S$. The other implication is a consequence of the relation $\varphi^{**} = \varphi$.

(1,27) *A mapping φ is reducible if and only if φ^* is reducible.*

Proof. If $\varphi(N) = 0$, then $\varphi^*(N) = 0$, too. If $S \subset N$ is different both from 0 and N, so is $T = N - S$. If $\varphi(S) \subset S$, we have $\varphi^*(T) \subset T$ according to (1,26).

(1,28) *A mapping φ is primitive if and only if φ^* is primitive.*

Proof. Let φ^* be primitive and let $\mathbf{T} = \{T, \varphi(T), \ldots, \varphi^{k-1}(T)\}$ be a cyclic partition with respect to φ. Since $\varphi^k(V) \subset V$ for each $V \in \mathbf{T}$, we have $(\varphi^k)^*(T) \subset \subset T$. Theorem (1,10) implies that $(\varphi^*)^k$ is irreducible; by (1,24) we obtain $(\varphi^*)^k = (\varphi^k)^*$ whence $T = N$, $k = 1$. It follows from (1,27) that φ is irreducible so that φ is primitive.

2. MATRICES AND THEIR COMBINATORIAL PROPERTIES

In the rest of the paper the symbols A, B will always denote matrices of order n with complex elements a_{ik}, b_{ik}. The letters x, y will denote vectors with n complex coordinates x_i, y_i.

For every matrix A we define a mapping $\varphi \in \mathbf{F}$ in the following manner:

if $S \subset N$, let $\varphi(S)$ be the set of those $j \in N$ for which there exists an $i \in S$ such that $a_{ij} \neq 0$. The fact that $\varphi \in \mathbf{F}$ is easily verified. Further it is easy to see that $\varphi^*(S)$ is the set of those $i \in N$ for which there exists a $j \in S$ such that $a_{ij} \neq 0$. We shall write $\varphi = F(A)$.

We shall adopt the following convention: the matrix A will be called reducible, irreducible or primitive if the corresponding mapping $F(A)$ is reducible, irreducible or primitive.

(2,1) *Suppose that the matrices A and B are non-negative. Then* $F(AB) = F(A)\,F(B)$. *Especially we have* $F(A^r) = (F(A))^r$ *for every natural r.*

Proof. Put $\alpha = F(A)$, $\beta = F(B)$, $\varphi = F(AB)$. Let $S \subset N$ and $k \in \varphi(S)$. There exists an $i \in S$ such that $\sum_j a_{ij}b_{jk} \neq 0$ so that $a_{il}b_{lk} \neq 0$ for some l. Clearly $l \in \alpha(S)$, $k \in \beta(\alpha(S))$ whence $\varphi(S) \subset \beta(\alpha(S))$. If, on the other hand, $k \in \beta(\alpha(S))$, then there exists an $i \in S$ and an $l \in \alpha(i)$ such that $k \in \beta(l)$; it follows that $\sum_j a_{ij}b_{jk} \geqq a_{il}b_{lk} > 0$ whence $k \in \varphi(i)$ so that $\beta(\alpha(S)) \subset \varphi(S)$.

(2,2) Notation. For every A and every x put $|A| = \max_i \sum_k |a_{ik}|$, $|x| = \max_i |x_i|$. We have thus $|Ax| \leqq |A| \,.\, |x|$, $|AB| \leqq |A| \,.\, |B|$. If $|A| \leqq 1$, let $P(A)$ be the set of those $i \in N$ for which $\sum_k |a_{ik}| = 1$. If $|x| \leqq 1$, we shall denote by $P(x)$ the set of those $i \in N$ for which $|x_i| = 1$; further, we put $Q(A) = N - P(A)$, $Q(x) = N - P(x)$.

(2,3) *Suppose that* $|A| \leqq 1$, $|x| \leqq 1$ *and put* $\varphi = F(A)$. *Then*

1° $P(Ax) \subset P(A)$,

2° $P(x) \supset \varphi(P(Ax))$,

3° $\varphi^*(Q(x)) \subset Q(Ax)$.

Proof. The first inclusion is obvious. Now let $i \in \varphi^*(Q(x))$; there is a $k \in N$ such that $|x_k| < 1$ and $a_{ik} \neq 0$. We have then $|\sum_j a_{ij}x_j| \leqq \sum_j |a_{ij}x_j| < \sum_j |a_{ij}| \leqq 1$ so that $i \in Q(Ax)$ which proves 3°. According to lemma (1,26), the second inclusion is a consequence of 3°.

(2,4) *Let* $|A| \leqq 1$ *and let M be the matrix consisting of elements* $|a_{ik}|$. *Let r be a natural number,* $\varphi = F(A^r)$, $\omega = F(M^r)$, $S \subset N$. *Then* $P(A^r) \subset P(M^r)$, $\varphi(S) \subset \omega(S)$. *If* $\varphi(i) \neq \omega(i)$ *for some i, then* $i \in Q(A^r)$.

Proof. It is easy to see that we have $|a_{ik}^{(r)}| \leqq m_{ik}^{(r)}$, where $a_{ik}^{(r)}$ and $m_{ik}^{(r)}$ are elements of A^r and M^r. Further, $\sum_k m_{ik}^{(r)} \leqq 1$ for every i. The rest is easy.

(2,5) *Let* $|A| \leqq 1$ *and let r be a natural number. If* $P(A^r) = N$ *then* $F(A^r) = (F(A))^r$.

Proof. According to (2,4), we have $F(A^r) = F(M^r)$, where M is the matrix with elements $|a_{ik}|$. Clearly $F(M) = F(A)$. It follows from (2,1) that $F(M^r) = (F(M))^r$ so that $(F(A))^r = F(A^r)$.

(2,6) *If* $|A| = |B| = 1$ *then* $P(AB) \subset P(A)$, $Q(AB) \supset Q(A)$.

Proof. If $i \in P(AB)$, there exists a vector x such that $|x| = 1$ and that the vector $y = ABx$ has $|y_i| = 1$. Putting $z = Bx$, we have $|z| \leqq 1$ and $i \in P(Az) \subset \subset P(A)$. The second inclusion is an immediate consequence of the first relation.

(2,7) *Let* $|A| = |B| = 1$, $\psi = (F(A))^*$. *Then* $\psi^r(Q(B)) \subset Q(A^rB)$ *for every non-negative integer* r.

Proof. Our assertion is trivial for $r = 0$. Suppose first that $r = 1$ and put $AB = W = (w_{ik})$. If $i \in \psi(Q(B))$, then there exists a k such that $a_{ik} \neq 0$, $k \in Q(B)$, whence $|a_{ik}| \sum_q |b_{kq}| < |a_{ik}|$. It follows that $\sum_q |w_{iq}| = \sum_{p,q} |a_{ip}b_{pq}| \leqq \leqq \sum_p |a_{ip}| \sum_q |b_{pq}| < \sum_p |a_{ip}| \leqq 1$ so that $i \in Q(W)$. The rest follows by induction.

(2,8) *Suppose that* $|A| = |B| = 1$, $AB = BA$ *and put* $\varphi = F(A)$. *Let* s *be the number of elements of* $P(B)$. *Then there exists a set* Z *such that* $\varphi(Z) \subset Z$, $P(A^sB) \subset Z \subset P(B)$.

Proof. Put $Q = Q(B)$, $\psi = \varphi^*$, $V = Q \cup \psi(Q) \cup \ldots \cup \psi^s(Q)$, $Z = N - V$. Evidently $Z \subset N - Q = P(B)$. Since Q has $n - s$ elements, it follows from (1,3) that $\psi(V) \subset V$ whence $\varphi(Z) \subset Z$ by (1,26). Lemma (2,6) implies that $Q(A^{k+1}B) = Q(A^kBA) \supset Q(A^kB)$ for every $k \geqq 0$; if $0 \leqq k \leqq s$, it follows from (2,7) that $\psi^k(Q) \subset Q(A^kB) \subset Q(A^sB)$, whence $V \subset Q(A^sB)$, $Z \supset P(A^sB)$ which completes the proof.

(2,9) *Let* $|A| = |B| = 1$, $AB = BA$ *and let* A *be irreducible. If* $P(B)$ *has* $s < n$ *elements, then* $|A^sB| < 1$.

Proof. By the preceding result, there exists a set Z such that $P(A^sB) \subset \subset Z \subset P(B)$ and $\varphi(Z) \subset Z$. Since φ is irreducible and $Z \subset P(B) \neq N$, we have $Z = 0$ whence $P(A^sB) = 0$.

3. NORMS AND PROPER VALUES

If $|A| \leqq 1$ and if α is a proper value of A, then there exists a vector x such that $|x| = 1$, $Ax = \alpha x$; it follows that $|\alpha| = |\alpha x| = |Ax| \leqq |A| \cdot |x| \leqq 1$. If $|\alpha| = 1$, we have $|A^r| \geqq |A^rx| = |\alpha^rx| = 1$ and, consequently, $|A^r| = 1$ for every r.

We say that the matrix A is *conservative*, if $|A| = 1$ and if $|\alpha| = 1$ for some proper value of A. We have then $|A^r| = 1$ for every r. On the other hand, the main result of this section asserts that a matrix A of order n is conservative if $|A| = |A^{n^2-n+1}| = 1$. In other words: if $|A| = 1$ and A is not conservative, we have $|A^{n^2-n+1}| < 1$ so that A^r converges to the zero matrix.

(3,1) *Let* s *be a natural number. If* $|A| = 1$ *and if* A^s *is conservative, then* A *is conservative as well.*

Proof. There exists an α such that $|\alpha| = 1$, $\det (A^s - \alpha^s E) = 0$. Let $\varepsilon_1, \ldots, \varepsilon_s$ be all s-th roots of 1. Then $0 = \det (A - \varepsilon_1 \alpha E) \ldots \det (A - \varepsilon_s \alpha E)$ so that $\det (A - \varepsilon_j \alpha E) = 0$ for some j.

(3,2) *Let A be irreducible; let $|\alpha| = 1$, $|x| = 1$, $Ax = \alpha x$. Then $P(A) = P(x) = N$, $a_{ik} = \alpha x_i \bar{x}_k |a_{ik}|$.*

Proof. By lemma (2,3), we have $0 \neq P(x) \supset \varphi(P(Ax)) = \varphi(P(x))$, so that $N = P(x) = P(Ax) \subset P(A)$. Since $|x_i| = 1$, $\sum_k |a_{ik}| = 1$, $\sum a_{ik} x_k = \alpha x_i$, we have $a_{ik} x_k = |a_{ik}| \alpha x_i$.

(3,3) Theorem. *Let A be an irreducible matrix. Then A is conservative if and only if $A = \alpha DMD^{-1}$ where α is a complex number with $|\alpha| = 1$, $D = (d_{ik})$ is a diagonal matrix with $|d_{ii}| = 1$ and $M = (m_{ik})$ is a non-negative matrix with $\sum_k m_{ik} = 1$ for $i = 1, 2, \ldots, n$.*

Proof. This is an immediate consequence of (3,2).

(3,4) Notation. If A is a matrix and if $V \subset N$, we shall denote by A_V the matrix with elements a_{ik}, where $i, k \in V$. If $\varphi(V) \subset V$, then obviously $(A_V)^r = (A^r)_V$ for every natural r.

(3,5) Theorem. *Let $|A| \leq 1$ and let $\{T_1, \ldots, T_r\}$ be the family of all minimal nonvoid $T \subset N$ such that $\varphi(T) \subset T$, where $\varphi = F(A)$. Put $V = N - \bigcup T_j$. Then*

1° *$T_i \cap T_j = 0$ for $i \neq j$,*
2° *the matrices A_{T_j} are irreducible,*
3° *the relations $Z \subset V$, $\varphi(Z) \subset Z$ imply $Z = 0$,*
4° *each proper value α of A such that $|\alpha| = 1$ is a proper value of some A_{T_j},*
5° *if $V \neq 0$, then A_V is not conservative.*

Proof. If $T_i \cap T_j = T \neq 0$, then $\varphi(T) \subset \varphi(T_i) \cap \varphi(T_j) \subset T$, whence $T = T_i = T_j$, so that 1° is proved; 2° and 3° are obvious. Let now $|\alpha| = 1$, $|x| = 1$, $Ax = \alpha x$. Since $P(x) \supset \varphi(P(Ax)) = \varphi(P(x))$ by (2,3), we have $T_j \subset P(x)$ for some j; it follows that α is a proper value of A_{T_j} which proves 4°. Now suppose that $V \neq 0$ and let β be a proper value of A_V with $|\beta| = 1$. According to what has just been proved there exists a non-void $W \subset V$ such that $V \cap \varphi(W) \subset W$, that A_W is irreducible and that β is a proper value of A_W. It follows from (3,2) that $P(A_W) = W$ so that $\varphi(W) \subset W$. We see that $W \supset T_j$ for some j in contradiction with the relation $W \subset V \subset N - T_j$. This completes the proof.

(3,6) Notation. If c is a complex number different from 0, put $\sigma(c) = \dfrac{c}{|c|}$.

(3,7) *Let $|A| = 1$, $|x| = |y| = 1$ and $P(Ax) = P(Ay) = N$. Let $\varphi = F(A)$ be irreducible. If $j\mathbf{E}(\varphi)k$[1]) and $x_j = \alpha y_j$ then $x_k = \alpha y_k$ and $|\alpha| = 1$.*

[1]) See definition (1,16).

Proof. We may clearly limit ourselves to the case when j and k are contained in some $\varphi(i)$; then $a_{ij}a_{ik} \neq 0$. Now $|\sum_r a_{ir}x_r| = |\sum_r a_{ir}y_r| = 1$; it follows that $|x_j| = |x_k| = |y_j| = |y_k| = 1$ whence $|\alpha| = 1$. Clearly $\sigma(a_{ij}x_j) = \sigma(a_{ik}x_k)$ and $\sigma(a_{ij}y_j) = \sigma(a_{ik}y_k)$. Further, $x_k\sigma(a_{ik}) = \sigma(a_{ik}x_k) = \sigma(a_{ij}x_j) = \alpha\sigma(a_{ij}y_j) = \alpha\sigma(a_{ik}y_k) = \alpha y_k\sigma(a_{ik})$ whence $x_k = \alpha y_k$ which completes the proof.

(3,8) Theorem. *Let A be primitive, $|A| = 1$, $|x| = |y| = 1$. If $P(A^{n-1}x) = P(A^{n-1}y) = N$, then there exists a complex number α such that $y = \alpha x$, $|\alpha| = 1$.*

Proof. Put $\omega = F(A^{n-1})$. It follows from the relation $P(A^{n-1}) \supset P(A^{n-1}x) = N$ and from (2,5) that $\omega = (F(A))^{n-1}$; since A is primitive, theorem (1,22) implies that there cannot be more than one class of the equivalence $\mathbf{E}(\omega)$. The rest follows from (3,7) applied to A^{n-1}.

(3,9) *Let $n > 2$ and let A be the matrix*

$$\begin{pmatrix} 0 & a_1 & 0 & \ldots & 0 \\ 0 & 0 & a_2 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & a_{n-1} \\ p & q & 0 & \ldots & 0 \end{pmatrix}$$

where $pq \neq 0$. Suppose that $|A| \leqq 1$, $|x| \leqq 1$ and $|y_2| = \ldots = |y_n| = 1$, where $y = A^n x$. Then A is conservative.

Proof. It is easy to see that $\det (tE - A) = t^n - \mu t - \lambda$ where $\lambda = a_1 \ldots a_{n-1}p$, $\mu = a_2 \ldots a_{n-1}q$. By the Hamilton-Cayley theorem, we have $A^n = \lambda E + \mu A$ whence

$$y_j = \lambda x_j + \mu a_j x_{j+1} \quad \text{for } j = 1, \ldots, n-1\,,$$
$$y_n = \lambda x_n + \mu p x_1 + \mu q x_2\,.$$

Now $1 = |y_2| \leqq |\lambda| + |\mu| \leqq |p| + |q| \leqq 1$ whence $|p| + |q| = 1$. Suppose that $|a_1 \ldots a_{n-1}| < 1$. Then $|\lambda| = |a_1 \ldots a_{n-1}| \cdot |p| < |p|$ whence $1 = |\lambda| + |\mu| < |p| + |q| = 1$; this contradiction shows that $|a_1| = \ldots = |a_{n-1}| = 1$. Since $1 = |y_j| = |\lambda x_j + \mu a_j x_{j+1}|$ for $j = 2, 3, \ldots, n-1$, we have $|x_3| = \ldots = |x_n| = 1$. Since $1 = |y_n| = |\lambda x_n + \mu p x_1 + \mu q x_2|$, we have $|x_1| = |x_2| = 1$ as well so that $P(x) = N$. It is easy to see that $y_j = \sigma(\lambda x_j) = \sigma(\lambda)\, x_j$ for $j = 2, 3, \ldots, n$; since $y_n = \lambda x_n + \mu p x_1 + \mu q x_2$, we infer that $y_n = \sigma(\mu p x_1) = \sigma(\mu q x_2)$. It follows that $\sigma(p x_1) = \sigma(q x_2)$. Now $y_1 = \lambda x_1 + \mu a_1 x_2 = a_1 \ldots a_{n-1}(p x_1 + q x_2)$; hence $|y_1| = |p x_1 + q x_2| = |p x_1| + |q x_2| = |p| + |q| = 1$ so that $y_1 = \sigma(\lambda)\, x_1$ as well. We have thus $y = \sigma(\lambda)\, x$ which implies that A^n is conservative. The rest follows from lemma (3,1).

(3,10) *Let A be a primitive matrix of order 2 such that $|A| = |A^3| = 1$. Then A is conservative.*

Proof. We have $a_{12} a_{21} \neq 0$ and at least one of the diagonal elements is different from zero; consider the case $a_{11} \neq 0$. There exists an x such that $|x| = 1$ and $|A^3x| = 1$. Put $y = Ax$, $z = Ay$. Since $|Az| = |Ay| = 1$ and $a_{11}a_{21} \neq 0$, we have $|y_1| = |z_1| = 1$. Now $a_{11}a_{12} \neq 0$ and $|a_{11}y_1 + a_{12}y_2| = |a_{11}x_1 + a_{12}x_2| = 1$. Clearly this is impossible unless $y = \alpha x$ for some α of modulus 1.

(3,11) Notation. For every non-negative integer r put $q(r) = r^2 - r + 1$.

(3,12) *Let A be a primitive matrix such that $|A| = |A^{q(n)}| = 1$. Then A is conservative.*

Proof. If $n = 2$, then our assertion reduces to the preceding result. Now let $n > 2$ and put $s = (n - 1)^2$, $\varphi = F(A)$, $B = A^s$. According to (2,9), we have $P(B) = N$ so that $F(B) = \varphi^s$ by lemma (2,5). Take a vector x such that $|x| = 1$ and $|A^{q(n)}x| = |BA^nx| = 1$. We shall distinguish two cases.

1° We have $\varphi^s(i) = N$ for every $i \in N$. From the relation $F(B) = \varphi^s$ it follows that $b_{ik} \neq 0$ for all i, k. Since $|BA^{n-1}x| = |BA^nx| = 1$, we see that $P(A^{n-1}x) = P(A^{n-1}Ax) = N$. According to theorem (3,8) there exists an α such that $|\alpha| = 1$, $Ax = \alpha x$.

2° The set $\varphi^s(j)$ is different from N for some j. In virtue of theorem (1,15) we can suppose that A has the form described in (3,9). We have then $\varphi^s(i) = N$ for $i = 2, 3, \ldots, n$, while $\varphi^s(1) = \{2, 3, \ldots, n\}$ so that $b_{ik} \neq 0$ for every pair of indexes except $i = k = 1$. Since $|By| = 1$ for $y = A^nx$, we have $|y_2| = \ldots = |y_n| = 1$ and A is conservative by lemma (3,9).

(3,13) *For $d = 1, \ldots, n$ we have*

$$2n - 1 \leqq \frac{n(n-1)}{d} + d \leqq q(n)\,.$$

Proof. For $t > 0$ put $g(t) = \dfrac{n(n-1)}{t} + t$. We have $g(1) = q(n)$, $g(n) = 2n - 1$; if $n > 1$ then $g(n - 1) = 2n - 1$, too. Our assertion is thus proved for $n = 1$. If $n \geqq 2$ and $1 \leqq t \leqq n - 1$ then $g'(t) = -\dfrac{n(n-1)}{t^2} + 1 < 0$, whence $g(n) = g(n - 1) \leqq g(t) \leqq g(1)$ which completes the proof.

(3,14) Theorem. *Let A be an irreducible matrix of order n; let d be the index of imprimitivity of $F(A)$. If $|A^r| = 1$ for every integer r such that $1 \leqq r \leqq \dfrac{n(n-1)}{d} + d$, then A is conservative.*

Proof. Put $B = A^d$, $\varphi = F(A)$, $\omega = F(B)$. Since $n - 1 + d \leqq \dfrac{n}{d}(n - 1) + d$, we have $|BA^{n-1}| = |A^{d+n-1}| = 1$. In accordance with lemmas (2,9) and (2,5) we obtain $P(B) = N$, $\omega = \varphi^d$. Let $\{M_1, \ldots, M_d\}$ be the finest cyclic partition with respect to φ. We have $\omega(M_j) = M_j$ for $j = 1, \ldots, d$ and it follows from theorem (1,9) that the matrices $B_j = B_{M_j}$ are primitive. Let n_j be the order of B_j. There exists a j such that $n_j \leqq \frac{n}{d}$; put $s = q(n_j)$. Suppose that $|B_j^s| < 1$. Then $P(B^s)$ has at most $n - n_j$ elements, whence $|A^{ds+n-n_j}| < 1$ by lemma (2,9). On the other hand, we have $ds + n - n_j \leqq \frac{n}{n_j}(n_j^2 - n_j + 1) + (n - n_j) = f(n_j)$, where $f(t) = (n-1)\,t + \frac{n}{t}$. According to lemma (3,13) we have $f(1) = 2n - 1 \leqq \frac{n(n-1)}{d} + d = f\left(\frac{n}{d}\right)$; since $f''(t) > 0$ for $t > 0$, we obtain $f(n_j) \leqq \max\left(f(1), f\left(\frac{n}{d}\right)\right) = \frac{n(n-1)}{d} + d$, whence $|A^{ds+n-n_j}| = 1$. This contradiction proves that $|B_j^s| = 1$. By lemma (3,12) B_j is conservative. It follows that $B = A^d$ is conservative; by lemma (3,1) A is conservative as well.

(3,15) *Let A be an irreducible matrix such that $|A| = |A^{q(n)}| = 1$. Then A is conservative.*

Proof. This is an easy consequence of lemma (3,13) and theorem (3,14).

(3,16) Theorem. *Let A be a reducible matrix of order n such that $|A| = |A^{q(n-1)+1}| = 1$. Then A is conservative.*

Proof. Let $T_1, \ldots, T_r, V$ be the sets from theorem (3,5). We intend to show that at least one of the matrices A_{T_j} is conservative. Suppose not. Let n_j be the number of elements of T_j; put $r = \max t_j$, $B = A^{q(r)}$. Let $P(B)$ have s elements. We infer from lemma (3,15) that $|A_{T_j}^{q(r)}| < 1$ for all j so that $P(B) \subset N - \bigcup T_j$, whence $s \leqq n - r$. It follows from (2,8) that there exists a $Z \subset N$ such that $\varphi(Z) \subset Z$, $P(A^sB) \subset Z \subset P(B)$. Since $P(B) \subset V$, theorem (3,5) implies that $0 = Z = P(A^sB)$, whence $|A^{q(r)+s}| = |A^sB| < 1$. On the other hand, we have $s \leqq n - r$, so that $q(r) + s \leqq q(r) + n - r = (r-1)^2 + n \leqq (n-2)^2 + n = q(n-1) + 1$ and hence $|A^{q(n-1)+1}| < 1$. This contradiction proves that at least one of the matrices A_{T_j} is conservative; consequently, the matrix A itself is conservative.

(3,17) Theorem. *Let A be a matrix of order n such that $|A| = |A^{n^2-n+1}| = 1$. Then A is conservative.*

Proof. If A is irreducible we make use of lemma (3,15). If A is reducible we have $n > 1$ so that $q(n-1) < q(n)$ and we may apply theorem (3,16).

(3,18) *Let $n > 1$; put*

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & \ldots & 1 \\ p & q & 0 & \ldots & 0 \end{pmatrix}$$

where $|p| + |q| = 1$, $pq \neq 0$ *and* $\sigma(p^{n-1}) \neq \sigma(q^n)$.[2]) *Then* $|A| = |A^{n^2-n}| = 1$ *and* $|A^{n^2-n+1}| < 1$.

Proof. Since $\det(tE - A) = t^n - qt - p$, we have $A^n = pE + qA$ so that $A^{n(n-1)} = \sum_{r=0}^{n-1} \binom{n-1}{r} q^r p^{n-1-r} A^r$. Let $a_i^{(r)} = [a_{i1}^{(r)}, \ldots, a_{in}^{(r)}]$ be the i-th row of A^r. It is evident that $a_j^{(1)} = a_{j-1}^{(2)} = \ldots = a_1^{(j)}$; if $r < n$, we have therefore $a_{1,r+1}^{(r)} = 1$, $a_{1,k}^{(r)} = 0$ for $k \neq r + 1$. It follows that $\sum_{j=1}^{n} |a_{1,j}^{(n(n-1))}| = \sum_{r=0}^{n-1} |a_{1,r+1}^{(n(n-1))}| = \sum_{r=0}^{n-1} \binom{n-1}{r} |q^r p^{n-1-r}| = (|p| + |q|)^{n-1} = 1$, whence $|A^{n(n-1)}| = 1$. Suppose now that $|A^{n^2-n+1}| = 1$. Theorem (3,17) implies that there exists an α such that $|\alpha| = 1$ and $\alpha^n - q\alpha - p = \det(\alpha E - A) = 0$. Since $\alpha^n = q\alpha + p$, we have $\sigma(p) = \sigma(q\alpha)$ whence $\alpha = \sigma(pq^{-1})$, $\sigma(p^n q^{-n}) = \alpha^n = \sigma(p)$ and $\sigma(p^{n-1}) = \sigma(q^n)$ contrary to our assumption. The proof is complete.

(3,19) Theorem. *For every natural number n there exists a matrix A of order n such that* $|A| = |A^{n^2-n}| = 1$ *and* $|A^{n^2-n+1}| < 1$.

Proof. Our assertion is trivial for $n = 1$. If $n > 1$, we can take the matrix A from (3,18) where we put $p = q = -\frac{1}{2}$.

4. MATRICES WITH DIAGONAL ELEMENTS DIFFERENT FROM ZERO

(4,1) *Let A be irreducible. Let* $a_{ii} \neq 0$ *and* $\sigma(a_{ii}) = \alpha$ *for* $i = 1, 2, \ldots, n$. If $|A^n| = 1$, *then there exists a vector x such that* $|x| = 1$, $P(x) = N$ *and* $Ax = \alpha x$.

Proof. Put $\varphi = F(A)$. We note first that $\varphi(S) \supset S$ for every $S \subset N$. Since $|A^n| = 1$, there exists a vector x such that $|x| = 1$ and $|A^n x| = 1$. For $r = 0, 1, \ldots, n$ put $P_r = P(A^r x)$. We have $P(A^{r-1}x) \supset \varphi(P(A^r x)) \supset P(A^r x)$ by (2,3) so that $P_n \subset P_{n-1} \subset \ldots \subset P_1 \subset P_0$. Since $P_n \neq 0$, there exists a natural r such that $P_r = P_{r-1}$. Now $P_{r-1} \supset \varphi(P_r) \supset P_r$, whence $\varphi(P_r) = P_r$. Since $P_r \supset P_n \neq 0$ and φ is irreducible, we have $P_r = N$ so that $P_1 = P_0$. It follows that $P(x) = P(y) = N$ where $y = Ax$. Now $y_i = \sum_k a_{ik} x_k$, $|y_i| = 1$ and $\sum_k |a_{ik}| \leqq 1$, whence $a_{ik} x_k = |a_{ik}| \, y_i$ for every i and k. Especially, $a_{ii} x_i = |a_{ii}| \, y_i$ for every i. Since $a_{ii} = \alpha \, |a_{ii}| \neq 0$, it follows that $y = \alpha x$ which completes the proof.

(4,2) *Let A be irreducible,* $|A| \leqq 1$. *Suppose that* $a_{ii} \neq 0$ *for* $i = 1, 2, \ldots, n$ *but that* $\sigma(a_{pp}) \neq \sigma(a_{qq})$ *for some p and q. Then* $P(A^2)$ *has at most* $n - 2$ *elements and* $|A^n| < 1$.

[2]) If $n = 2$ we have $A = \begin{pmatrix} 0 & 1 \\ p & q \end{pmatrix}$.

Proof. Let us denote by S the set of those $j \in N$ for which $\sigma(a_{jj}) = \sigma(a_{11})$ so that $1 \in S$. At the same time $S \neq N$. Let $T = N - S$. The mapping $\varphi = F(A)$ being irreducible, we have $\varphi(T) \cap S \neq 0 \neq \varphi(S) \cap T$ so that there exist points t, s, v, w such that $t \in T$, $s \in S$, $s \in \varphi(t)$ and $v \in S$, $w \in T$, $w \in \varphi(v)$. Further, $\sigma(a_{tt}) \neq \sigma(a_{ss}) = \sigma(a_{vv}) \neq \sigma(a_{ww})$. Since $a_{ts} \neq 0$, we have $\sigma(a_{ts}a_{ss}) \neq \sigma(a_{tt}a_{ts})$ so that $|b_{ts}| = |\sum_j a_{tj}a_{js}| < \sum_j |a_{tj}| \cdot |a_{js}|$ where $B = A^2$. Now $\sum_k |b_{tk}| < \sum_k \sum_j |a_{tj}| \cdot |a_{jk}| \leqq 1$ whence $t \notin P(B)$. In a similar manner, it is possible to show that $v \notin P(B)$. Since $t \in T$ and $v \in S$, the set $P(B)$ contains at most $n - 2$ elements so that $|A^n| = |A^{n-2}B| < 1$ by lemma (2,9).

(4,3) Theorem. *Let $A = (a_{ik})$ be a matrix of order n such that $a_{ii} \neq 0$ for $i = 1, 2, \ldots, n$. If $|A| = |A^n| = 1$, then A is conservative.*

Proof. If A is irreducible, our assertion is an immediate consequence of the preceding results. If A is reducible, we may apply the method used in the proof of theorem (3,16); we write, of course, r instead of $q(r)$ and we make use of theorem (4,3) for irreducible matrices instead of lemma (3,15).

Let us conclude with the following remark as an illustration.

Let B be a complex matrix of order n with elements b_{ik} such that $\det B \neq 0$ and $|b_{ii}| \geqq \sum_{k \neq i} |b_{ik}|$ for $i = 1, 2, \ldots, n$. Consider the system of equations

$$Bx = y \tag{4}$$

and the equivalent system

$$x = Ax + y \tag{5}$$

where $A = E - B$.

Clearly we may suppose that $0 < b_{ii} < 1$ for all i. We have then $\sum_k |a_{ik}| = 1 - b_{ii} + \sum_{k \neq i} |b_{ik}| \leqq 1$, so that $|A| \leqq 1$ and, moreover, $a_{ii} = 1 - b_{ii} > 0$ for all i. Let α be a proper value of A. There exists a vector $x = [x_1, \ldots \ldots, x_n]$ such that $Ax = \alpha x$, $|x_i| \leqq 1$ for all i and $x_j = 1$ for some j. It follows that $\alpha = \alpha x_j = \sum_k a_{jk}x_k$, whence $|\alpha| \leqq 1$. Since $\det(E - A) = \det B \neq 0$, we have $\alpha \neq 1$; further we observe that $a_{jj}x_j > 0$. If $a_{jk}x_k \geqq 0$ for all k, then $0 < \alpha < 1$, so that $|\alpha| < 1$; otherwise $|\alpha| = |\sum_k a_{jk}x_k| < \sum_k |a_{jk}| \leqq 1$, so that $|\alpha| < 1$ again. It follows that the iterative method $x^{(r+1)} = Ax^{(r)} + y$ converges and, according to the results of the fourth section, $|A^n| < 1$.

Резюме

НОРМЫ, СПЕКТРЫ И КОМБИНАТОРНЫЕ СВОЙСТВА МАТРИЦ

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Пусть A — матрица порядка n с комплексными элементами a_{ik} и с нормой $|A| = \max\limits_i \sum\limits_k |a_{ik}|$. Рассмотрим уравнение $x = Ax + y$. Для того, чтобы процесс итерации $x_{n+1} = Ax_n + y$ был сходящимся для любого начального вектора x_0 и любого y, необходимо и достаточно, чтобы ряд $E + A + A^2 + \ldots$ сходился. Хорошо известно, что для этого необходимо и достаточно, чтобы имело место неравенство $|\lambda| < 1$ для любого собственного числа λ матрицы A. Это условие, очевидно, выполняется, если $|A| < 1$ или, более общо, если $|A^p| < 1$ для какого-либо p. Итак, мы приходим к следующей проблеме:

Пусть дана матрица A с нормой $|A| = 1$ и построим последовательность $A, A^2, A^3, \ldots$. Очевидно, будет $|A| \geqq |A^2| \geqq |A^3| \geqq \ldots$, так что возможны следующие два случая: или $|A^r| = 1$ для каждого натурального r или $|A^p| < 1$ для какого-либо p; тогда A^r, очевидно, стремится к нулевой матрице. Итак, возникает вопрос, сколько членов последовательности $|A|$, $|A^2|$, … нужно рассмотреть, чтобы решить, какая из указанных двух возможностей имеет место.

Главным результатом настоящей работы является следующая теорема:

Пусть A — комплексная матрица порядка n с элементами a_{ik}; пусть $|A| = 1$ и предположим, что выполняется одно из следующих трех условий:

1° $|A^{n^2-n+1}| = 1$;

2° *матрица A разложима и* $|A^{n^2-3n+4}| = 1$;

3° $a_{ii} \neq 0$ *для* $i = 1, \ldots, n$ *и* $|A^n| = 1$.

Тогда $|A^r| = 1$ для всех r.

Если положим

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & \ldots & 0 \end{pmatrix},$$

то $|A^{n^2-n}| = 1$ *и* $|A^{n^2-n+1}| < 1$.

Сущностью этого результата является исследование комбинаторной структуры матриц. Доказательство главной теоремы разделяется на сорок восемь частичных результатов, из которых большинство имеет самостоятельное значение. Чисто комбинаторные результаты собраны в первом параграфе; при надлежащем истолковании этот параграф содержит существенную часть теории неотрицательных матриц.